Interpretable groups are definable

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We prove that in an arbitrary o-minimal structure, every interpretable group is definably isomorphic to a definable one. We also prove that every definable group lives in a cartesian product of one-dimensional definable group-intervals (or one-dimensional definable groups). We discuss the general open question of elimination of imaginaries in an o-minimal structure.

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1. Introduction

Elimination of imaginaries, namely the ability to associate a definable set to every quotient of another definable set by a definable equivalence relation, plays a major role in modern model theory. In the study of o-minimal structures this issue is often avoided by making the auxiliary assumption that the structure expands an ordered group. Indeed, this assumption resolves the matter because o-minimal expansions of ordered groups eliminate imaginaries in a very strong form (see [1, Proposition

6.1.2]), namely every A-definable equivalence relation has an A-definable set of representatives, after naming a non-zero element of the group. In particular, the structure has definable Skolem functions. Since most interesting examples of o-minimal structures do expand ordered groups and even ordered fields, this assumption seems reasonable for most purposes.

Recently, following the work on Pillay's Conjecture, it was shown in [2, Corollary 8.7] that even when starting with a definable group G in expansions of real closed fields, the group G/G^{00} is definable in an o-minimal structure over $\mathbb{R}$ which is not known *a priori* to expand an ordered group.

In this paper we are concerned with the two issues raised above. First, we are interested to know to what extent o-minimal structures in general eliminate imaginaries. Second, we show that in many cases the assumption that the underlying structure expands a one-dimensional group is indeed harmless because such a group is already definable in our structure (actually, we might need several different such groups).

Let us be more precise now. Definable equivalence relations can be treated either within the many-sorted structure $\mathcal{M}^{eq}$, or explicitly, as definable objects in $\mathcal{M}$. In order to apply o-minimality, we mostly work in the latter context, so we clarify some definitions. We assume we work in an arbitrary o-minimal structure.

Definition 1.1. Let X, Y be definable sets, E_1, E_2 two definable equivalence relations on X and Y, respectively. A function $f : X/E_1 \to Y/E_2$ is called *definable* if the set $\{\langle x, y\rangle \in X \times Y : f([x]) = [y]\}$ is definable.

Definition 1.2. Let E be a definable equivalence relation on a definable set X, where both X and E are defined over a parameter set A. We say that the quotient X/E *can be eliminated over* A if there exists an A-definable injective map $f : X/E \to M^k$, for some k. We say in this case that f *eliminates* X/E *over* A.

It was already observed in [3] that quotients cannot in general be eliminated in o-minimal structures, over arbitrary parameter sets. Indeed, consider the expansion of the ordered real numbers by the equivalence relation on $\mathbb{R}^2$ given by: $\langle x, y\rangle \sim \langle z, w\rangle$ if and only if $x - y = z - w$. This quotient cannot be eliminated over $\emptyset$. However, once we name any element a, the map $f(\langle x, y\rangle/\sim) = a + x - y$ is definable and eliminates this quotient (over a).

It is therefore reasonable to ask:

Question 1.3. Given an o-minimal structure $\mathcal{M}$ and a definable equivalence relation E on a definable set X, both defined over a parameter set A, is there a definable map which eliminates X/E, possibly over some $B \supseteq A$?

It was shown in [3, Proposition 3.2] that this question has a positive answer when $\mathcal{M}$ is modular or the dcl operation yields a substructure. Here, we give a positive answer when $\dim(X/E) = 1$ (see Corollary 7.8), but the general question remains open. We also note that recent work [4] shows that VC-minimal theories have elimination of imaginaries for one-dimensional sets, though this is a standard fact in the o-minimal context.

Definition 1.4. An *interpretable group* is a group whose universe is a quotient X/E of a definable set X by a definable equivalence relation E, and whose group operation is a definable map.

As we will see below, we prove in this paper that interpretable groups can be eliminated.

1.1. *Groups in o-minimal structures*

The analysis of definable groups in o-minimal structures depends to a large extent on a theorem of Pillay, [5], about the existence of a definable basis for a group topology. The theorem holds for definable groups, but until now it was not clear how to treat interpretable groups. In [6, Proposition 7.2], Edmundo was able to circumvent part of this problem by showing that if a group G is already definable in an o-minimal structure $\mathcal{M}$ then $\mathcal{M}$ has elimination of imaginaries for definable subsets of G. This makes it possible to handle interpretable groups which are obtained as quotients of one definable group by another. But interpretable groups in general remained out of reach (see Appendix in [2] for the technical difficulties which may arise).

The following theorem, which we prove here, reduces the study of interpretable groups to definable ones. (See Theorem 8.23(1) below.)

Theorem 1. *Every interpretable group is definably isomorphic to a definable group.*

In order to describe the main ideas of the proof we need to return to the second problem mentioned at the beginning of this introduction.

1.2. *Group-intervals*

As we already remarked, it is often convenient to assume that the o-minimal structure expands an ordered group. Beyond the experimental fact that most examples have this property, there is another justification for this assumption, related to the Trichotomy Theorem ([7]), as we discuss next.

Recall that a point $x \in M$ is called *nontrivial* if there exist open nonempty intervals $I, J \subseteq M$, with $x \in I$, and a definable continuous function $F : I \times J \to M$ such that F is strictly monotone in each variable separately (the original definition required $I = J$ but it is easy to see that the two are equivalent).

The Trichotomy Theorem implies that if $x \in M$ is non-trivial then there exists an open interval $I' \ni x$ that can be endowed with a definable partial group operation $+$, making I' into a *group-interval* (a technical definition will appear in Definition 3.1 below, but for now, we may think of a group-interval as an open interval $(-a, a)$ in an ordered divisible abelian group, endowed with the partial group operation). One complication is that the definition of the group operation on I' may require additional parameters.

Consider for example the expansion of the ordered real numbers by the ternary operation $x+y-z$, defined for all x, y, z with $|x-y|, |y-z|, |x-z| \leqslant 1$. In this structure and in elementarily equivalent ones every point a is non-trivial and contained in an a-definable group-interval. It can be shown that all definable group-intervals in elementarily equivalent structures have finite length. Moreover, the group-intervals can be 'far apart', meaning that there are no definable bijections between them.

In our current paper we propose a systematic treatment of the group-intervals which arise from the Trichotomy Theorem and suggest a technique of "stretching" these intervals as much as possible. We call an interval *group-short*, also written as *gp-short* (Definition 4.1) if it can be written as a finite union of points and open intervals, each of which endowed with the structure of a group-interval. After [9], we develop a pre-geometry based on the closure relation: $a \in cl(A)$ if there is a gp-short interval containing a whose endpoints are in $\mathrm{dcl}(A)$. Our main theorem here (Theorem 6.7) is:

Theorem 2. *Let $I, J \subseteq M$ be open intervals and assume that there exists a definable $F : I \times J \to M$ which is continuous in both variables and strictly monotone in each variable separately. Then either I or J (but possibly not both) is group-short.*

Let us make a few remarks on the above theorem:
1. The Trichotomy Theorem implies, under the above assumptions, that every point in I (and in J) is contained in a definable group-interval. However, each such interval may be defined over different parameters and furthermore there is no simple way to patch these potentially very small intervals into one "large" group-interval covering I (or J).
2. It is possible in the above theorem that I and J will not both be gp-short. Consider a nonstandard real closed field R, and in it an interval of infinite length $J = (-\alpha, \alpha)$ and an interval of standard length $I = (-1, 1)$. Now expand the linearly ordered structure $\langle R, < \rangle$ by the restriction of $+$ to $I \times J$. In this structure one can endow I with the structure of a group-interval, and moreover every interval of finite length is group-short, but the interval J is not group-short.

Let us now sketch the proof of Theorem 1 using Theorem 2. We start with an interpretable group G. Using elimination of one-dimensional quotients (Corollary 7.8) and Theorem 2, we first prove that every one-dimensional subset of G is group-short (Theorem 8.2). We also show, in Proposition 7.11, that there are definable maps $f^i : G \to M$, $i = 1, \dots, k$ and a definable set $X \subseteq \Pi_{i=1}^{k} f^i(G)$ such that G is in definable bijection with X/E for some definable equivalence relation E. Our final goal is to prove that each definable set $f^i(G)$ is a finite union of group-short intervals (in which case we can eliminate X/E).

To achieve that, we endow G with a group topology with a definable basis. This is done by identifying a neighborhood of a generic point in G with an open subset of $M^{\dim(G)}$. Next, just as with definable groups, we can use the distinction between definably compact interpretable groups and those which are not definably compact.

In the first case, we prove in Theorem 8.20 Definable Choice for definable subsets of G using Edmundo's ideas [6]. As a result, it follows that each $f^i(G)$ is group-short. This finishes the definably compact case. In the general case, we use induction on dimension, together with the standard analysis of groups definable in o-minimal structures as quotients of semisimple groups, torsion-free abelian groups, etc. This finishes our final goal and the proof of Theorem 1.

At the end of the argument we show not only that G is in definable bijection with a definable group, but also prove (Theorem 8.23(2)):

Theorem 3. *If G is an interpretable group then there is a definable injection $f : G \to \Pi_{i=1}^k J_i$, where each $J_i \subseteq M$ is a definable group-interval.*

There are also definable one-dimensional groups $H_1, \ldots, H_k$ and a definable set-injective map $h : G \to \Pi_{i=1}^k H_i$ (with no assumed connection between the group operations of G and of the H_i's).

Note that the group-intervals (or the groups) in the above result are not assumed to be orthogonal to each other, namely, there could be definable maps between some of them. However, the theorem might help in reducing problems about definable groups, such as Pillay's Conjecture, to structures which expand ordered groups, or at least group-intervals. As a first attempt, it would be interesting to see if one can prove, using Theorem 3, an analogue of the Edmundo-Otero theorem, [10], on the number of torsion points in definably compact abelian groups in arbitrary o-minimal structures.

Theorem 3 answers positively a question which Hrushovski asked the second author in past correspondence.

On the structure of the paper: In Section 2 we recall the Marker-Steinhorn theorem and apply it for our purposes. In Sections 3 and 4 we study various properties of group-intervals and then use these, in Section 5, to develop the pre-geometry of the gp-short closure. In Section 6 we prove Theorem 2 and in Section 7 we discuss quotients and their various properties. Finally, in Section 8 we analyze interpretable groups and prove Theorem 1 and Theorem 3.

On notation To avoid confusion we use $\langle a, b\rangle$ to denote an ordered pair and use (a, b) to denote an open interval.

2. Model theoretic preliminaries

Fix $\mathcal{M} = \langle M, <, \ldots\rangle$ an arbitrary (dense) o-minimal structure, with or without endpoints.

Fact 2.1. Assume that $M_1 \neq \emptyset$ is a subset of M with the following properties:

(i) $\mathrm{dcl}_{\mathcal{M}}(M_1) = M_1$.

(ii) The restriction of $<$ to M_1 is a dense linear ordering.

(iii) If M_1 has a maximum (minimum) point then $\mathcal{M}$ has a maximum (minimum) point.

Then $\mathcal{M}_1 = \langle M_1, < \ldots \rangle$ (with all restrictions of functions and relations from $\mathcal{M}$) is an elementary substructure of $\mathcal{M}$.

Proof. By (i), $\mathcal{M}_1$ is a substructure of $\mathcal{M}$. We need to show that every nonempty definable set in $\mathcal{M}$, over parameters in M_1, has a point in M_1. By induction, it is sufficient to show it for definable sets in one variable. Assume then that $X \subseteq M$ is definable over M_1. It is enough to handle the case where $X = (a, b)$ is an open interval with $a, b \in M_1 \cup \{\pm\infty\}$. If both a and b are in M_1 then by the density assumption $(a, b) \cap M_1 \neq \emptyset$. If $b = +\infty$ and $a \in M_1$ then there are two possibilities. If $a = \max(M_1)$ then by assumption (ii), $b = \max(M) \in \mathrm{dcl}(\emptyset)$, so $b \in M_1$, contradiction. If a is not the maximum of M_1 then there is an element $a' \in M_1$ which belongs to $(a, +\infty)$ and we are done. We handle the other cases similarly. □

Proposition 2.2. *Assume that for all $a, b, c \in M$, there is no definable bijection between intervals of the form (a, b) and $(c, +\infty)$, and there is also no definable bijection between intervals of the form $(-\infty, a)$ and $(b, +\infty)$. Let $\mathcal{M} \prec \mathcal{N}$ and let $M_1 = \{x \in N : \exists m \in M \ m \geqslant x\}$ be the "downward closure" of $\mathcal{M}$ in $\mathcal{N}$. Then M_1 is a substructure of $\mathcal{N}$, and*

1. *$\mathcal{M}_1 \prec \mathcal{N}$.*
2. *If $X \subseteq N^k$ is an $\mathcal{N}$-definable set then $X \cap M_1^k$ is $\mathcal{M}_1$-definable.*

Proof. (1) By the choice of M_1 as the downward closure of an elementary substructure, M_1 satisfies (ii) and (iii) of Fact 2.1. It is therefore sufficient to prove that $\mathrm{dcl}_{\mathcal{N}}(M_1) = M_1$. The proof is similar to [11, Lemma 2.3].

As in [11], induction allows us to treat only the case of $b \in \mathrm{dcl}_{\mathcal{N}}(a)$, for $a \in M_1$. We must show that $b \in M_1$, so it is sufficient to find an element $m \in M$, with $b \leqslant m$. If $b \in \mathrm{dcl}_{\mathcal{N}}(\emptyset)$ then it is already in M so we are done. Otherwise, there is a $\emptyset$-definable, continuous, strictly monotone function $f : (a_1, a_2) \to M$, for $a_1, a_2 \in M \cup \{\pm\infty\}$, such that $a \in (a_1, a_2)$ and $b = f(a)$.

Assume first that f is strictly increasing on (a_1, a_2) and consider two cases: If $a_2 = +\infty$ then, by our construction of M_1, there exists $m \in (a_1, +\infty) \cap M$ with $a \leqslant m$. Hence $b = f(a) \leqslant f(m) \in M$. If $a_2 \in M$ then, by our assumptions, the limit $\ell = \lim_{t \to a_2^-} f(t)$ is in M (otherwise, there is a bijection of (a_1, a_2) and some interval $(a', +\infty)$) so we have $b \leqslant \ell$.

Assume now that f is strictly decreasing. Then, again by our assumptions on $\mathcal{M}$, the limit $\ell = \lim_{t \to a_1^+} f(t)$ is not $+\infty$. It follows that $\ell \in M$ and by monotonicity, $b \leqslant \ell$. We therefore showed that $\mathrm{dcl}_{\mathcal{N}}(M_1) = M_1$.

(2) Since $\mathcal{M}_1$ is convex in $\mathcal{N}$ it is clearly Dedekind complete in $\mathcal{N}$ and hence we can apply the Marker-Steinhorn theorem, [12], on definability of types which says exactly what we need. □

3. Group-intervals

Definition 3.1. By a *positive group-interval* $I = \langle (0,a), 0, +, < \rangle$ we mean an open interval with a binary partial continuous operation $+ : I^2 \to I$, such that

(i) $x + y = y + x$ (when defined), $(x + y) + z = x + (y + z)$ when defined, and $x < y \to x + z < y + z$ when defined.
(ii) For every $x \in I$ the domain of $y \mapsto x + y$ is an interval of the form $(0, r(x))$.
(iii) For every $x \in I$, we have $\lim_{x' \to 0} x' + x = x$ (this replaces the statement $0 + x = x$) and $\lim_{x' \to r(x)} x + x' = a$ (this replaces $x + r(x) = a$).

We say that I is a *bounded positive group-interval* if the operation $+$ is only partial. Otherwise we say that it is *unbounded* (in which case the interval is actually a semigroup). Note that the term *unbounded* only refers to the group-interval operation and does not mean that the interval is unbounded in the sense of the whole structure $\mathcal{M}$.

We similarly define the notion of a *negative group-interval* $\langle (a,0), +, < \rangle$ and also a *group-interval* $\langle (-a,a), +, < \rangle$ (in this case we also require that for every $x \in (-a,a)$ there exists a group inverse). We say that an open interval I is a *(bounded) generalized group-interval* if it is one of the above possibilities.

Our use of the symbols $0, a, -a$ is only suggestive. The endpoints of the interval can be arbitrary elements in $M \cup \{\pm\infty\}$, so when we write that an interval (b,c) is, say, a bounded group-interval, we think of the elements b and c as a and $-a$, respectively, from the definition.

Note 3.2. If the interval (a,b) can be endowed with a definable $+$ which makes it into a generalized group-interval then there is an ab-definable family of such operations (we just take the operation $+$ and vary the parameters which defined it, and further require the domain to be (a,b) and the operation to satisfy (i), (ii) and (iii) from the definition).

The following is easy to verify:

Fact 3.3.
(i) If (a,b) can be endowed with the structure of a bounded group-interval (either positive or negative) then we can also endow it with a structure of a bounded positive group-interval (making a into 0).
(ii) Conversely, if (a,b) can be endowed with a structure of a bounded positive group-interval then it can also be endowed with the structure of a bounded group-interval.
(iii) If I is a generalized group-interval then any nonempty open subinterval of I can be endowed with the structure of a generalized group-interval.

Theorem 3.4. *Assume that $\mathcal{M}$ is an o-minimal structure and let $I_t = (a_0, a_t)$, $t \in T$, be a definable family of intervals, all with the same left endpoint. Let $I =$*

$(a_0, a) = \bigcup_t I_t$. *If each interval* I_t *can be endowed with the structure of a generalized group-interval then there is* a_1, $a_0 \leqslant a_1 < a$ *such that* (a_1, a) *admits the structure of a generalized group-interval. In particular, we can write the interval* (a_0, a) *as a union of two generalized group-intervals.*

Proof. First note that if there exists some $a_1 \in [a_0, a)$ and a definable continuous injection sending (a_1, a) onto a subinterval $(a_2, a_3) \subseteq (a_0, a)$, with $a_2 < a_3 < a$, then (a_2, a_3) is contained in one of the intervals I_t and hence, by 3.3(iii), it inherits a structure of a generalized group-interval itself. Clearly then (a_1, a) can also be endowed with such a structure. We assume then that there is no such definable injection in $\mathcal{M}$.

Consider now the structure $\mathcal{I}$ which $\mathcal{M}$ induces on the interval $I = (a_0, a)$. By that we mean that the $\emptyset$-definable sets in $\mathcal{I}$ are the intersection of $\emptyset$-definable subsets of M^n with I^n. By [7, Lemma 2.3], every $\mathcal{M}$-definable subset of I^n is definable in $\mathcal{I}$ (the result is proved for closed intervals but the result for open intervals immediately follows). The points a_0 and a are now identified with $-\infty$ and $+\infty$ in the sense of $\mathcal{I}$, respectively. We may assume from now on that $\mathcal{M} = \mathcal{I}$.

Our above assumptions on $\mathcal{I}$ translate to the fact that $\mathcal{M}$ satisfies the assumptions of Proposition 2.2. Namely, that there are no $-\infty \leqslant a_1, a_2 < +\infty$ and $a_3 \in M$ for which $(a_1, +\infty)$ is in definable bijection with an interval of the form (a_2, a_3).

Using Note 3.2, we may assume that there is a $\emptyset$-definable family of (partial) operations $+_t : I_t \times I_t \to I_t$ making each I_t into a generalized group-interval. Indeed, to see that, we use the note to "blow up" each I_t to a t-definable family of group-intervals $\{I_{s,t} = \langle I_t, +_{s,t} \rangle : s \in S_t\}$, all of them with domain I_t. By compactness, we can show that as we vary $t \in T$ the family of S_t's and $+_{s,t}$ can be given uniformly. We now replace the original family $\{I_t\}$, with the family $\{I_{s,t} : t \in T \mathbin{\&} s \in S_t\}$, on which the group operations are given uniformly. Furthermore, we may assume that all intervals are either positive group-intervals, negative group-intervals, or group-intervals *uniformly* (we partition the family into the various sets and choose one whose union is still of the form $(-\infty, +\infty)$). For simplicity we still denote the intervals by I_t and the parameter set by T.

We first consider the case where each I_t is a positive group-interval (bounded or unbounded).

Each interval $I_t = (-\infty, a(t))$ is a positive group-interval (recall that in $\mathcal{M}$ the point $-\infty$ plays the role of 0). Furthermore, we have $\bigcup_{t \in T} I_t = (-\infty, \infty)$. Consider now a sufficiently saturated elementary extension $\mathcal{N}$ of $\mathcal{M}$ and take $a' < +\infty$ in N such that $a' > m$ for all $m \in M$. By our assumptions, there is $t_0 \in T(\mathcal{N})$ such that $(-\infty, a') \subseteq I_{t_0}$ and therefore there is a positive group-interval operation $+_{t_0}$ on the interval $(-\infty, a')$, which is definable in $\mathcal{N}$.

We now let $\mathcal{M}_1$ be the downward closure of M in $\mathcal{N}$ as in Proposition 2.2. By the same proposition, the intersection of the graph of $+_{t_0}$ with M_1^3, call it G, is a definable set in the structure $\mathcal{M}_1$.

Let's see first that in $\mathcal{M}_1$, the set G is the graph of a positive group-interval operation on $(-\infty, \infty)$ (with $-\infty$ playing the role of 0).

(1) G is the graph of a partial function from M_1^2 into M_1: this is clear since for every $(x, y) \in N^2$ there is at most one $z \in N$ such that $(x, y, z) \in G$. Call it $+_G$.

(2) $+_G$ is continuous, since the order topology of M_1 is the subspace topology of N and M_1 is convex in N.

(3) $+_G$ is associative and commutative when defined, as inherited from $\mathcal{N}$.

(4) $+_G$ respects order: again, inherited from $\mathcal{N}$.

(5) For every $x \in (-\infty, \infty)$, the domain of $y \mapsto x +_G y$ is a convex set in M_1 of the form $(-\infty, r_G(x))$: Indeed, the domain of $y \mapsto x +_{t_0} y$ in $\mathcal{N}$ is an interval $(-\infty, r_{t_0}(x))$. Hence, the domain of $y \mapsto x +_G y$ is the intersection of $(-\infty, r_{t_0}(x))$ with M_1. Since M_1 is closed downwards in $\mathcal{N}$, this intersection is $(-\infty, r_G(x))$, where $r_G(x) = +\infty$ if $r_{t_0}(x)$ is greater than all elements of M_1 and otherwise it is some element of M_1.

(6) Consider $\lim_{x' \to 0} x' +_G x$. Since this limit was $-\infty$ in $\mathcal{N}$ (i.e. 0 in the original structure), it remains so in $\mathcal{M}_1$, because M_1 was downwards closed in $\mathcal{N}$. It is left to see that $\lim_{x' \to r_G(x')} x +_G x' = \infty$ (i.e. a in the original structure). This follows from the fact that for every t we have $\lim_{x' \to r_t(x')} x +_t x' = a(t)$, and $\sup_t a(t) = \infty$.

We therefore showed that $+_G$ makes $(-\infty, +\infty)$ a positive group-interval in the structure $\mathcal{M}_1$.

Since $\mathcal{M} \prec \mathcal{M}_1$ we can now write down the (first-order) properties which make $+_G$ into an operation of a positive group-interval in M_1 and obtain an operation $+$ on M, which is definable in $\mathcal{M}$. This completes the case where each I_t is a positive group-interval.

Assume now that each I_t is a group-interval. If each I_t is bounded then, as we noted earlier we can transform it into a positive bounded group-interval and finish as above. If I_t is unbounded then $a_0 = -\infty_t$ and $a(t) = +\infty_t$. We can now fix some $a_1 \in (a_0, a)$ and restrict our attention to those t's for which $a_1 \in I_t$. For each such t we can endow $(a_1, a(t))$ with the structure of an unbounded positive group-interval, and then finish as above.

Finally, if each I_t is a negative group-interval (so $I_t = (a_0, a(t)) = (-\infty, a(t))$), then we can again assume that there is an a_1 which belongs to all I_t, and replace each I_t with the interval $(a_1, a(t))$, endowed with the structure of a bounded positive group-interval, and finish as above.

This ends the proof of the main part of the theorem, the last clause is easy to verify. $\square$

Note: We don't claim that the operation $+_G$ that we obtain in $\mathcal{M}_1$ belongs to the family $\{+_t : t \in T\}$ that we started with. E.g., in the structure $\langle \mathbb{R}, <, + \rangle$, take $+_t$ to be the restriction of the usual $+$ in $\mathbb{R}$ to an interval $I_t = (0, t)$. Each I_t is a bounded positive group-interval but the union $(0, +\infty)$ can only be endowed with the structure of an unbounded positive group-interval.

We end this section with an observation about group-intervals and definable groups.

Lemma 3.5. *Let $\langle I, +\rangle$ be a generalized group-interval. Then there exists a definable one-dimensional group $\langle H, \oplus\rangle$ and a definable injective $\sigma : I \to H$, such that $\sigma(x + y) = \sigma(x) \oplus \sigma(y)$, when $x + y$ is defined. Said differently, every generalized group-interval can be embedded into a definable one-dimensional group.*

If I is a bounded generalized group-interval, then H is definably compact and if I is unbounded then H is linearly ordered.

Proof. Assume that $I = (0, \infty)$ is an unbounded positive group-interval. Then we let $H = I \times \{-1\} \cup \{0\} \cup I \times \{+1\}$ (with $-1, 0, +1$ suggestive symbols for elements in M). We define $a \oplus 0 = a$ for every $a \in H$ and define $\langle x, i\rangle \oplus \langle y, j\rangle$ to be $\langle x + y, i\rangle$ if $i = j$. If $i \neq j$ and $x < y$ we let $\langle x, i\rangle \oplus \langle y, j\rangle = \langle z, j\rangle$, with $z \in I$ the unique element such that $x + z = y$. If $y < x$ then $\langle x, i\rangle \oplus \langle y, j\rangle = \langle z, i\rangle$, with z the unique element in I such that $y + z = x$. The group H we obtain is linearly ordered and torsion-free. Obviously I is embedded in H.

Assume now that $I = (0, a)$ is a positive bounded group-interval and let $a/2 \in I$ denote the unique element in I such that $\lim_{t\to a/2} t + t = a$. We consider H the half-open interval $[0, a/2)$ with addition "modulo $a/2$". Namely, for $x, y \in [0, a/2)$,

$$x \oplus y = \begin{cases} x + y & \text{if } x + y \in [0, a/2) \\ x + y - a/2 & \text{if } x + y \geqslant a/2 \end{cases}$$

The group H is a one-dimensional definably compact group. To see that I is embedded in H, consider the map $x \mapsto x/4$ sending I into $(0, a/4)$ (by $x/4$ we mean the unique element $y \in I$ such that $y + y + y + y = x$). It is easy to check that this is an embedding of I into H. □

In addition to the above we note that by [8, Proposition 5.2] (see there the preceding proof), every group-interval can be embedded into an ordered group which itself can be obtained as the direct limit of definable sets, so in general not definable.

4. Gp-short and gp-long intervals

4.1. *Definitions and basic properties*

We assume here that $\mathcal{M}$ is an arbitrary sufficiently saturated o-minimal structure.

Definition 4.1. An interval $I \subseteq M$ is called a *group-short (gp-short) interval* if it can be written as a finite disjoint union of points and open intervals, each of which can be endowed with the structure of a generalized group-interval. An interval which is not group-short is called a *group-long (gp-long) interval.*

Although there is no global notion of distance in M, in abuse of notation we say that the *distance between $a, b \in M$ is gp-short* if either $a = b$, or the interval (a, b) (or (b, a)) is gp-short. Otherwise, we say that this distance is gp-long.

Note that points are gp-short intervals according to the above.

Definition 4.2. A definable set $S \subseteq M^n$ is called a *gp-short set* if there are gp-short intervals $I_1, \ldots, I_k$ and a finite set F such that S is in definable bijection with union of F and a subset of $\Pi_j I_j$.

Remark Note that if S in the above definition is finite and there exists at least one generalized group interval in $\mathcal{M}$, call it I, then S in obviously in definable bijection with a subset of I (over parameters), so there is no need for F. Hence, the only reason to include F in the definition is to handle finite sets in case there are no generalized group intervals in $\mathcal{M}$.

Fact 4.1. (1) If X is an infinite gp-short set then X is in definable bijection, possibly over new parameters, with a subset of a cartesian product of generalized group-intervals (and not only gp-short intervals).
(2) A finite union of gp-short sets is a gp-short set.

Proof. (1) Since X is infinite there exists at least one generalized group interval. By definition, it is sufficient to prove the result for X an infinite gp-short interval. More generally, we will prove the result for X a finite union of gp-short intervals. Without loss of generality we can assume that the intervals are pairwise disjoint. We demonstrate the proof in the case that $X = I \sqcup J$, where I and J are generalized group intervals.

Fix $a \neq a' \in I$ and $b \neq b' \in J$. and embed $I \cup J$ into $I \times J$ by sending every $x \in I$ to $\langle x, b\rangle$, every $y \neq b \in J$ to $\langle a, y\rangle$ and by sending b to $\langle a', b'\rangle$. The proof is similar for a larger number of intervals and points.

(2) If each set in the union is finite then we get a finite set and we are done by definition. Assume then that at least one of the set gp-short sets is infinite. As remarked above we can now assume that each of the gp-short sets is a subset of some cartesian product of generalized group-intervals. So, now by (1), if X is a finite union of gp-short sets it can be definably embedded (injectively) into a finite union of cartesian products of generalized group-intervals. This union can be definably embedded into a cartesian product of finite unions of group-short intervals (e.g, $(I_1 \times I_2) \cup (J_1 \times J_2 \times J_3)$ can be embedded into $(I_1 \cup J_1) \times (I_2 \cup J_2) \times J_3$. By (1), this can be replaced by a finite cartesian product of generalized group-intervals, thus X is gp-short. □

Note 4.3.

1. It is not hard to see that an interval which is also a gp-short set can be written as a finite union of points and generalized group-intervals, hence it is a gp-short interval according to our earlier definition.
2. As in the case of generalized group-intervals, if (a, b) is a gp-short interval then it can be endowed with an ab-definable family of subintervals, with operations on them, witnessing the fact that (a, b) is gp-short. Indeed, we start with particular parameterically definable such witnesses and let the parameters (including the end points of the sub-intervals) vary.

3. It is of course possible that the only gp-short sets in $\mathcal{M}$ are finite, namely there are no definable generalized group-intervals in $\mathcal{M}$. The Trichotomy Theorem, [7], tells us that in this case the definable closure is trivial and every point in $\mathcal{M}$ is trivial. This is equivalent to the fact ([13]) that $\mathcal{M}$ has quantifier elimination down to $\emptyset$-definable binary relations.
4. Clearly, if I is a gp-short interval and $f : I \to M$ is a definable continuous injection then $f(I)$ is also a gp-short interval.

Fact 4.4. If $I_1, \ldots, I_k$ are gp-short intervals then, after fixing finitely many parameters A, the product $X = \Pi_j I_j$ has Definable Choice. Namely, if $\{S_t : t \in T\}$ is a B-definable family of subsets of X then there is an AB-definable function $\sigma : T \to X$ such that for every $t \in T$, we have $\sigma(t) \in S_t$ and if $S_{t_1} = S_{t_2}$ then $\sigma(t_1) = \sigma(t_2)$.

Proof. We write each I_j as a finite union of points and generalized group-intervals (possibly over extra parameters), and then repeat the standard proof of Definable Choice in expansions of ordered groups (see [1, Proposition 6.1.2]), using the group operations on each interval. □

Fact 4.5. Assume that $S \subseteq M^n$ is a gp-short set and $f : S \to M^k$ is a definable map. Then $f(S)$ is also a gp-short set.

Proof. If S is finite then we are done. Otherwise, by Fact 4.1, we may assume that $S \subseteq \Pi_j I_j$, for $I_1, \ldots, I_k$ generalized gp-intervals intervals. By Fact 4.4, there is a definable set $X_0 \subseteq S$ such that $f|X_0$ is a bijection between X_0 and $f(X_0) = f(S)$. By definition, $f(S)$ is a gp-short set. □

We now collect a list of important properties.

Fact 4.6. Let $\{I_t : t \in T\}$ be a definable family of intervals. Then

(i) The set of all $t \in T$ such that I_t is gp-long is type-definable
(ii) The set of all $t \in T$ such that I_t is gp-short is $\bigvee$-definable.
(iii) If $\bar{a} \in M^m$ is a tuple and the formula $\varphi(\bar{x}, \bar{a})$ defines a gp-short set in M^n, then there is a $\emptyset$-definable formula $\psi(\bar{x})$ such that $\psi(\bar{a})$ holds and if $\psi(\bar{b})$ holds then the set defined by $\varphi(\bar{x}, \bar{b})$ is gp-short.

Proof. For every natural number K and for every definable family of K functions $F_1(x, y, \bar{w}_1), \ldots, F_K(x, y, \bar{w}_K)$, we can write a formula which says: For every possible writing of I_t as a union of K intervals $I_1, \ldots, I_K$ and for every $\bar{w}_1, \ldots, \bar{w}_K$, it is not the case that $F_1(-, -, \bar{w}_1), \ldots, F_K(-, -, \bar{w}_K)$ are operations making $I_1, \ldots, I_K$, respectively, into group-intervals (here we need to go through the various possibilities of positive, negative group-intervals etc). When varying over all possible K's and all possible families, we obtain a type-definable definition for the set of t's for which I_t is gp-long. The complement of this set is $\bigvee$-definable.

For (iii), note that if (c, d) is a gp-short interval, then there is a formula $\rho(c, d)$ saying that (c, d) is the finite union of points and intervals, each of which is a generalized group-interval. Let $\theta(\bar{x}, \bar{x}', \bar{e})$ be an $\bar{e}$-definable bijection between $\varphi(M^n, \bar{a})$ and $\Pi_j I_j$ for some gp-short intervals I_j's. Let ρ_j be the formula witnessing that I_j is gp-short for each $j = 1, \ldots, m$. Then the desired formula $\psi(\bar{y})$ says that there exist parameters $\bar{w}$ such that $\theta(\bar{x}, \bar{x}', \bar{w})$ defines a bijection between $\varphi(M^n, \bar{y})$ and $\Pi_j(c_j, d_j)$ for some gp-short intervals (c_j, d_j), witnessed by formulas ρ_j for $j = 1, \ldots, m$. □

Theorem 4.7. *Let $\{S_t : t \in T\}$ be a definable family of gp-short, definably connected subsets of M^n and assume that there is $a_0 \in M^n$ such that for every $t \in T$, $a_0 \in \mathrm{Cl}(S_t)$. Then $S = \bigcup_t S_t$ is a gp-short set.*

Before we prove the result we note that the requirement about a_0 is necessary: Consider the structure on $\mathbb{R}$ with the restriction of the graph of $+$ to all $\langle a, b\rangle \in \mathbb{R}^2$ such that $|a - b| \leqslant 1$. In this structure (and in elementary extensions) there is a group-interval around every point so the whole structure is a definable union of gp-short intervals. However, the union (i.e. the universe) is not gp-short.

Proof. Let $\pi_i : M^n \to M$ be the projection onto the i-th coordinate. It is sufficient to show that each $\pi_i(S)$ is gp-short. Because S_t is definably connected, its projection $\pi_i(S_t) = I_t$ is an interval, which by Fact 4.5 is gp-short. Furthermore, $\pi_i(a_0) \in \mathrm{Cl}(I_t)$. Hence, we may assume from now on that $S_t = I_t$ is a gp-short interval in M and $a_0 \in \mathrm{Cl}(I_t)$ for every t. It is sufficient to prove that $I = \bigcup_t \mathrm{Cl}(I_t)$ is gp-short, so by replacing I_t with $\mathrm{Cl}(I_t)$ (still gp-short) we may assume that $a_0 \in I_t$ for all t. Let $a < b$ be the endpoints of I.

Claim 4.8. *There is $b_1 < b$ such that the interval (b_1, b) is gp-short.*

Proof. Since each I_t is a gp-short interval the type $p(t)$, which says that I_t is gp-long (see Fact 4.6), is inconsistent. It follows that there exists a fixed number K such that every I_t can be written as the union of at most K generalized group-intervals and K many points.

We write $I_t = I_{t,1} \cup \cdots \cup I_{t,K} \cup F_t$, such that each $I_{t,i} = (a_{t,i}, a_{t,i+1})$, $i = 1, \ldots, K$, can be endowed with the structure of a generalized group-interval, and F_t finite. The end points of the I_t's are definable functions of t and $b = \sup_t a_{t,K+1}$.

Let $a' = \sup_t a_{t,K}$ and assume first that $a' < b$.

We can restrict ourselves to those $t \in T$ such that $a_{t,K+1} > a'$ and consider each interval $(a', a_{t,K+1})$ as a sub-interval of $(a_{t,K}, a_{t,K+1})$. We already noted that $(a', a_{t,K+1})$ admits the structure of a generalized group-interval. So, we write a_t for $a_{t,K+1}$ and consider the family of all generalized group-intervals (a', a_t). By Theorem 3.4 there exists $b_1 < b$ such that (b_1, b) admits an operation of a generalized group-interval.

Assume now that $a' = \sup a_{t,K} = b$. In this case, we can replace each I_t by $I'_t = I_{t,1} \cup \cdots \cup I_{t,K-1}$, and still have $\bigcup_t I'_t = I$, and finish by induction on K. □

Just as we found b_1 above, we can find $a_1 > a$ such that (a, a_1) admits a definable generalized group-interval. Choose t_1 such that $I_{t_1} \cap (a, a_1) \neq \emptyset$ and t_2 such that $I_{t_2} \cap (b_1, b) \neq \emptyset$. Since $a_0 \in I_{t_1} \cap I_{t_2}$, the union of the two intervals is again an interval, containing (a_1, b_1), and therefore (a_1, b_1) is gp-short. We can conclude that (a, b) is gp-short.

As a corollary we obtain:

Corollary 4.9. *Let (a, b) be an interval which is gp-short.*

1. *Assume that $c \in (a, b)$. Then there exists a c-definable interval $I \supset (a, b)$ such that I is gp-short (possibly witnessed by extra parameters).*
2. *There is an a-definable (b-definable) interval $I \supseteq (a, b)$ which is gp-short (possibly witnessed by extra parameters).*

Proof. (1) By Note 4.3(2), (a, b) belongs to a $\emptyset$-definable family of gp-short intervals. Using the parameter c, we obtain a c-definable family of gp-short intervals, all containing c. By Theorem 4.7, their union is gp-short (and clearly definable over c).

(2) We do the same, but now obtain an a-definable (b-definable) family of intervals all with the same left-endpoint a (right endpoint b). We now use again Theorem 4.7. □

Lemma 4.10. *Let $\{S_t : t \in T\}$ be a definable family of gp-short sets and assume that T is a gp-short subset of M^k. Then the union $S = \bigcup_{t \in T} S_t$ is gp-short.*

Proof. We may assume that T is definably connected. By partitioning each S_t, uniformly in t, into its definably connected components we can also assume that each S_t is definably connected. It is enough to see that the projection of S onto each coordinate is gp-short. Let $\pi_1 : M^n \to M$ be the projection onto the first coordinate and let $I_t = \pi_1(S_t)$. By Fact 4.5, each I_t is a gp-short interval, so it is enough to prove that $\bigcup_{t \in T} I_t$ is gp-short. Write $I_t = (a_t, b_t)$ with a_t and b_t definable functions of t. Again, after a finite partition, we may assume that $t \mapsto a_t$ and $t \mapsto b_t$ are continuous on T.

Let $(a, b) = \bigcup_t I_t$, let $a_1 = \sup_t a_t$ and $b_1 = \inf_t b_t$. The image of T under $t \mapsto a_t$ is an interval I_1 and the image of T under $t \mapsto b_t$ is another interval I_2 (since T is definably connected and the functions are continuous). The interval (a, b) equals, up to finitely many points, $I_1 \cup [a_1, b_1] \cup I_2$.

If $a_1 < b_1$ then the interval (a_1, b_1) is gp-short since it is contained in all I_t's. By Fact 4.5, I_1 and I_2 are gp-short, hence (a, b) is gp-short. We therefore showed that $\pi_1(S)$ is gp-short, and prove similarly that each $\pi_i(S)$ is gp-short. □

5. Gp-short closure and gp-long dimension

5.1. *Defining short closure*

We follow here ideas from [9].

Definition 5.1. For $a \in M$ and $A \subseteq M$ we say that *a is in the gp-short closure of A*, written as $a \in \mathrm{shcl}(A)$, if either $a \in \mathrm{dcl}(A)$ or there is $b \in \mathrm{dcl}(A)$ such that the distance between a and b is gp-short (see Definition 4.1). Namely, the closed interval $[a, b]$ (or $[b, a]$) is gp-short.

Note that $\mathrm{dcl}(A) \subseteq \mathrm{shcl}(A)$.

Clearly, if $\mathcal{M}$ expands an ordered group then $M = \mathrm{shcl}(\emptyset)$, so our definition really aims for those o-minimal structures which do not expand ordered groups.

Fact 5.2. For every $a \in M$ and $A \subseteq M$, $a \in \mathrm{shcl}(A)$ if and only if there exists an A-definable, closed, gp-short interval containing a.

Proof. The "if" direction is clear, so we only need to prove the "only if". Assume that we have $[a, b]$ gp-short with $b \in \mathrm{dcl}(A)$. By Corollary 4.9(2), there is a b-definable gp-short interval $[c, b]$ which contains $[a, b]$, so $a \in [c, b]$. □

Lemma 5.3. *The gp-short closure is a pre-geometry. Namely:*

(i) $A \subseteq \mathrm{shcl}(A)$.
(ii) $A \subseteq B \Rightarrow \mathrm{shcl}(A) \subseteq \mathrm{shcl}(B)$.
(iii) $\mathrm{shcl}(A) = \cup\{\mathrm{shcl}(B) : B \subseteq A$ *finite* $\}$.
(iv) $\mathrm{shcl}(\mathrm{shcl}(A)) = \mathrm{shcl}(A)$.
(v) (Exchange) $a \in \mathrm{shcl}(bA) \setminus \mathrm{shcl}(A) \;\rightarrow\; b \in \mathrm{shcl}(aA)$.

Proof. (i) (ii) (iii) are clear from the definition. (iv) Assume that $a_i \in \mathrm{shcl}(A)$ for $i = 1, \ldots, n$. By Fact 5.2, for every i, there is an A-definable gp-short interval I_i containing a_i. Assume now that $b \in \mathrm{shcl}(a_1, \ldots, a_n)$. We want to show that $b \in \mathrm{shcl}(A)$. Let $S = I_1 \times \cdots \times I_n$.

By 5.2, there is a gp-short interval $J_{\bar{a}} \ni b$, defined over $a_1, \ldots, a_n$, which we may assume belongs to a $\emptyset$-definable family of gp-short intervals $\{J_{\bar{a}'} : \bar{a}' \in T\}$ gp-short (and contains b). Since S is A-definable so is J.

(v) Assume that $a \in \mathrm{shcl}(bA) \setminus \mathrm{shcl}(A)$. Then there is an Ab-definable gp-short interval $[b_1, b_2]$ containing a. Since $a \notin \mathrm{shcl}(A)$, it follows that $b_i \notin \mathrm{dcl}(A)$ for $i = 1, 2$, so Exchange for dcl implies that $b \in \mathrm{dcl}(b_i A)$. By 4.9, there is an a-ndefinable gp-short interval containing $[b_1, b_2]$ and hence $b_1, b_2 \in \mathrm{shcl}(aA)$. By transitivity of shcl, proved in (iv), we have $b \in \mathrm{shcl}(aA)$. □

5.2. *Gp-long dimension of tuples*

Definition 5.4. A set $B \subseteq M$ is called shcl-*independent over* $A \subseteq M$ if for every $a \in B$, we have $a \notin \mathrm{shcl}(B \cup A \setminus \{a\})$. For $(a_1, \ldots, a_n) \in M^n$ and $A \subseteq M$ we let the *gp-long dimension of $\bar{a}$ over A*, $\mathrm{lgdim}(\bar{a}/A)$, be the maximal $m \leqslant n$ such that $\bar{a}$ contains a tuple of length m which is shcl-independent over A.

Note 5.5.

1. We have $\mathrm{lgdim}(a/A) \leqslant \dim(a/A)$.
2. Because the gp-long dimension is based on a pre-geometry we have the dimension formula

$$\mathrm{lgdim}(a, b/A) = \mathrm{lgdim}(a/bA) + \mathrm{lgdim}(b/A).$$

3. If $\bar{a}, \bar{b}$ realize the same type over A then $\mathrm{lgdim}(\bar{a}/A) = \mathrm{lgdim}(\bar{b}/A)$.
4. If $\mathcal{M}$ is an expansion of an ordered group then the whole universe is gp-short and therefore $\mathrm{lgdim}(a/A) = 0$ for every $a \in M$, $A \subseteq M$. On the other end, it is possible that no group-intervals are definable in $\mathcal{M}$. In this case, $\mathrm{shcl}(A) = \mathrm{dcl}(A)$ and by the Trichotomy Theorem, [7], the resulting pre-geometry is trivial.

Definition 5.6. For $I = (a, b)$ and $c \in I$, we say that c is *long-central in I* if both (a, c) and (c, b) are gp-long.

Fact 5.7.
Let $A \subset M$ be smaller than the saturation of M.

1. If I is a definable gp-long interval, then there is $a \in I$ such that $a \notin \mathrm{shcl}(A)$.
2. Let $a \in M^n$, $\mathrm{lgdim}(a/A) = k$, and $p(x) = \mathrm{tp}(a/A)$. Then for every $B \supseteq A$ there exists $b \models p$ such that $\mathrm{lgdim}(b/B) = k$.
3. Let $I = (d_1, d_2)$ be a gp-long interval and $a \in I$ long-central. Given any $\bar{b} \in M^n$, there exist c_1, c_2, $d_1 \leqslant c_1 < c_2 \leqslant d_2$, such that a is long-central in (c_1, c_2) and $\mathrm{lgdim}(\bar{b}/A) = \mathrm{lgdim}(\bar{b}/Ac_1c_2)$.

Proof.
(1) Consider the type over A:

$$p(x) = \{x \in I\} \cup \{x \notin (a_1, a_2) : \ a_1, a_2 \in \mathrm{dcl}(A) \ \&(a_1, a_2) \text{ gp-short } \}$$

(note that in the definition of the type we are just going over all $a_1, a_2 \in \mathrm{dcl}(A)$ such that (a_1, a_2) is gp-short. We don't claim any uniformity here).

If $p(x)$ is inconsistent then I is contained in a finite union of gp-short intervals, which is impossible.

(2) We prove the result for $a \in M$, with $\mathrm{lgdim}(a) = 1$. The case of M^n is done by induction. The set $p(M)$ can be written as the intersection of open intervals, defined over A, which are necessarily gp-long. By (1), each such interval contains a point $b \notin \mathrm{shcl}(B)$. By compactness we can find $b \models p$ with $b \notin \mathrm{shcl}(B)$.

(3) Write $I = (d_1, d_2)$. Using (1), we first choose $c_1 \in (d_1, a)$ such that $c_1 \notin \mathrm{shcl}(A\bar{b}a)$. In particular, (c_1, a) is gp-long. Next, choose $c_2 \in (a, d_2)$ such that $c_2 \notin \mathrm{shcl}(Ac_1\bar{b}a)$. It follows that $\mathrm{lgdim}(c_1c_2/A\bar{b}) = 2$ and therefore, by the dimension formula, $\mathrm{lgdim}(\bar{b}/Ac_1c_2) = \mathrm{lgdim}(\bar{b}/A)$. □

5.3. *Gp-long dimension of definable sets*

Definition 5.8. For $X \subseteq M^n$ definable over a small $A \subseteq M$, we let

$$\mathrm{lgdim}_A(X) = \max\{\mathrm{lgdim}(a/A) : a \in X\}.$$

By Fact 5.7(2), if X is definable over A and $A \subseteq B$ then $\mathrm{lgdim}_B(X) = \mathrm{lgdim}_A(X)$, so we can let $\mathrm{lgdim}(X) := \mathrm{lgdim}_A(X)$ for any A over which X is definable. We call $\mathrm{lgdim}(X)$ the *gp-long dimension* of X.

We say that $a \in X$ is *gp-long generic over* A if $\mathrm{lgdim}(a/A) = \mathrm{lgdim}(X)$.

An immediate corollary of the definition and the above observation is:

Corollary 5.9. *If* $X = \bigcup_{i=1}^{n} X_i$ *is a finite union of definable sets then* $\mathrm{lgdim}(X) = \max_i \mathrm{lgdim}(X_i)$.

Fact 5.10. A definable $X \subseteq M^n$ is gp-short if and only if $\mathrm{lgdim}(X) = 0$.

Proof. Without loss of generality X is definably connected, defined over $\emptyset$. If X is gp-short then its projection on each coordinate is gp-short so every tuple in X is contained in $\mathrm{shcl}(\emptyset)$. Conversely, if some projection of X is gp-long then, by Fact 5.7(1), this projection contains an element of gp-long dimension 1, so X contains a tuple of positive gp-long dimension over $\emptyset$. □

Definition 5.11. A k-long box is a cartesian product of k gp-long open intervals.

If $B = \Pi_{i=1}^{n}(c_i, d_i)$ is an n-long box in M^n, we say that $\bar{a} = (a_1, \ldots, a_n) \in B$ is *long-central in* B if for every $i = 1, \ldots, n$, a_i is long-central in (c_i, d_i).

Clearly, if B is an n-long box defined over A, $a \in B$ and $\mathrm{lgdim}(a/A) = n$ then a is long-central in B.

The following is easy to verify:

Fact 5.12. Let $B \subseteq M^n$ be an n-long box and let a be long-central in B. If $C \subseteq B$ is some A-definable, definably connected, gp-short set containing a, then the topological closure of C in M^n is contained in B.

Fact 5.13. Assume that $X \subseteq M^n$ is an A-definable set, $a \in X$ and $\mathrm{lgdim}(a/A) = n$. Then there exists $A_1 \supseteq A$ and an A_1-definable n-long box B, such that $a \in B$, $\mathrm{Cl}(B) \subseteq X$ and $\mathrm{lgdim}(a/A_1) = n$. In particular, $X \subseteq M^n$ has gp-long dimension n if and only if it contains an n-long box.

Proof. We use induction on n.

For $n = 1$, if $X \subseteq M$ is A-definable then a belongs to one of its definably connected components, which is an A-definable interval containing a. Since $\mathrm{lgdim}(a/A) = 1$, a must be long-central in it. We can then apply Fact 5.7(3).

For $a \in M^{n+1}$, we may assume that X is an $n+1$-cell and let $\pi : M^{n+1} \to M^n$ be the projection onto the first n coordinates. We let $f, g : \pi(X) \to M$ be the A-definable boundary functions of the cell X, with $f < g$ on $\pi(X)$. Because $\mathrm{lgdim}(a) = n+1$, the interval $(f(\pi(a)), g(\pi(a)))$ is gp-long (otherwise $a_{n+1} \in \mathit{shcl}(\pi(a))$). Applying 5.7(3), for $\bar{b} = a$, we can find e_1, e_2, with $f(\pi(a)) < e_1 < a_{n+1} < e_2 < g(\pi(a))$, such that a_{n+1} is long-central in (e_1, e_2) and such that $\mathrm{lgdim}(a/Ae_1e_2) = n+1$. Consider the first order formula over Ae_1e_2, in the variables $x = (x_1, \ldots, x_n)$, which says that $f(x) < e_1 < e_2 < g(x)$. This is an Ae_1e_2-definable property of $\pi(a)$, so by induction there exists an n-long box $B \subseteq \pi(X)$, defined over $A_1 \supseteq A$, and containing $\pi(a)$, with $\operatorname{lgdim}(\pi(a)/A_1e_1e_2) = n$, such that for all $x \in B$, we have $f(x) < e_1 < e_2 < g(x)$. The box $B \times (e_1, e_2)$ is the desired $n+1$-long box. □

Corollary 5.14. *Let $X \subseteq M^n$ be a definable set. Then $\operatorname{lgdim}(X) \geqslant k$ if and only if there exists a projection π of X onto k coordinates such that $\pi(X)$ contains a k-long box.*

We can now conclude:

Fact 5.15. Assume that $\operatorname{lgdim}(a/A) = n$, for $a \in M^n$ and let $p(x) = \operatorname{tp}(a/A)$. Then there exists an n-long box $B \subseteq M^n$, defined over $A_1 \supseteq A$, such that $a \in B \subseteq p(M)$, and $\operatorname{lgdim}(a/A_1) = n$.

Proof. Write the type $p(x)$ as the collection of A-formulas $\{\phi_i(x) : i \in I\}$ and let $X_i = \phi_i(\mathcal{M})$. We let $B(x, y) = \Pi_{j=1}^{n}(x_j, y_j)$ be a variable-dependent n-box, and consider the type $q(x, y)$ which is the union:

$$\{\operatorname{Cl}(B(x, y)) \subseteq X_i : i \in I\} \cup \text{“}B(x, y) \text{ is an } n\text{-long box”} \cup \text{“}\operatorname{lgdim}(a/xyA) = n''.$$

By Fact 4.6, $q(x, y)$ is indeed a type over A. By Fact 5.13, the type is consistent, so we can find a box as needed. □

Lemma 5.16. *Assume that $\{X_t : t \in T\}$ is a definable family of subsets of M^n, with $X = \bigcup_{t \in T} X_t$. Assume that $\operatorname{lgdim}(T) \leqslant \ell$ and for every $t \in T$, we have $\operatorname{lgdim}(X_t) \leqslant k$. Then $\operatorname{lgdim}(X) \leqslant k + \ell$.*

Proof. Without loss of generality, X is $\emptyset$-definable. Take $x \in X$ with $\operatorname{lgdim}(x/\emptyset) = \operatorname{lgdim}(X)$, and choose $t \in T$ so that $x \in X_t$. We then have

$$\operatorname{lgdim}(xt/\emptyset) = \operatorname{lgdim}(x/t) + \operatorname{lgdim}(t) = \operatorname{lgdim}(t/x) + \operatorname{lgdim}(x).$$

By our assumptions, $\operatorname{lgdim}(x/t) \leqslant k$ and $\operatorname{lgdim}(t) \leqslant \ell$, hence $\operatorname{lgdim}(t/x) + \operatorname{lgdim}(x) = \operatorname{lgdim}(xt) \leqslant k + \ell$. It follows that $\operatorname{lgdim}(x) \leqslant k + \ell$ so $\operatorname{lgdim}(X) \leqslant k + \ell$. □

6. Functions on gp-long and gp-short intervals, and the main theorem

Lemma 6.1. *Let I be a gp-long interval, and assume that $f : I \to M$ is A-definable, continuous and strictly monotone. Let t_0 be long-central in I. For every $t \in M$, let*

$$\operatorname{Sh}(t) = \{x \in M : \textit{ the distance between } x \textit{ and } t \textit{ is gp-short}\}.$$

Then $\operatorname{Sh}(t_0) \subseteq I$ and $f(\operatorname{Sh}(t_0)) = \operatorname{Sh}(f(t_0))$.

Proof. First note that $Sh(t)$ is in general not a definable set but only $\bigvee$-definable. It is clear that $\operatorname{Sh}(t_0) \subseteq I$. Because f is continuous it sends pairs of elements whose distance is gp-short to pairs of gp-short distance, hence $f(\mathrm{Sh}(t_0)) \subseteq \mathrm{Sh}(f(t_0))$. Because f is strictly monotone, $J = f(I)$ is also gp-long and $f(t_0)$ is long-central in J. We now apply the same reasoning to $f^{-1}|J$ and conclude that $f(\mathrm{Sh}(t_0)) = \mathrm{Sh}(f(t_0))$. □

Lemma 6.2. *Assume that* $X \subseteq M^n$ *is definable with* $\mathrm{lgdim}(X) = \dim(X) = k > 0$*, and that* $f : X \to M$ *is an* A*-definable function. If* $f(X)$ *is gp-short then there are finitely many* $y_1, \ldots, y_m \in M$*, all in* $\mathrm{dcl}(A)$*, such that* $\mathrm{lgdim}(X \setminus f^{-1}(\{y_1, \ldots, y_m\}) < k$*. Equivalently,* f *is locally constant at every gp-long generic point in* X*.*

Proof. The set of all points in X at which f is locally constant is definable over A and has finite image, contained in $\mathrm{dcl}(A)$. It is therefore sufficient to prove that f is locally constant at every $a \in X$, with $\mathrm{lgdim}(a/A) = k$.

If $b = f(a)$ then $k = \mathrm{lgdim}(ab/A) = \mathrm{lgdim}(a/Ab) + \mathrm{lgdim}(b/A)$. But $b \in f(X)$, a gp-short set defined over A and therefore $\mathrm{lgdim}(b/A) = 0$. It follows that $\mathrm{lgdim}(a/Ab) = k$, so in particular, $\dim(a/Ab) \geqslant k$. Since $\dim(X) = k$, we must have $\dim(a/Ab) = k$, so a is generic in X over Ab. This implies that there is a neighborhood of a, in the sense of X, on which $f(x) = b$, so f is locally contstant at a. □

As a corollary we have:

Lemma 6.3. *Assume that* $X \subseteq M^{k+1}$ *is definable,* $\dim(X) = k + 1$*,* $\mathrm{lgdim}(X) = k$ *and the projection* $\pi(X)$ *onto the last coordinate is gp-short. Then* X *contains a definable set of the form* $B \times J$*, for* $B \subseteq M^k$ *a* k*-long box and* $J \subseteq M$ *an open gp-short interval.*

Proof. Without loss of generality, X is $\emptyset$-definable. Take $\langle a, b\rangle \in M^k \times M$ generic in X, with $\mathrm{lgdim}(a) = k$. Let $Y \subseteq M^k$ be the projection of X onto the first k coordinates. By assumption, we have $\mathrm{lgdim}(Y) = \dim(Y) = k$. Since $\dim(X) = k+1$ and $\langle a, b\rangle$ is generic in X, there are $\emptyset$-definable functions σ_1, σ_2 taking values in M, whose domain is a $\emptyset$-definable set $Y_1 \subseteq Y$ containing a, such that $\sigma_1(a) < b < \sigma_2(a)$ and for every $y \in Y_1$ we have $\{y\} \times (\sigma_1(y), \sigma_2(y)) \subseteq X$. In particular, the functions σ_1, σ_2 take values in the closure of $\pi(X)$, namely in a gp-short set. By Lemma 6.2, these functions are locally constant, with values $\sigma_1 \equiv j_1$, $\sigma_2 \equiv j_2$, on a $\emptyset$-definable set $Y' \subseteq M^k$ containing a. We can finish by Lemma 5.13, taking $J = (j_1, j_2)$. □

Here are our two main lemmas:

Lemma 6.4. *Assume that* $L \subseteq M^n$ *is* A*-definable with* $\mathrm{lgdim}(L) = \dim(L) = k$*,* $J \subseteq M$ *is a gp-short open interval and* $f : L \times J \to M$ *is definable over* A*. If* $f(L \times J)$ *is gp-short then there exist an* A*-definable set* $S \subseteq L$ *with* $\mathrm{lgdim}(S) < k$

and finitely many A-definable partial functions $g_1, \ldots, g_K : J \to M$ such that for all $\ell \in L \setminus S$, and all $x \in J$, we have

$$\bigvee_{i=1}^{K} f(\ell, x) = g_i(x).$$

Proof. For every $x \in J$, let $f^x : L \to M$ be defined by $f^x(\ell) = f(\ell, x)$. By Lemma 6.2, there exists an Ax-definable set $L_x \subseteq L$ such that f^x is locally constant on L_x (in particular, $f^x(L_x)$ is finite) and $\operatorname{lgdim}(L \setminus L_x) < k$. By o-minimality, there exists a uniform bound K on the size of $f^x(L_x)$, so we can define (possibly partial) functions $g_i : J \to M$, $i = 1, \ldots, K$, such that for every $\ell \in L_x$, we have $f(\ell, x) = g_i(x)$ for some $i = 1, \ldots, K$. If we let $S = \bigcup_{x \in J}(X \setminus L_x)$ then, by Lemma 5.16, we have $\operatorname{lgdim}(S) < k$. □

As a corollary we have:

Lemma 6.5. *Let $L \subseteq M^2$ be a definable set, with $\operatorname{lgdim}(L) = 2$, and $J = (a, b)$ a gp-short interval. Assume that $f : L \times J \to M$ is a definable function and that a_0 is generic in J (in the usual sense).*

Then there exist a 2-long box $B \subseteq L$, an open interval $J' \subseteq J$ containing a_0, and a definable partial, two-variable function $g : M^2 \to M$, such that for every $\ell \in B$ and $x \in J'$, we have

$$f(\ell, x) = g(f(\ell, a_0), x).$$

Proof. Assume that all data is definable over $\emptyset$. By Lemma 6.3, we may assume that f is continuous (we apply the lemma to the set of all points in $L \times J$ at which f is continuous).

For every $y \in f(L \times \{a_0\})$ let $L^y = \{\ell \in L : f(\ell, a_0) = y\}$. We divide the argument in two cases:

Case 1 There exists $y \in f(L \times \{a_0\})$ such that $\operatorname{lgdim}(L^y) = 2$.

Here we get a stronger version of our result. We fix y as assumed. By Fact 5.13, there exists a 2-long box $B \subseteq L^y$ containing a_0 which is definable over parameters independent from a_0. By the genericity of a_0 we obtain a $\emptyset$-definable interval $J' \subseteq J$ and a $\emptyset$-definable one-variable function $g : J' \to M$ such that for every $\ell \in B$ and $x \in J'$ we have $f(\ell, x) = g(x)$.

Case 2 For every $y \in f(L \times \{a_0\})$, $\operatorname{lgdim}(L^y) \leqslant 1$.

Let ℓ_0 be gp-long generic in L and let $y_0 = f(\ell_0, a_0)$. Clearly, $\operatorname{lgdim}(\ell_0 y_0 / a_0) = \operatorname{lgdim}(\ell_0 / a_0) = 2$. We then have:

$$2 = \operatorname{lgdim}(\ell_0 y_0 / a_0) = \operatorname{lgdim}(\ell_0 / y_0 a_0) + \operatorname{lgdim}(y_0 / a_0).$$

Because $\ell_0 \in L^{y_0}$ we have $\operatorname{lgdim}(\ell_0 / y_0 a_0) \leqslant 1$ and of course, we also have $\operatorname{lgdim}(y_0 / a_0) \leqslant 1$. Hence, $\operatorname{lgdim}(\ell_0 / y_0 a_0) = \operatorname{lgdim}(y_0 / a_0) = 1$ and therefore $\operatorname{lgdim}(L^{y_0}) = 1$.

By Fact 5.15 there exists an a_0-definable gp-long interval $I \ni y_0$ such that every $y \in I$ realizes the same type over a_0. It follows from Note 5.5 (3) that for all $y \in I$, $\mathrm{lgdim}(L^y) = 1$. By removing finitely many points from I we may also assume that for all $y \in I$, $\dim(L^y) = 1$.

We claim that for every $y \in I$ the set $f(L^y \times J)$ is gp-short. Indeed,

$$f(L^y \times J) = \bigcup_{\ell \in L^y} f(\{\ell\} \times J)$$

and each $f(\{\ell\} \times J)$ is a gp-short interval (since J is gp-short) which contains y. Hence, by Theorem 4.7 its union is gp-short.

We can therefore apply Lemma 6.4 to $f|L^y \times J$, for each $y \in I$. Hence, there exists a number k_y, definable functions $g_{1,y}(x), \ldots, g_{k_y,y}(x)$, and a definable set $S^y \subseteq L^y$ with $\mathrm{lgdim}\, S^y = 0$, such that for all $\ell \in L^y \setminus S^y$ and all $x \in J$ we have

$$\bigvee_{i=1}^{k_y} f(\ell, x) = g_{i,y}(x) = g_{i,f(\ell,a_0)}(x).$$

By o-minimality, the number k_y is bounded uniformly in $y \in I$, so we can find K, and definable partial two-variable functions $g_i(x, y)$, $i = 1, \ldots, K$, such that for every $y \in I$, $\ell \in L^y \setminus S^y$ and $x \in J$,

$$\bigvee_{i=1}^{K} f(\ell, x) = g_i(y, x) = g_i(f(\ell, a_0), x).$$

If we let $S = \bigcup_y S^y$ then, by Lemma 5.16, $\mathrm{lgdim}(S) \leqslant 1 < \mathrm{lgdim}\, L$ and therefore $L \setminus S$ contains a 2-long box B. Finally, for $i = 1, \ldots, K$, let X_i be all $(\ell, x) \in B \times J$ such that $f(\ell, x) = g_i(f(\ell, a_0), x)$. Each X_i is $\emptyset$-definable, $B \times J = \bigcup_i X_i$, so at least one of these X_i's contains a point $\langle \ell, a_0 \rangle$ with $\mathrm{lgdim}(\ell/a_0) = 2$. It follows that this X_i contains a box of the form $B' \times J'$ with $\mathrm{lgdim}(B') = 2$ and $J' \ni a_0$ an open interval. We now have, for every $\ell \in B'$ and $x \in J'$,

$$f(\ell, x) = g_i(x, f(\ell, a_0)). \qquad \square$$

Until now we did not use at all the Trichotomy Theorem for o-minimal structures. The next result requires it.

Corollary 6.6. *Assume that I_1, I_2, I_3 are gp-long intervals. Let $f : I_1 \times I_2 \times I_3 \to M$ be a definable function. Then there are gp-long intervals $I_1' \subseteq I_1$, $I_2' \subseteq I_2$ and $I_3' \subseteq I_3$ such that for all $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in I_1' \times I_2'$,*

$$\begin{array}{c} \exists x \in I_3' \, (f(a_1, b_1, x) = f(a_2, b_2, x)) \\ \Leftrightarrow \\ \forall x \in I_3' \, (f(a_1, b_1, x) = f(a_2, b_2, x)) \end{array} \tag{6.1}$$

Namely, the family of functions $f(a, b, -)|I_3'$, for $\langle a, b \rangle \in I_1' \times I_2'$, is at most 1-dimensional.

Proof. Without loss of generality, f is continuous. We assume that all data are definable over $\emptyset$. Fix $\langle a_1, a_2, a_0\rangle \in I_1 \times I_2 \times I_3$ with $\operatorname{lgdim}(a_1, a_2, a_0/\emptyset) = 3$.

Assume first that there is no gp-short open interval containing a_0. In this case, by the Trichotomy Theorem, a_0 is a trivial point, so the function $f(a_1, a_2, -)$ is either constant around a_0, namely equals some $g(a_1, a_2)$, or, if it is strictly monotone, then it equals some $\emptyset$-definable 1-variable function $h(-)$, independent of $\langle a_1, a_2\rangle$. In either case there are gp-long $I'_j \subseteq I_j$, $j = 1, 2, 3$, for which we either have $f(a'_1, a'_2, x) = g(a'_1, a'_2)$ or $f(a'_1, a'_2, x) = h(x)$, for all $(a'_1, a'_2, x) \in I'_1 \times I'_2 \times I'_3$. In either case the second clause of (6.1) holds.

We can therefore assume that there is some gp-short interval J around a_0. By Lemma 6.5, there are gp-long intervals $I'_1 \subseteq I_1$ and $I'_2 \subseteq I_2$ such that

($\star$) *for every* $\ell_1, \ell_2 \in I'_1 \times I'_2$,
the functions $f(\ell_1, -) = f(\ell_2, -)$ *agree in some neighborhood of* a_0 *if and only if* $f(\ell_1, a_0) = f(\ell_2, a_0)$.

By Fact 5.7(3), we can choose the intervals $I'_1 = (a', b')$ and $I'_2 = (a'', b'')$ so that $a_0 \notin \operatorname{shcl}(a'b'a''b'')$. Because a_0 is gp-long generic in I_3, and (*) is a first order formula about a_0 over $a'b'a''b''$, there is a gp-long interval $I'_3 \subseteq I_3$ containing a_0 such that for all $x \in I'_3$ we have:

If $\ell_1, \ell_2 \in I'_1 \times I'_2$, then $f(\ell_1, -)$ and $f(\ell_2, -)$ agree on a neighborhood of x if and only if $f(\ell_1, x) = f(\ell_2, x)$.

But now, by continuity and definable connectedness of I'_3 if $f(\ell_1, -)$ and $f(\ell_2, -)$ agree anywhere in I'_3 then they must agree everywhere on I'_3. □

We now reach our main theorem of this section:

Theorem 6.7. *Let* $f : I \times J \to M$ *be a definable function which is strictly monotone in each variable separately. Then either* I *or* J *is gp-short.*

Proof. We start by assuming, for contradiction, that both I and J are gp-long. Write $I = (a, b)$ and $J = (c, d)$. The general idea is that outside of subsets of $I \times J$ of gp-long dimension smaller than 2, we have a phenomenon similar to local modularity (every definable family of curves is one-dimensional) and therefore we can apply the standard machinery of local modularity to produce a definable group.

For $x \in I$, we write $f_x(y) := f(x, y)$. By partitioning $I \times J$ into finitely many sets, and by applying Fact 5.13, we may assume that f is continuous and for every $x \in I$, f_x is strictly monotone, say increasing.

Claim 6.8. *There exists a gp-long interval* K *and a gp-long interval* $I_1 \subseteq I$ *such that for all* $x \in I_1$, *we have* $K \subseteq f_x(J)$.

Proof. Take $x_0 \in I$ to be gp-long generic. The interval $f(x_0, J)$ is gp-long so we can find y_0 in it which is gp-long generic over x_0, and so $\operatorname{lgdim}(x_0, y_0/\emptyset) = 2$. The set $\{(x,y) \in I \times M : y \in f(x,J)\}$ is $\emptyset$-definable and contains $\langle x_0, y_0 \rangle$. By Fact 5.13, it contains a cartesian product $I_1 \times K$ of two gp-long intervals. □

To simplify notation, we assume that for all $x \in I$, we have $K \subseteq f_x(J)$. We can now consider the family of functions $\{f_x f_y^{-1}|K : x, y \in I\}$ as a collection of continuous functions from K into M. Let

$$F(x,y,t) = f_x f_y^{-1}(t).$$

The function F is a map from $I \times I \times K$. We apply Corollary 6.6, and find $I_1 \subseteq I$, $I_2 \subseteq I$, and $I_3 \subseteq K$ all gp-long such that for every $x, x' \in I_1$, $y, y' \in I_2$ and $t \in I_3$, if $F(x,y,t) = F(x',y',t)$ then for all $t' \in K$ we have $F(x,y,t') = F(x',y',t')$. Namely, for all $x_1, x_1' \in I_1$ and $x_2, x_2' \in I_2$,

$$\exists t \in I_3 \; f_{x_1} f_{x_2}^{-1}(t) = f_{x_1'} f_{x_2'}^{-1}(t) \Leftrightarrow \forall t \in I_3 \; f_{x_1} f_{x_2}^{-1}(t) = f_{x_1'} f_{x_2'}^{-1}(t) \qquad (6.2)$$

We now fix $\langle x_0, y_0, t_0 \rangle$ gp-long generic in $I_1 \times I_2 \times I_3$ and let $w_0 = f_{x_0} f_{y_0}^{-1}(t_0)$. We also let $a_0 = f_{y_0}^{-1}(t_0)$. By Lemma 6.1, if $t \in \mathrm{Sh}(t_0)$ then we have $f_{y_0}^{-1}(t) \in \mathrm{Sh}(a_0)$, hence by the same lemma, the map $y \mapsto f_y^{-1}(t)$ sends $\mathrm{Sh}(y_0)$ bijectively onto $\mathrm{Sh}(a_0)$. Similarly, for every $y \in \mathrm{Sh}(y_0)$, the map $t \mapsto f_y^{-1}(t)$ sends $\mathrm{Sh}(t_0)$ bijectively onto $\mathrm{Sh}(a_0)$ and for every $x \in \mathrm{Sh}(x_0)$, the map $a \mapsto f_x(a)$ sends $\mathrm{Sh}(a_0)$ bijectively onto $\mathrm{Sh}(w_0)$. Thus, if $x_1, x_2 \in \mathrm{Sh}(x_0)$ then the function $f_{x_1}^{-1} f_{x_2}$ is a permutation of $\mathrm{Sh}(a_0)$.

Claim 6.9. *(1) For every $x_1, x_2 \in \mathrm{Sh}(x_0)$ there is a unique $x_3 \in \mathrm{Sh}(x_0)$ such that $f_{x_2}^{-1} f_{x_3} = f_{x_0}^{-1} f_{x_1}$, as functions from $\mathrm{Sh}(a_0)$ to $\mathrm{Sh}(a_0)$.*

(2) For every $x_1, x_3 \in \mathrm{Sh}(x_0)$ there exists a unique $x_2 \in \mathrm{Sh}(x_0)$ such that $f_{x_2}^{-1} f_{x_3} = f_{x_0}^{-1} f_{x_1}$.

Proof. We prove (1) – the proof of (2) is similar. Consider first $f_{x_1} f_{y_0}^{-1}(t_0) \in \mathrm{Sh}(w_0)$. By the above observations, there exists a unique $y_1 \in \mathrm{Sh}(y_0)$ such that

$$f_{x_1} f_{y_0}^{-1}(t_0) = f_{x_0} f_{y_1}^{-1}(t_0).$$

By the same reasoning, there exists a unique $x_3 \in \mathrm{Sh}(x_0)$, such that

$$f_{x_3} f_{y_0}^{-1}(t_0) = f_{x_2} f_{y_1}^{-1}(t_0).$$

By (6.2), the above two equalities at the point t_0 translate to equality of functions on $\mathrm{Sh}(t_0)$. Using composition and substitution we obtain

$$f_{x_2}^{-1} f_{x_3} = f_{x_0}^{-1} f_{x_1}.$$ □

Our first goal is to define a group-operation on $\mathrm{Sh}(x_0)$ with identity x_0. First note that $\mathrm{Sh}(x_0)$ cannot be a definable set. Indeed, this is clearly a convex set so if it were definable it would be a subinterval $I' \subseteq I$. By Theorem 4.7, I' is necessarily gp-short, so must be of the form $[a', b']$. Because I itself is gp-long, at least one of the end points, say a', belongs to $\mathrm{Int}(I)$. However, every point in $\mathrm{Int}(I)$ is nontrivial, because of f, hence by the Trichotomy Theorem a' is contained in some open group-interval J'. But then $I' \cup J' \subseteq \mathrm{Sh}(x_0)$, contradicting the fact that a' was the end point of $\mathrm{Sh}(x_0)$.

The definition of the group operation on $\mathrm{Sh}(x_0)$ goes as follows:

For $x_1, x_2, x_3 \in \mathrm{Sh}(x_0)$,

$$(\star)\ \ x_1 + x_2 = x_3 \Leftrightarrow f_{x_2}^{-1} f_{x_3} = f_{x_0}^{-1} f_{x_1}, \text{ as functions on } \mathrm{Sh}(a_0).$$

The identity element is clearly x_0, Claim 6.9 (1) implies that $+$ is associative, Claim 6.9(2) guarantees an inverse (with $x_3 = x_0$), and commutativity follows from one-dimensionality of $\mathrm{Sh}(x_0)$ and o-minimality. Furthermore, the monotonicity of f in both variables ensures that $+$ is monotone in both variables so the group we get is an ordered group.

Notice that for every $x \in \mathrm{Sh}(x_0)$, the restriction of $+$ to $(x_0, x) \times (x_0, x)$ (or to $(x, x_0) \times (x, x_0)$) endows the interval (x_0, x) (or (x, x_0)) with the structure of a generalized group-interval.

Let $R \subseteq I^3$ be the relation defined by $(\star)$ and consider now the following definable set

$$S = \{x \in I : \ R \cap (x_0, x)^3 \text{ or } R \cap (x, x_0)^3 \text{ defines a generalized group-inerval}\}.$$

Clearly, every $x \in S$ belongs to $\mathrm{Sh}(x_0)$ (as witnessed by R), hence $S \subseteq \mathrm{Sh}(x_0)$. On the other hand, as pointed out above, $\mathrm{Sh}(x_0) \subseteq S$. But then, $S = \mathrm{Sh}(x_0)$, contradicting our earlier observation that $\mathrm{Sh}(x_0)$ is not definable here.

So returning to our original assumptions, either I or J must be gp-short.This ends the proof of Theorem 6.7. □.

The following is a generalization we will require later:

Corollary 6.10. *Let $f : I_1 \times \cdots \times I_{n+1} \to M^n$ be a definable function which is injective in each variable separately (namely for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ in $I_1, \ldots, I_{i-1}, I_{i+1}, \ldots, I_n$, respectively, the map $f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) : I_i \to M^n$ is injective).*

Then at least one of the intervals I_j is gp-short.

Proof. Assume towards contradiction that all intervals are gp-long.

Lemma 6.11. *If $h : I_1 \times \cdots \times I_m \to M$ is a definable function on a product of gp-long intervals, then there exist gp-long subintervals $J_i \subseteq I_i$, $i = 1, \ldots, m$, there exists $j \in \{1, \ldots, m\}$, and a definable $g : J_j \to M$, such that for every $x = (x_1, \ldots, x_m) \in J_1 \times \cdots \times J_m$, we have $h(x) = g(x_j)$.*

Proof. We take $a = (a_1, \ldots, a_m) \in I_1 \times \cdots \times I_m$ which has gp-long dimension m (over the parameters defining everything). Then there is a gp-long interval $J_m \subseteq I_m$ containing a_m and defined over parameters A so that $\mathrm{lgdim}(a/A) = m$, and such that $h(a_1, \ldots, a_{m-1}, x)$ is either constant for $x \in J_m$ or strictly monotone.

In the first case (h constant in the last coordinate) we can find, using the genericity of a, gp-long intervals $J_i \subseteq I_i$, $i = 1, \dots, m-1$, such that for every $(a'_1, \dots, a'_{m-1}) \in J_1 \times \cdots \times J_{m-1}$, the function $h(a'_1, \dots, a'_{m-1}, x)$ is constant in x, call its value $g(a'_1, \dots, a'_{m-1})$. By applying induction to g we finish the proof.

In the second case (h strictly monotone in the last coordinate) we prove that $h(x_1, \dots, x_m)$ must depend on x_m only, on an appropriate neighborhood: Using the same reasoning as before, we find a gp-long interval $J_{m-1} \subseteq I_{m-1}$ on which the map $h(a_1, a_2, \dots, a_{m-2}, t, a_m)$ is either constant or strictly monotone for $t \in J_{m-1}$. But now, this second possibility is impossible, or else the map $h(a_1, \dots, a_{m-2}, t, x)$ is strictly monotone in both variables on $J_{m-1} \times J_m$, implying, using Theorem 6.7, that one of these intervals is gp-short, contradiction. We are then left with the first case, and finish by induction as before. □

We now return to our proof of the corollary and to the assumption that all intervals $I_1, \dots, I_n$ are gp-long. We write $f(a) = (f_1(a), \dots, f_n(a))$. If we apply the last lemma to f_1 then we can find gp-long intervals $J_i \subseteq I_i$, $i = 1, \dots, n+1$, and a definable one-variable function $h(x)$ such that for every $(x_1 \dots, x_{n+1}) \in J_1 \times \cdots \times J_{n+1}$, we have, without loss of generality, $f_1(x_1, \dots, x_{n+1}) = h(x_1)$. But now, fix any $a_1 \in J_1$ and consider the function

$$F(x_2, \dots, x_{n+1}) = (f_2(a_1, x_2 \dots, x_{n+1}), \dots, f_n(a_1, x_2, \dots, x_{n+1}))$$

from $J_2 \times \cdots \times J_{n+1}$ into M^{n-1}. It is easy to see that the function F is still injective in each coordinate so by induction, one of the J_i, $i = 2, \dots, n+1$, is gp-short, contradiction. We then conclude that one of the I_j's must be gp-short.

Here is a first observation about definable groups.

Corollary 6.12. *Let $G \subseteq M^n$ be a definable group in an o-minimal structure. Then every definable 1-dimensional subset of G is group-short.*

Proof. Let $J \subseteq G$ be a definable 1-dimensional set, identified with a finite union of intervals in M, and apply Corollary 6.10 to the map $f : J^{n+1} \to G$ given by the group product

$$f(g_1, \dots, g_{n+1}) = g_1 g_2 \cdots g_{n+1}.$$

It follows that J must be gp-short. □

Although Theorem 3 from the Introduction will be proved in Theorem 8.23(2) below, the case where G is definable can be deduced at this point.

Corollary 6.13. *Let $G \subseteq M^n$ be a definable group, and let $\pi_i : M^n \to M$ be the projection onto the i-th coordinate. If $I_i = \pi_i(G)$ is infinite then I_i is gp short.*

In particular, the underlying set G is group-short, so in definable bijection with a subset of a finite cartesian product of generalized group-intervals.

Proof. Assume that $I_i = \pi_i(G)$ is infinite. By Edmundo's result, [6], we have Definable Choice for definable families of subsets of G and therefore there exists a one-dimensional subset $J_i \subseteq G$ which is in definable bijection with I_i. By Corollary 6.12, J_i is gp-short and hence so is I_i. □

7. $\mathcal{M}$-quotients

7.1. *Dimension of elements in $\mathcal{M}^{eq}$*

Definition 7.1. Let $X_1, \ldots, X_n$ be pairwise disjoint definable subsets of $M^{k_1}, \ldots, M^{k_n}$, respectively, and let $X = X_1 \sqcup \cdots \sqcup X_n$. We call a subset $W \subseteq X$ *definable* if $W \cap X_i$ is definable for every $i = 1, \ldots, n$. For $\dim W$ we take the maximum of all $\dim(W \cap X_i)$.

Note that X^k can be similarly written as a finite pairwise disjoint union of cartesian products of the X_i's, and a subset of X^k is called *definable* accordingly. If E is a definable equivalence relation on X then we say that X/E is an $\mathcal{M}$*-quotient*.

A subset of X/E is called *definable* (we should say "interpretable" but it sounds awkward) if it is the image of a definable subset of X under the quotient map.

Note that X above is in definable bijection with an actual definable subset of some M^k, after naming parameters, but it is often more natural to consider it as as the finite union of definable sets in various M^k's.

For $A \subseteq \mathcal{M}^{eq}$ a small set of parameters and $a \subseteq M$, the closure operation $\mathrm{dcl}(aA)$ still defines a pre-geometry on M so $\dim(a/A)$ makes sense.

Definition 7.2. Let $A \subseteq \mathcal{M}^{eq}$ be a small set, $X \subseteq M^k$ an A-definable set and E an A-definable equivalence relation on X.

For $g \in X/E$, we define $\dim(g/A)$ to be the maximum among $\dim(x/A) - \dim([x])$, as x varies in the class g (to avoid ambiguity, here $\dim([x])$ is the set dimension of the equivalence class of x). For $Y \subseteq X/E$ definable over A, we let

$$\dim Y = \max\{\dim(g/A) : g \in Y\}.$$

If for $g \in Y$ we have $\dim(Y) = \dim(g/A)$ then g is called a *generic element of Y over A*.

An equivalent definition of dimension was given in [14]. Also, the definiton in [15, Definition 3.2] is equivalent to ours (see (3) on p. 322 there).

One can show that the above definition does not depend on A, namely if we calculate the dimension of Y with respect to a larger set of parameters $B \supseteq A$ then we obtain the same result. Here are some more basic properties.

Fact 7.3.

1. For $g, h \in X/E$ and $A \subseteq \mathcal{M}^{eq}$, we have
$$\dim(g, h/A) = \dim(g/hA) + \dim(h/A).$$
(Here we think of the pair $\langle, g, h\rangle$ as a member of $X \times X/E \times E$).

2. Assume that $g = [a]$ for $a \in X$, and $\dim(a/A) = k$. Then $\dim(g/A) \leqslant k$.
3. Let $\{X_t/E_t : t \in T\}$ be a definable family of quotients (i.e. E_t is an equivalence relation on X_t and the families $\{X_t : t \in T\}$ and $\{E_t : t \in T\}$ are definable), with T possibly in $\mathcal{M}^{eq}$. Then for every $e \in \mathbb{N}$, the set $\{t \in T : \dim(X_t/E_t) = e\}$ is definable.

Proof. (1) is in [15, Proposition 3.4].

(2). Since $\dim(g/aA) = 0$, the dimension formula implies that $\dim(a/gA) + \dim(g/A) = \dim(a/A) = k$ and hence $\dim(g/A) \leqslant k$.

(3) This can be easily seen via the equivalent definition, given in [14]. □

The following is a direct corollary of the dimension formula.

Claim 7.4. *Let $T \subset M^{eq}$ be a definable set, and let $\{X_t : t \in T\}$ be a definable family of pairwise disjoint definable sets in $\mathcal{M}^{eq}$. If the dimension of each X_t is r and $\dim T = e$ then $\dim \bigcup_{t \in T} X_t = r + e$.*

Recall:

Definition 7.5. For X, Y definable sets and E_1, E_2 definable equivalence relations on X and Y, respectively, a function $f : X/E_1 \to Y/E_2$ is called *definable* if the set $\{\langle x, y\rangle \in X \times Y : f([x]) = ([y])\}$ is definable.

We are going to need the fact below about definable equivalence relations. Before stating it, a clarification:

A remark about disjointness. When we say below that Y can be partitioned into finitely many open sets $U_1, \ldots, U_m$, we allow for example the possibility that Y is the **disjoint** union of U_1 and U_2, with $U_1 = U_2$. By that we formally mean that $Y = V_1 \cup V_2$, where $V_1 = U_1 \times \{a\}$ and $V_2 = U_1 \times \{b\}$, for $a \neq b \in M$. When our structure expands a real closed field this can be avoided by finding definably homeomorphic open sets which are pairwise disjoint, but in the general case this is unavoidable.

Claim 7.6. *Let $X \subseteq M^k$ be an A-definable set and E an A-definable equivalence relation on X. Then there exists an $\mathcal{M}$-quotient Y/E', defined over A, and an A-definable bijection $f : X/E \to Y/E'$ such that Y can be partitioned into finitely many definable sets $U_1, \ldots, U_m$ with the following properties:*

1. *Each U_i is an open subset of M^{k_i}.*
2. *Each E'-class is contained in a single U_i.*
3. *For each $i = 1, \ldots, m$, there exists $d_i \in \mathbb{N}$ such that every E'-class in U_i is a set of dimension d_i, and restriction of the projection $\pi_{d_i} : M^{k_i} \to M^{d_i}$ onto the first d_i coordinates is a homeomorphism.*

Proof. We prove the result by induction on $n = \dim X$.

First, partition X into a finite union of sets, in each of which every E-class has the same dimension, and such that every E-class is contained in exactly one of these sets. It is sufficient to prove the result separately for each of these sets. Thus, we may assume that all classes have the same dimension d.

Let $\mathcal{C} = \{C_1, \dots, C_r\}$ be a cell decomposition of X. For each cell $C_i \in \mathcal{C}$ there exists a projection homeomorphism $\pi_i : C_i \to Z_i$ onto an open $Z_i \subseteq M^{n_i}$ for some $n_i \leq n$. By replacing each C_i with Z_i we may think of X as *the disjoint union* of $Z_1, \dots, Z_r$ (see above remark). We now consider the restriction of E to Z_1 and prove the claim for Z_1 (note that after doing that we plan to discard all elements in $X \setminus Z_1$ whose classes intersect Z_1).

If $\dim(Z_1) < n$, then the claim holds for it by induction. Otherwise, $\dim(Z_1) = n$, and thus necessarily, $\dim(Z_1/E) = n - d$. Let

$$Z_1' = \{x \in Z_1 : \dim([x] \cap Z_1) < d\}.$$

Since $\dim(Z_1'/E) \leqslant n - d$ and every $[x] \cap Z_1'$ has dimension smaller than d, it follows from Claim 7.4 that $\dim Z_1' < n$. Moreover, for every x, either $[x] \cap Z_1 \subset Z_1'$ (in case $\dim([x] \cap Z_1) < d$) or $[x] \cap Z_1' = \emptyset$, so proving the claim for Z_1' and $\tilde{Z}_1 = Z_1 \setminus Z_1'$ is sufficient. By induction, the claim holds for Z_1', we are left with $\tilde{Z} = \{x \in Z_1 : \dim([x] \cap Z_1) = d\}$.

By Lemma 9.1, we can uniformly partition all the equivalence classes in $\tilde{Z}_1$ into cells, then choose a d-dimensional cell from each equivalence class in $\tilde{Z}_1$, and replace $\tilde{Z}_1$ by the union of these cells (still calling it $\tilde{Z}_1$). Note that omitting the remaining part of each class does not change the quotient. Next, we partition $\tilde{Z}_1$ into finitely many sets, so that in a single set, the cell of each class is of the same type (by that we mean that the projection onto the same d coordinates is a homeomorphism). Since the partition respects the classes, we may deal with each part separately.

Any set in this partition with dimension less than n is handled by induction, so we may only consider the sets of dimension n. We assume then that $\tilde{Z}_1$ is an n-dimensional union of d-dimensional cells, all of the same type. By permutation of variables, we can suppose that projection onto the first d coordinates is a homeomorphism of each class onto an open subset of M^d.

Now let $\mathcal{D}$ be a cell decomposition of $\tilde{Z}_1$, and let B be the union

$$\bigcup_{D \in \mathcal{D},\, \dim D < n} D.$$

Because $\tilde{Z}_1 \subseteq M^n$ and $\dim \tilde{Z}_1 = n$, the union of all n-dimensional cells in $\mathcal{D}$ is an open subset of M^n, so $\tilde{Z}_1 \setminus B$ is still open in M^n. Thus, for each $x \in \tilde{Z}_1$, if the set $[x] \cap (\tilde{Z}_1 \setminus B)$ has dimension smaller than d then it must be empty (here we use the fact that $[x] \subseteq \tilde{Z}_1$ is a d-cell). Hence, a class $[x]$ which intersects $\tilde{Z}_1 \setminus B$ might not be a cell anymore, but it is still true that its projection onto the first d coordinates is a homeomorphism onto an open subset of M^d. Hence $\tilde{Z}_1 \setminus B$ satisfies the claim. We now remove from B all classes which are already represented in $\tilde{Z}_1 \setminus B$ (we still call the new set B) and handle B/E by induction on dimension. We therefore showed that the claim holds for $\tilde{Z}_1$ and hence also for Z_1.

Note that the above argument only used the fact that Z_1 was an n-dimensional subset of M^n (and that every class in X has dimension d).

Next, remove all classes from $Z_2, \ldots, Z_r$ with representatives in Z_1. We still use $Z_2, \ldots, Z_r$ for the remaining sets. Clearly, each class which is contained in the new $Z_2 \cup \cdots \cup Z_n$ still has dimension d (since it did not inersect Z_1 at all). If $\dim Z_2 = n$ then we handle it exactly as we handled Z_1, and if $\dim Z_2 < n$ then we apply induction. We proceed in the same manner until handle all Z_i's and thus prove the claim for X. □

7.2. *Elimination of one dimensional quotients*

Our goal here is to show that one-dimensional quotients can be eliminated. The argument is through a certain induction which naturally suggests the following terminology.

Let $\mathcal{F} = \{X_t : t \in T\}$ be a definable family of sets. We say that $f : T \to M^n$ is an $\mathcal{F}$-map if for every $t, s \in T$, if $X_t = X_s$ then $f(t) = f(s)$. We say that f is $\mathcal{F}$-injective if in addition, whenever $f(t) = f(s)$ we have $X_t = X_s$.

We will use the following fact [16, Claim 1.1]:

DEQ: *If E is an A-definable equivalence relation on a A-definable X, $A \subseteq M^{eq}$, such that X/E is finite, then every E-class is A-definable. Said differently,* $\mathrm{acl}^{eq}(A) = \mathrm{dcl}^{eq}(A)$.

Theorem 7.7. *Let $\mathcal{F} = \{X_t : t \in T\}$ be a definable family of subsets of M^k, with $T \subseteq \mathcal{M}^{eq}$ and $\dim T = 1$. Then there exists a definable $\mathcal{F}$-injective map $f : T \to M^m$, for some m, possibly over parameters.*

Proof. We may assume that $dcl(\emptyset)$ contains at least two elements so that every finite set can be coded in some M^n. Note that if $T = T_1 \sqcup T_2$ is a partition and we let $\mathcal{F}_i = \{X_t : t \in T_i\}$, $i = 1, 2$, then it is enough to obtain $\mathcal{F}_i$-injective maps for $i = 1, 2$. Also, if we can uniformly partition the X_t's into $Y_{1,t} \cup \cdots \cup Y_{m,t}$, $t \in T$, then it is sufficient to find $\mathcal{F}_i$-injective functions for each of the families $\mathcal{F}_i = \{Y_{i,t} : t \in T\}$, $i = 1, \ldots, m$.

We go by induction on k (recall, all X_t are in M^k).

Let us consider first the case $k = 1$. Each X_t is a finite union of points and open intervals so by a uniform partition we may assume that each X_t is either a point or an open interval. In the first case each X_t is coded by itself while in the second case it is coded by the pair of endpoints of the interval. Note that there is no need for additional parameters in this case. We now consider $k > 1$.

Let $X = \bigcup_t X_t$. If $\dim(X) < k$, then take a cell decomposition $C_1, \ldots, C_m$ of X. We now consider $\mathcal{F}_i = \{X_t \cap C_i : t \in T\}$, for $i = 1, \ldots, m$. As mentioned above, it is sufficient to handle each of the $\mathcal{F}_i$'s. Since each C_i is in definable bijection with a subset of M^{k-1} we can finish by induction. Thus, we may suppose that $\dim(X) = k$.

By replacing T with $T/\sim$, with $t \sim s$ if and only if $X_t = X_s$, we may assume that $X_t = X_s$ if and only if $t = s$, and still $\dim T = 1$ (if $\dim T = 0$ then we are done coding T in $dcl(\emptyset)$).

By Lemma 9.1, we can uniformly partition each set X_t into a disjoint union of cells $X_t^1, \ldots, X_t^m$ (in particular, the partition depends only on the set X_t and not on t), and let $\mathcal{F}_i = \{X_t^i : t \in T\}$, $i = 1, \ldots, m$. It is sufficient to handle each of the $\mathcal{F}_i$'s, so we assume that each X_t is a cell.

By a further partitioning of T, we may assume that all X_t's are cells in M^k of the same dimension r. We continue the proof by induction on both r and k.

Case 1. $r = k$. Namely, each X_t is an open cell.

We may assume that each of the cells is of the same form, namely either all are bounded above and below in the last coordinate, or all bounded below and unbounded above, etc. Assume first that each X_t is bounded above and below, in the last coordinate, by two $k-1$ cells Y_t^1, Y_t^2, of dimension $k-1$. Notice that the pair $\langle Y_t^1, Y_t^2\rangle$ determines the set X_t. By considering the families $\{Y_t^1 : t \in T\}$ and $\{Y_t^2 : t \in T\}$ and applying induction we will be done. We similarly handle the other forms of r-cells.

Case 2. $r < k$.

For $x \in X$, let $T(x) = \{t \in T : x \in X_t\}$ and let

$$X^0 = \{x \in X : T(x) \text{ is finite}\}\ ;\ X^1 = \{x \in X : T(x) \text{ is infinite}\}.$$

By definability of dimension (Fact 7.3 (3)), X^0, X^1 are definable. As before it is sufficient to handle each of the families $\mathcal{F}^i = \{X_t \cap X^i : t \in T\}$, $i = 0, 1$, separately.

Let us consider first the family $\mathcal{F}^0$. To simplify notation we replace X^0 by X and $\mathcal{F}^0$ by $\mathcal{F}$. Namely, we assume that for each $x \in X$, the set $T(x)$ is finite. By o-minimality there is a bound $\ell \in \mathbb{N}$ such that for every $x \in X$, $|T(x)| \leqslant \ell$. By a further partitioning of X we may assume that the size of $T(x)$ is uniform, namely for every $x \in X$, we have $|T(x)| = \ell$. Note that we may still assume that $X_t = X_s$ if and only if $t = s$ (if needed we replace T by $T/\sim$ as above). Our next reduction is to the case where the X_t's are pairwise disjoint.

By DEQ, for every $x \in X$ each element in $T(x)$ is also in $\mathrm{dcl}^{eq}(x)$. By compactness, we can find $\emptyset$-definable functions $f_i : X \to T$, $i = 1, \ldots, \ell$, such that for every $x \in X$, $T(x) = \{f_1(x), \ldots, f_\ell(x)\}$. In particular for $i \neq j$, we have $f_i(x) \neq f_j(x)$. It follows that for every $t \in T$, the set X_t can be written as the disjoint union $f_1^{-1}(t) \sqcup \cdots \sqcup f_\ell^{-1}(t)$. It is therefore sufficient to handle each of the families $\{f_i^{-1}(t) : t \in T\}$ separately for $i = 1, \ldots, \ell$. This is clearly a family of pairwise disjoint sets, so from now on we assume, by abuse of notation, that the X_t's are pairwise disjoint.

Because $\dim X_t < k$, we can assume by uniformly partitioning the X_t's into cells, that each X_t is a cell of dimension $r < k$ and that the projection of X_t onto the first r-coordinates is a homeomorphism. But then each X_t is the graph of a continuous function $f_t : C_t \to M$, where $C_t \subseteq M^{k-1}$ is an r-cell.

Let $\mathcal{F}_{\text{proj}} = \{C_t : t \in T\}$. By induction, there exists an injective $\mathcal{F}_{\text{proj}}$-function $g : T \to M^s$ for some s. We divide $\mathcal{F}_{\text{proj}}$ into two sub-families: Those $C_s \in \mathcal{F}_{\text{proj}}$ for which only finitely many distinct X_t's project onto C_s and those for which there are infinitely many such X_t's.

Consider the first case. Abusing notation we assume that for every $s \in T$, there are finitely many $t \in T$ such that $\pi(X_t) = C_s$ (the map π is the projection onto the first $k-1$ coordinates). We can define an equivalence relation E on T by t_1Et_2 if and only if $\pi(X_{t_1}) = \pi(X_{t_2})$ and our assumption implies that every E-class is finite. By o-minimality, we may assume that that all E-classes are of the same size ℓ. As we already did above, we can use DEQ to produce definable functions $h_i : T \to T$, $i = 1, \ldots, \ell$, such that for every $t \in T$, the set $\{h_1(t), \ldots, h_\ell(t)\}$ equals the E-class of t. Now fix arbitrary elements $a_1, \ldots, a_\ell \in M$ (preferably but not necessarily in $\text{dcl}(\emptyset)$) and let $f : T \to M^s \times M$ be the map $f(t) = \langle g(t), a_i\rangle$ if and only if $h_i(g(t)) = t$. The map f is an $\mathcal{F}$-injective map.

We are left to consider those $C_s \in \mathcal{F}_{\text{proj}}$ for which there are infinitely many X_t's which project onto it. Because T is one-dimensional, there are at most finitely many such distinct C_s's (this step fails in higher dimension). By handling each one separately, we can assume that all X_t project onto the same $C = C_s$. We now choose an arbitrary point $\bar{a} \in C$ and define $g(t) = f_t(\bar{a})$ (recall that X_t is the graph of f_t). Because we assumed the X_t's to be pairwise disjoint, this is an $\mathcal{F}$-injective map, defined over $\bar{a}$, thus ending the case where $\mathcal{F} = \mathcal{F}^0$.

We assume now that $\mathcal{F} = \mathcal{F}^1$ and that $X = X^1$. Namely, for every $x \in X$ we asume that $T(x) = \{t \in T : x \in X_t\}$ is infinite. As before we may assume that all X_t's have the same dimension $r < k$.

Claim $\dim(X) \leqslant r < k$.

Indeed, let $x \in X$ be generic and let $t \in T$ be generic in $T(x)$ over x. By the dimension formula we have

$$\dim(x/t) + \dim(t) = \dim(t/x) + \dim(x).$$

However, because $T(x)$ is infinite and $\dim(T) = 1$ we have $\dim(t/x) = \dim(t) = 1$ and hence $\dim(x) = \dim(x/t) \leqslant r$ (this right-most inequality is due to the fact that $\dim(X_t) = r$). It follows that $\dim(X) = \dim(x) \leqslant r$, thus proving the claim.

As we already saw at the very beginning of the proof, when $\dim(X) < k$ the proof can be finished by induction. We have thus finished the proof of Theorem 7.7. □

Corollary 7.8. *If $\dim(X/E) = 1$ then X/E is in definable bijection, over parameters, with a definable set.*

Here is a simple corollary that we are not going to use.

Corollary 7.9. *If $\dim Y = 2$ and E is a definable equivalence relation on Y then Y/E is in definable bijection (possibly, over parameters) with a definable set.*

Proof. We may assume that all classes have the same dimension. If the classes are finite then we can definably choose representatives. If the classes have dimension 1 then $\dim(Y/E) = 1$ and we are done by the previous corollary. If the classes have dimension 2 then there are only finitely many classes. □

7.3. *A general observation about $\mathcal{M}$-quotients*

Definition 7.10. Let X, Y be two definable sets in $\mathcal{M}^{eq}$. The set Y is said to be *internal to* X if there exists a finite set A such that $Y \subseteq \mathrm{dcl}^{eq}(A, X)$ (in a sufficiently saturated structure). Equivalently, there exists $n \in \mathbb{N}$, and a definable surjective function $f : X^n \to Y$, possibly over additional parameters.

Note that if Y is a subset of X then it is clearly internal to it.

Proposition 7.11. *Let X/E be an infinite $\mathcal{M}$-quotient. Then there exists an $\mathcal{M}$-quotient X'/E' which is in definable bijection with X/E, possibly over additional parameters, and there exist intervals $I_1, \dots, I_k \subseteq M$ such that:*

1. $X' \subseteq I_1 \times \cdots \times I_k$.
2. Each I_j is internal to X/E.

Before proving the result we point out that it is equivalent to the following statement: ($\star$) Assume that X/E is infinite. Then there are intervals $J_1, \dots, J_t \subseteq M$, all internal to X/E, such that X/E is internal to $J_1 \cup \cdots \cup J_t$.

Indeed, assume that the proposition is true. Then X' is internal to $I_1 \times \cdots \times I_k$, where all I_j are themselves internal to X/E. But then clearly X'/E' (so also X/E) is internal to $I_1 \times \cdots \times I_k$ which is itself internal to the union of these intervals.

As for the converse, assume that ($\star$) holds. Since X/E is internal to $J_1 \cup \ldots \cup J_t$ there exists a definable surjection from a product $\prod_{j=1}^{r} I_j$ onto X/E, where each I_j equals one of the J_i's. We can now take $X' = \prod_{j=1}^{r} I_j$ and let $xE'y$ if $g(x) = g(y)$. Now, X'/E' is in definable bijection with X/E so the proposition holds since by assumption each I_j is internal to X/E.

Proof. Note first that if X/E is the union of two definable quotients then by ($\star$) is it enough to prove the proposition for each of these separately.

We now start the proof. By uniformly partitioning each equivalence class into cells (see Lemma 9.1) and choosing a cell from each class we may assume that all E-classes are cells and all have the same dimension.

Assume that $X \subseteq M^n$ and use induction on n. When $n = 1$ then necessarily each cell is a point (otherwise X/E is finite), in which case $X = X/E$ is a finite union of points and intervals so clearly ($\star$) holds.

Consider the general case, and for every $i = 1, \dots, n$ let $\pi_i : M^n \to M$ be the projection onto the i-th coordinate. We define $\sigma_1^+ : X/E \to M \cup \{+\infty\}$ by: $\sigma_1^+([x])$ is the supremum of $\pi_1([x])$, and let $J_1^+, \dots, J_k^+$ be the definably connected components of the image of X/E under σ_1^+. Similarly, we let $\sigma_1^-([x]) \in M \cup \{-\infty\}$ be the infimum of $\pi_1([x])$, and let $J_1^-,\dots,J_r^-$ be the definably connected components of the image of X/E under σ_1^-. Note that $\pi_1([x])$ is contained in the closed interval $[\sigma^-([x]),\sigma^+([x])]$. For $1\leqslant i\leqslant k$ and $1\leqslant j\leqslant r$, we let $X_{i,j}=\{x\in X : \sigma_1^+([x])\in J_i^+$ and $\sigma_1^-([x])\in J_j^-.\}$

This is a partition of X which is compatible with E, namely if $x\in X_{ij}$ and $x'Ex$ then also $x'\in X_{ij}$. It is sufficient to prove the result for each X_{ij} separately. We consider two cases:

(a) $J_i^-\cup J_j^+$ is not definably connected.

In this case every element in J_i^- is smaller than every element in J_j^+, and we can fix an arbitrary element a_{ij} with $J_i^- < a_{ij} < J_j^+$. By the definition of σ_1^+ and σ_1^-, and since $[x]$ is definably connected, the element a_{ij} is contained in $\pi_1([x])$, for every $x\in X_{ij}$. For each $x\in X_{ij}$ we can now replace $[x]$ with $[x]\cap\pi_1^{-1}(a_{ij})$. The union of all these new classes, call it X'_{ij}, is contained in $\pi_1^{-1}(a_{ij})$. If we let E'_{ij} be the restriction of E to X'_{ij} then X_{ij}/E and X'_{ij}/E'_{ij} are in definable bijection. However, X'_{ij} can be identified with a subset of M^{n-1}, so by induction we can find X',E' as needed for the quotient X'_{ij}/E'_{ij}.

(b) $J_i^-\cup J_j^+$ is definably connected.

In this case, we let $I_1=J_i^-\cup J_j^+$. This is an interval which is internal to X/E. Because $\pi_1([x])\subseteq[\sigma_1^-([x]),\sigma_1^+([x])]$ the set X_{ij} is a subset of $J_{ij}\times M^{n-1}$.

Next, we consider the coordinate x_2 and let $\pi_2 : X\to M$ be the projection onto this coordinate. As before, we define $\sigma_2^-,\sigma_2^+ : X/E\to M\cup\{\pm\infty\}$ by

$$\sigma_2^-([x])=\inf(\pi_2([x])\ ;\ \sigma_2^+([x])=\sup(\pi_2([x])).$$

We again consider the images of these maps, partition them into intervals and proceed as before, until we can replace in each of these sets the coordinate x_2 with an element of an interval I_2 which is itself internal to X/E. We continue in this manner until we go through all coordinates $x_1,\dots,x_n$ and put each one of them in intervals which are internal to X/E. □

Here are two examples which demonstrate the above result and its proof:

Example 1 Let $X=\mathbb{R}^2$ and let E be the equivalence relation whose classes are the lines $y=x+c$, as c varies in $\mathbb{R}$. In this case for every $[x]\in X/E$, $\sigma^-([x])=-\infty$ and $\sigma^+([x])=+\infty$. So the the union of the images of these maps is the disconnected set $\{\pm\infty\}$. If we now fix the point 0 in between then each class intersects the line $X'=\{0\}\times\mathbb{R}$ in a single point. So, we can replace X/E with this line. The set X' can be identified with the interval $\mathbb{R}$ which is internal to X/E (via $g(y=x+c)=c$), so we are done.

Example 2 Consider the same relation E, restricted to the closed unit disc which we call X. Now the images of X/E under σ^- and σ^+, respectively, are the intervals $J_1^-=[-1,\sqrt{2}]$ and $J_1^+=[-\sqrt{2},1]$. The union of the intervals is a connected set $[-1,1]$ which is internal to X/E. We can now repeat the process with respect to the second coordinate and obtain X as a subset of the product $[-1,1]\times[-1,1]$, and each of these intervals is internal to X/E.

7.4. *Gp-long dimension and definable quotients*

Before our next, technical lemma, we recall the following.

Fact 7.12. If $X \subseteq M^n$ is a definable closed and bounded set, then X contains a point x_0 which is invariant under any automorphism of $\mathcal{M}$ which preserves X set-wise. Namely, $x_0 \in \mathrm{dcl}(\{X\})$, where X is now considered as an element of $\mathcal{M}^{eq}$. More generally, if $X \subseteq M^n$ is definable and $Cl(X)$ is bounded then $\mathrm{dcl}(\{X\}) \cap Cl(X) \neq \emptyset$.

Proof. This is the same as showing that every definable family of closed and bounded sets has Definable Choice. For $X \subseteq M$, we just take x_0 to be $minX$, and for $X \subseteq M^n$ we use induction. The last clause follows since the set $Cl(X)$ is definable over the parameter $\{X\}$. □

Lemma 7.13. *Let $X \subseteq M^n$ be an A-definable set such that $\mathrm{lgdim}(X) = n$. Assume that E is an A-definable equivalence relation in X, such that every equivalence class is gp-short. Then $\dim(X/E) = n$. Moreover, if $a \in X$ is gp-long generic over A then $[a]$ must be finite.*

Proof. First note why the result holds when X/E can be eliminated. Indeed, if we have a definable bijection between X/E and a definable set $Y \subseteq M^r$ then we obtain a definable surjection $g : X \to Y$ such that the preimage of every y is an E-class. Because every class is gp-short it follows from the dimension formula that $\mathrm{lgdim}(X) = \mathrm{lgdim}(Y)$ and in particular $\dim Y = n$.

We now return to our setting. Let $a \in X$ be gp-long generic over A. If we show that $\dim([a]/A) = n$ (here we view $[a]$ as an element of $\mathcal{M}^{eq}$) then clearly, $\dim(X/E) \geqslant n$, so we must have $\dim(X/E) = n$. Without loss of generality $A = \emptyset$.

Using uniform cell decomposition, we partition X into finitely many $\emptyset$-definable sets $X_1, \ldots, X_m$, such that the intersection of every class with each X_i is definably connected (possibly empty). The element a belongs to one of these X_i, in which case $\mathrm{lgdim}\, X_i = n$. We therefore may assume that each E-class in X is definably connected.

Let $p(x) = \mathrm{tp}(a/\emptyset)$. By Fact 5.15, there exists an n-long-box B with $\mathrm{Cl}(B) \subseteq p(M)$, such that a is long-central in B. Since the class $[a]$ is gp-short, we have $\mathrm{Cl}([a]) \subseteq \mathrm{Cl}(B) \subseteq p(M)$, so in particular the set $\mathrm{Cl}([a])$ is closed and bounded in M^n. By Fact 7.12, there exists $y \in \mathrm{Cl}([a])$ such that $y \in \mathrm{dcl}(\{[a]\})$.

But then, by the dimension formula $\dim(y/\emptyset) \leqslant \dim([a]/\emptyset)$ and since $y \models p$, we have $\dim(y/\emptyset) = n$ (here we use the fact that a was a generic element of M^n). Hence, $\dim([a]/\emptyset) = n$, so we are done.

To see that $[a]$ must be finite consider the set $Y \subseteq X$ of all $x \in X$ such that $[x]$ is infinite. By definition of dimension we must have $\dim(Y/E) < \dim Y \leqslant n$. Hence, by what we have just showed, $\mathrm{lgdim}(Y) < n$. It follows that Y cannot contain any gp-long generic element of X. □

8. Interpretable groups

We assume now that G is an interpretable group (as in Definition 1.4).

8.1. *One dimensional sets in interpretable groups*

Definition 8.1. Let Y/E be an $\mathcal{M}$-quotient such that $\dim(Y/E) = 1$. Then we call Y/E *gp-short* if it is in definable bijection with a definable gp-short subset of M^n. Otherwise, we call it *gp-long*.

Theorem 8.2. *Let $G = X/E$ be an interpretable group. Then every one-dimensional subset of G is gp-short.*

Proof. Without loss of generality, G is defined over $\emptyset$. We let $I \subseteq G$ be a $\emptyset$-definable one-dimensional set. By Corollary 7.8, I is in bijection with a definable subset of M, and so we identify I with this definable subset and assume that the intersection of every E-class with $I \subseteq X$ is a singleton. We suppose towards contradiction that I is gp-long.

For every k we let $f_k : I^k \to G$ be the function defined by $f(x_1, \dots, x_k) = x_1 \cdots x_k$ (multiplying in G).

We take $k \geqslant 1$ maximal such that on some k-long box $B \subseteq I^k$ the function f_k is finite-to-one on B. Notice that a maximal k exists because we assume f_k to be finite-to-one and hence $k = \dim(B) = dim(f_k(B)) \leqslant \dim G$. By taking a sub-box of B we may assume that f_k is injective on B. We also assume that B is definable over $\emptyset$ and let $\bar{a} \in B$ be gp-long generic in B.

Claim 8.3. *Let a_{k+1} be gp-long generic in I over $\bar{a}$. Then there is a $k+1$-long box $B' \subseteq B \times I$ containing $a' = \langle \bar{a}, a_{k+1} \rangle$, such that $f_{k+1}(B')$ is contained in $f_{k+1}(B \times \{a_{k+1}\}) \subseteq f_k(B) \cdot a_{k+1}$.*

Proof. Define on $B \times I$ the equivalence relation $xE'y$ iff $f_{k+1}(x) = f_{k+1}(y)$. By the maximality assumptions on k, the union of all finite E-classes must have gp-long dimension smaller than $k+1$ (otherwise f_{k+1} is a finite-to-one map on a $k+1$-long box. Therefore, since $\mathrm{lgdim}(\langle a, a_{k+1} \rangle / \emptyset) = k+1$, the E'-class of $a' = \langle a, a_{k+1} \rangle$ is infinite.

We claim that $\mathrm{lgdim}[a'] > 0$. Indeed, assume towards contradiction that $[a']$ is gp-short. By Fact 4.6(iii), there is a formula $\psi(y)$ over $\emptyset$ such that $\psi(a')$ holds and if $\psi(b)$ holds then $[b]$ is gp-short. Thus, there exists, by Fact 5.13, a $k+1$-long box $B_0 \subseteq B \times I$ containing a' such that for every $x \in B_0$, the E'-class $[x]$ is infinite and gp-short. This contradicts Lemma 7.13.

We therefore showed that the E'-class of a' is not gp-short. A similar argument can show a stronger statement, namely that the definably connected component of $[a']$ which contains a', call it $[a']^0$, is also not gp-short.

Because $f_k|B$ was injective the projection of each E'-class on the $k+1$-coordinate is injective as well. It follows that the image of $[a']$ under this projection is gp-long, call it J.

By Fact 5.7(3), we may replace J by a possibly smaller gp-long interval and so assume that the gp-long dimension of a' over the parameter set A' defining J is still $k+1$. Let $p(x) = \mathrm{tp}(a'/A')$. By 5.15, there exists a $k+1$-long box $B' \subseteq p(M)$, in which a' is long-central. Because $B' \subseteq p(M)$, for every $x \in B'$ the projection of $[x]^0$ onto the last coordinate contains J. In particular, this projection contains the point a_{k+1}. This means that every $x' \in B'$ has an E'-equivalent element of the form $\langle x, a_{k+1}\rangle$, with $x \in B$, and hence $f_{k+1}(x') = f_k(x)a_{k+1}$.

This ends the proof of the claim. □

Let's recall what we have so far: (i) The restriction of f_k to B is an injective map and (ii) $f_{k+1}(B') \subseteq f_k(B) \cdot a_{k+1}$.

Since $f_{k+1}(x, a_{k+1}) = f_k(x)a_{k+1}$, (i) implies that the restriction of f_{k+1} to $B \times \{a_{k+1}\}$ is also injective. Therefore, we have a definable bijection

$$\sigma : f_k(B)a_{k+1} \to B$$

(given by $\sigma(y) = f_k^{-1}(ya_{k+1}^{-1})$) (where a_{k+1}^{-1} is the group inverse in G of a_{k+1}).

By (ii), we have a map from the $k+1$-long box B' into B, defined by $h(x_1, \dots, x_{k+1}) = \sigma(f_{k+1}(x_1, \dots, x_{k+1}))$. Notice that because f_{k+1} is group multiplication and σ is injective, the map h is injective in each coordinate separately. By Corollary 6.10, at least one of the intervals which make up B' must be gp-short, contradicting the fact that B' was a $k+1$-long box. This shows that I is gp-short, thus ending the proof of Theorem 8.2.

8.2. *Endowing interpretable groups with a topology*

A fundamental tool in the theory of definable groups in o-minimal structures is Pillay's theorem, [5], on the existence of a definable basis for a group topology on a definable group G, a topology which agrees with the subspace M^n-topology at every generic point of G (with $G \subseteq M^n$). Moreover, this topology can be realized using finitely many charts, each definably homeomorphic to an open subset of M^k (with $k = \dim G$).

Here we prove an analogous result for interpretable groups but the topology we obtain will initially not have finitely many charts.

We start with some preliminary definitions and results.

Definition 8.4. A definable set $Y \subseteq G$ is called *large in G* if $\dim(G \setminus Y) < \dim G$.

Fact 8.5. Let G be an interpretable group, $Y \subseteq G$ a definable set over A. If Y is large in G and $\dim G = n$ then G can be covered by $\leqslant n+1$ translates of Y.

Proof. This is standard, by induction on $\dim(G \setminus Y)$. We have $\dim(G \setminus Y) \leqslant n-1$, hence every generic element of G (over A) is in Y. We take g generic in G over A

and $h \in G$ independent from g over A (i.e. $\dim(h/gA) = \dim(h/A) = m$). Then, by the dimension formula, $\dim(hg^{-1}/hA) = \dim(g/A)$ and hence hg^{-1} is generic in G over A. It follows that $hg^{-1} \in Y$ and therefore $h \in Yg$.

We showed that every element in G which is independent from g over A belongs to $Y \cup Yg$. It follows that $\dim(G \setminus (Y \cup Yg)) < \dim(G \smallsetminus Y)$, so $\dim(G \setminus (Y \cup Yg)) \leqslant n - 2$. We proceed by induction. □

Defining the topology

We first obtain $U_1, \ldots, U_k$ as in Claim 7.6. Namely, each U_i is an open subset of M^{k_i} and each class in U_i has dimension d_i and projects homeomorphically onto the first d_i coordinates. For each i, write $x = \langle x', x'' \rangle \in M^{d_i} \times M^{k_i - d_i}$, with $x' = \pi_{d_i}(x)$. Since every E-class projects bijectively into M^{d_i}, the set $U_i(x) = \pi^{-1}(x') \cap U_i$ has a single representative for each E-class (if $d_i = 0$ then $x = x'' \in M^{k_i}$). It is contained in the set $\{x'\} \times M^{k_i - d_i}$ and because U_i is open the set $U_i(x)$ can be identified with an open subset of $M^{k_i - d_i}$. We say that $V \subseteq U_i(x)$ is an $M^{k_i - d_i}$-open set, if under this identification V is open. We have an obvious x-definable injection of each $U_i(x)$ into G. For $\langle x', x'' \rangle \in U_i$, let $[x', x'']$ denote the element of G by $[\langle x', x'' \rangle]$.

Fact 8.6. In the above setting,

1. For $g \in U_i$ and $x = \langle x', x'' \rangle$ generic in the class g (over the element $[g] \in \mathcal{M}^{eq}$ and $A \subseteq M^{eq}$), we have $\dim(x'/gA) = d_i$ and $x'' \in \mathrm{dcl}(x'g)$.
2. Assume that $x = \langle x', x'm \rangle$ is generic in U_i. Then x'' is generic in $U_i(x)$ over x'.
3. If $x = \langle x', x'' \rangle$ is generic in U_i, $h \in G$, and $y = \langle y', y'' \rangle \in U_j \subseteq M^{d_j} \times M^{k_j - d_j}$ is generic in the class h over the elements x, h then $\dim(x''/x'y') = \dim(x''/\emptyset) = k_i - d_i$.

Proof. (1), (2) are immediate from the fact that each class in U_i projects bijectively onto the first d_i coordinates. For (3), let $y \in U_j$ and note that by genericity, $\dim(y/x, h) = \dim([y]) = d_j$ and therefore by (1), $\dim(y'/x, h) = d_j$. But then, since $y' \in M^{d_j}$ we clearly must have

$$\dim(y'/x') = \dim(y'/x) = d_j.$$

By the dimension formula

$$\dim(y'/x'x'') + \dim(x''/x') + \dim(x'/\emptyset) = \dim(x''/x'y') + \dim(y'/x') + \dim(x'/\emptyset).$$

Since $\dim(y'/x'x'') = \dim(y'/x')$, we have $\dim(x''/x'y') = \dim(x''/x') = k_i - d_i$. □

We assume from now on that for $i = 1, \ldots, r$, we have $\dim(U_i/E) = k_i - d_i = \dim G = n$ and for $i = r+1, \ldots, k$ we have $\dim(U_i/E) < \dim G$. Let $U = U_1$.

Fact 8.7. Let $f : G \to G$ be a partial A-definable function. Let $x = \langle x', x'' \rangle$ be a generic element of U over A and let $g = [x]$. Let $h = f(g)$ and choose $y = \langle y', y'' \rangle$ a generic element of the class $h = f(g)$ over x.

If $y \in U_j$, for some $j = 1, \ldots, k$, then there is an $Ax'y'$-definable open M^n-neighborhood $V \subseteq U(x)$ such that for every $z \in V$ there is a unique element $w \in U_j(y)$, with $f([x', z]) = [y', w]$. We denote this local map $z \mapsto w$, defined over $x'y'$, by f^*. The map f^* is continuous at x'', as a function from an open subset of M^n into M^n (recall $n = k_j - d_j = k_i - d_i$).

Proof. By Fact 8.6(3), x'' is generic in $U(x)$ over $x'y'$. We now consider the formula $\phi(z)$, over the parameters $Ax'y'$, which says that there is a unique element $w \in U_j(y)$ such that $[y', w] = f([x', z])$. The formula $\phi(z)$ holds for x'', which is generic in $U(x)$ over $x'y'$. It follows that there exists an M^n-neighborhood $V \subseteq U(x)$ of x'' such that every $z \in V$ satisfies ϕ. We therefore obtain a function, definable over $Ax'y'$, from V into $U_j(y)$, and by genericity of x'', this function is continuous near x''. □

Theorem 8.8. *Let $x_0 = \langle x', x'' \rangle$ be a generic element in U and let $\{V_t : t \in T\}$ be a definable basis of (sufficiently small) M^n-neighborhoods of x'', all contained in $U(x_0)$.*

Then the family

$$\mathcal{B} = \{gV_t : g \in G,\ t \in T\}$$

is a basis for a topology on G, making G into a topological group.

Proof. Consider $g_0 = [x_0]$ and the family $\{g_0^{-1}V_t : t \in T\}$. Just like the proof of Lemma 2.12 in [17], we will prove that this family forms a basis of neighborhoods of $1 \in G$, for a group-topology on G whose basis is $\mathcal{B}$. Indeed, it is not hard to see that $\mathcal{B}$ is a basis for some topology, call it the *t-topology on G*. To see that this topology makes G into a topological group, we first prove:

Claim 8.9. *Let g be generic in G over g_0 and let $y = \langle y', y'' \rangle$ be generic in the class g over g, g_0. Then there is an open M^n-neighborhood $W \subseteq U(y)$ of y'' and a t-neighborhood $V \subseteq G$ of g such that the canonical embedding of $U(x)$ into G induces a homeomorphism of W and V.*

Roughly speaking, we say that the t-topology coincides with the M^n-topology in some neighborhood of g.

Proof. Consider the map $\sigma(h) = gg_0^{-1}h$. It is definable over the element gg_0 and takes g_0 to g. Consider the first-order formula $\phi(z)$ over x', y' which says that $z \in U(x)$ and there is a unique $w \in U(y)$ such that $\sigma(z) = w$ (here we identify $U(x)$ and $U(y)$ with subsets of G). The formula ϕ holds for x''. By Fact 8.6 (3), x'' is generic in $U(x)$ over x', y' hence there exists an M^n-neighborhood $V \subseteq U(x)$ of x'' such that for every $z \in V$, $\sigma(z) \in U(y)$. Hence σ defines a function from V into $U(y)$ sending x'' to y''. By the genericity of x'', we can choose such V so that σ is continuous, as a map from M^n into M^n. Because σ is invertible, we can use the same argument to find $W \subseteq U(y)$ for which σ^{-1} is also continuous, as a map into $U(y)$. Shrinking V and W if needed we may assume that $\sigma : V \to W$ is a homeomorphism with respect to the M^n-topology. Since left multiplication leaves $\mathcal{B}$ invariant, σ is also a homeomorphism with respect to the t-topology. It follows that the t-topology agrees with the M^n-topology on W. □

Claim 8.10. *Assume that $f : G \to M^d$ is an Ag_0-definable partial function and g is generic in G over Ag_0. Then f is continuous at g (with respect to the t-topology on G and the standard topology on M^d).*

Proof. Choose $y = \langle y', y'' \rangle$ generic in the class g over g, g_0. Consider now the map f, as a function from $U_j(y)$ into M^d. Since y'' is generic in $U_j(y)$ over $Ay'g_0$, this map is continuous near y'', with respect to the M^n-topology of $U_j(y)$, and hence, by Claim 8.9, also with respect to the t-topology. □

We also have:

Claim 8.11. *We fix g_0 as above. If $f : M^k \to G$ is a g_0-definable function then f is continuous at every point z generic in its domain (with respect to the M^k-topology and the t-topology),*

Proof. Let $h = f(z)$ and take g_1 generic in G over g_0, h, z. Instead of considering the map f we consider $\sigma(w) = g_1 h^{-1} f(w)$, which sends z to g_1. Since left multiplication is a homeomorphism (as it preserves the family $\mathcal{B}$) it is sufficient to show that σ is continuous at z. Using Claim 8.9, we can reduce the problem to a map from M^k into $U_j(y)$, with $[y] = [y', y''] = g_1$ and y generic in the class g_1 over all parameters. After noting that $\dim(z/g_0, g_1, y') = \dim(z/g_0, g_1)$, so z is still generic in the domain of f over $g_0 g_1 y'$, the result now follows from the theory of definable maps from M^k into M^n. □

The above results allow us to replace in many cases the t-topology by the M^n-topology, so we can follow the arguments from [17] and conclude in the same way that $\mathcal{B}$ defines a group-topology on G.

Since the t-topology has basis for neighborhoods given by open subsets of M^n, it means that, at least locally, many properties of the o-minimal topology still hold for the t-topology. A straightforward claim helps here:

Claim 8.12. *Let $Y \subseteq G$ be A-definable and let g be generic in Y over Ag_0. Then for every definable t-open set $V \ni g$, we have $\dim(Y \cap V) = \dim(Y)$.*

Proof. Replace V with a neighborhood $W \subseteq V$ of g which is definable over parameters B, with $\dim(g/B) = \dim(g/Ag_0)$. Indeed, this is possible to do: We may assume that $V = gg_0^{-1}V_t$ for $t \in T$, and we have $\dim(g/g_0, A) = \dim(g/gg_0^{-1}, A)$. We now replace V_t by $V_s \subseteq V_t$, with s generic in T over all parameters. We therefore have $\dim(g/gg_0^{-1}, s, A) = \dim(g/g_0, A)$.

The neighborhood $W = gg_0^{-1}V_s$ is the desired neighborhood of g. Since the dimension of g over the parameters defining $Y \cap W$ equals $\dim(g/A) = \dim(Y)$, we have $\dim(Y \cap W) = \dim(Y)$. □

Fact 8.13.

1. If $Y \subseteq G$ is a definable set then $\dim(\mathrm{Cl}(Y) \setminus Y) < \dim Y$ (the closure here is taken with respect to the t-topology.
2. If H is a definable subgroup of G then H is closed in G.

Proof. We prove (1) – the proof of (2) is as in [5, Corollary 2.8]. Assume towards contradiction that $\dim(\mathrm{Cl}(Y) \setminus Y) \geqslant \dim Y$. In particular, $\dim \mathrm{Cl}(Y) = \dim(\mathrm{Cl}\, Y \setminus Y)$. Let g be generic in both $\mathrm{Cl}(Y)$ and $\mathrm{Cl}(Y) \setminus Y$, let $h \in U$ be generic in G over g, and let V be a neighborhood of g small enough that every element of $hg^{-1}V$ is represented in an M^n neighborhood of h inside $U(h)$. Using Claim 8.12, $\dim((\mathrm{Cl}(Y) \setminus Y) \cap V) = \dim(\mathrm{Cl}(Y) \setminus Y)$ and $\dim(\mathrm{Cl}(Y) \cap V) = \dim(\mathrm{Cl}(Y))$. Translating by the t-homeomorphism $x \mapsto hg^{-1}x$, we get $Y' = hg^{-1}(Y \cap V)$, and $\mathrm{Cl}(Y') \setminus Y' = hg^{-1}((\mathrm{Cl}(Y) \setminus Y) \cap V)$. These sets are in definable bijection with definable sets in an M^n neighborhood of h inside $U(h)$, for which the closure operation is the standard one, so $\dim(Y') > \dim(\mathrm{Cl}(Y') \setminus Y')$. However, translation is dimension-preserving so we reach a contradiction. □

Although we cannot obtain at this point a finite atlas on G, we have an approximation to it: Let $\mathcal{U}$ be the disjoint union $U_1 \sqcup \cdots \sqcup U_r$. We say that $W \subseteq \mathcal{U}$ is open if $W \cap U_i$ is open for every $i = 1, \ldots, r$. We say that $X \subseteq \mathcal{U}$ is large in $\mathcal{U}$ if $X \cap U_i$ is large in U_i for every $i = 1, \ldots, r$. Note that if $W \subseteq \mathcal{U}$ is large in $\mathcal{U}$ then its image in G is large in G (since $\mathcal{U}$ is large in G).

As we showed above, if $y = \langle y', y'' \rangle$ is generic in U_i, for $i = 1 \ldots, r$, then the t-topology agrees with the M^n-topology on $U(y)$, near y''. This property of y is first order, so the set $\mathcal{U}_0$ of all $y \in \mathcal{U}$ for which the t-topology agrees with the M^n-topology on $U_i(y)$ near y, is definable and contains every generic element in $\mathcal{U}$. Hence, this set is large in $\mathcal{U}$.

Let $\pi : \mathcal{U}_0 \to G$ be the quotient modulo E. By definition of $\mathcal{U}_0$, the map $\pi : \mathcal{U}_0 \to G$ is open, when $\mathcal{U}_0$ is endowed with the o-minimal topology and G has the t -topology (indeed, for every open $V \subseteq \mathcal{U}$ ans $x \in V$, the set $U(x) \cap V$ is open in M^n hence $\pi(U(x) \cap V)$ is open in G). Next, we can apply Claim 8.11 and replace $\mathcal{U}_0$ by a large open subset, call it $\mathcal{U}_0$ again, on which π is continuous, and still open. Let $W = \pi(\mathcal{U}_0)$, and note as above that W is large in G. By Fact 8.5, finitely many G-translates of W, $h_1W, \ldots, h_mW$, cover G. We can now conclude:

Proposition 8.14. *There are finitely many t-open definable sets $W_1, \ldots, W_k \subseteq G$ whose union covers G. There exist a definable set $\mathcal{U}_0$ which is a finite disjoint union of definable open subsets of M^{r_i}'s and for each $i = 1, \ldots, k$ a definable surjective (but in general not injective) map $\pi_i : \mathcal{U}_0 \to W_i$, such that each π_i is continuous* *and open (with respect to the o-minimal topology in the domain and the t-topology in the image).*

As a corollary we have:

Corollary 8.15. *Every definable subset of G has finitely many definably connected components with respect to the t-topology.*

Proof. Fix $W_1, \ldots, W_k$ as above. Take $Y \subseteq G$ definable, It is enough to see that each $Y \cap W_i$ has finitely many definably connected components. As we saw, there is a definable and continuous map from U_0 onto W_i. The pre-image of $Y \cap W_i$ is a definable subset of U_0 so has finitely many definably connected components (with respect to the o-minimal topology). By continuity, $Y \cap W_i$ also has finitely many components. □

We can now also prove, just as in the definable case (see [3]):

Lemma 8.16. *For G interpretable, and H a definable subgroup of G, the following are equivalent:*

1. *H has finite index in G.*
2. *$\dim H = \dim G$.*
3. *H contains an open neighborhood of the identity.*
4. *H is open in G.*

Exactly as in the case of definable groups, we can deduce the descending chain condition:

Corollary 8.17. *Every descending chain of definable subgroups of G is finite.*

8.3. *Definable compactness*

Below, all limits in G are taken with respect to the t-topology

Our goal now is to review briefly several fundamental notions and results in the theory of definable groups and to verify that these results hold for interpretable G as well. The intention is to collect just those results which will allow us to prove that G is definably isomorphic to a definable group.

Recall that every definable one-dimensional subset of G is in definable bijection with finitely many points and open intervals (Corollary 7.8).

Definition 8.18. We say that G is *definably compact* if for every definable f from an open interval (a, b) into G, the limits of $f(x)$ as x tends to a and to b exist in G.

As in the case of definable groups ([18]) we have:

Lemma 8.19. *If G is not definably compact then it contains a definable, torsion-free one-dimensional subgroup $H \subseteq G$.*

Proof. We review briefly the proof as suggested in [19]. Assume that the limit $\lim_{x\to b} f(x)$ does not exist in G. By Lemma 8.11, we may assume that f is continuous on (a,b). The group H is defined to be the set of all possible limits of $f(t)f(s)^{-1}$, as t and s tend to b in the interval (a,b). More precisely, *H is the collection of all $h \in G$ such that for every t-neighborhood $V \ni h$ and every $a_0 \in (a,b)$ there exist $x, x' \in (a_0, b)$ for which $f(x)f(x')^{-1} \in V$.*

Since G has a definable basis for the t-topology, H is definable. Note that by o-minimality, if $h \in H$, $V \ni h$ and $a_0 \in (a,b)$, then *for every $x' \in (a_0,b)$* sufficiently close to b there exist $x \in (a_0,b)$ with $f(x)f(x')^{-1} \in V$.

To see that H is a subgroup, take $g,h \in H$ and show that $gh^{-1} \in H$: Fix a t-neighborhood $V \ni gh^{-1}$ and find t-neighborhoods $V_1 \ni g$ and $V_2 \ni h$ such that $V_1V_2^{-1} \subseteq V$. By the above, there exists $x' \in (a_0,b)$ sufficiently close to b and there are $x_1, x_2 \in (a_0,b)$ such that both $f(x_1)f(x')^{-1} \in V_1$ and $f(x_2)f(x')^{-1} \in V_2$. It follows that $f(x_1)f(x_2)^{-1} \in V_1V_2^{-1} \subseteq V$ as required, so $gh^{-1} \in H$.

The proof that H has dimension at least one is similar to the proof in [18, Lemma 3.8] because the identity element of G has a neighborhood R homeomorphic to a rectangular open subset of M^n: For every $a_0 \in (a,b)$ we have $f(a_0)f(a_0)^{-1} \in R$ and since $f(x)$ has no limit in G as x tends to b, for all $x' \in (a_0,b)$ close enough to b, we have $f(a_0)f(x')^{-1} \notin R$, if R is chosen sufficiently small. It follows that there exists $x'' \in (a_0,b)$ with $f(a_0)f(x'')^{-1} \in \mathrm{bd}(R)$. Because $\mathrm{bd}(R)$ is definably compact, as a_0 tends to b, the set of all of these points in $\mathrm{bd}(R)$ has a limit point which belongs to H. We therefore showed that every sufficiently small rectangular box $R \ni 1$ has a point from H on its boundary, so $\dim(H) \geqslant 1$.

Let's see that $\dim(H) \leqslant 1$: The set $D = \{\langle x, x', f(x)f(x')^{-1}\rangle \in (a,b)^2 \times G\}$ has dimension two and therefore its frontier $\mathrm{fr}(D) \subseteq [a,b]^2 \times G$ has dimension at most 1. The group H is contained in the projection of $\mathrm{fr}(D)$ onto the G-coordinate, so its dimension is at most 1. The fact that H is torsion-free is proved similarly to [18].□

On the definably compact side we need:

Theorem 8.20. *If G is definably compact then it has Definable Choice (possibly over a fixed set of parameters) for subsets of G definable in $\mathcal{M}^{eq}$. Namely, there is a fixed set $B \subseteq M$ such that if $\{Y_t : t \in T\}$ is a $\emptyset$-definable family of subsets of G, with T definable in M^{eq}, then there is a B-definable map $\sigma : T \to G$ such that for each $t \in T$, we have $\sigma(t) \in Y_t$, and if $Y_t = Y_s$ then $\sigma(t) = \sigma(s)$.*

Equivalently, if $Y \subseteq G$ is definable over $A \subseteq \mathcal{M}^{eq}$ then $\mathrm{dcl}(AB) \cap Y \neq \emptyset$.

Proof. Let us note why the two statements are indeed equivalent. Assume that we proved Definable Choice over B for families parameterized by a definable subset of M^{eq} and assume that Y is definable over a finte $a \subseteq \mathcal{M}^{eq}$. In this case there is a $\emptyset$-definable family of sets $\{Y_t : t \in T\}$, for some $\emptyset$-definable set $T \subseteq \mathcal{M}^{eq}$, with $a \in T$ and $Y_a = Y$. Definable Choice implies that $Y \cap \mathrm{dcl}(aB) \neq \emptyset$. As for the converse, assume that we are given the family $\{Y_t : t \in T\}$ and consider the equivalence relation on T given by $s \sim t$ if and only if $Y_s = Y_t$. We now obtain a new family $\{Y_{[t]} : [t] \in T/\sim\}$, with $Y_{[t]} = Y_t$. By our assumption, for every $[t]$, we have $Y_{[t]} \cap \mathrm{dcl}(B[t]) \neq \emptyset$. But for each $t \in T$, $[t] \in \mathrm{dcl}(Bt)$, and therefore $Y_{[t]} \cap \mathrm{dcl}(Bt) \neq \emptyset$. Definable Choice over B follows by compactness.

We now prove the theorem. The strategy of our proof is taken from Edmundo's [6].

Lemma 8.21. *For $G = X/E$ definably compact, let $Y \subseteq G$ be a definable set over $A \subseteq M^{eq}$. Then $\mathrm{dcl}(A) \cap \mathrm{Cl}(Y) \neq \emptyset$.*

Proof. First, note that $\mathrm{Cl}(Y)$ is also definably compact.

We are going to prove a slightly different statement: *For every A-definable set $Y^* \subseteq M^k$ (for some k) and for every A-definable function $g : Y^* \to G$, we have* $\mathrm{dcl}(A) \cap \mathrm{Cl}(g(Y^*)) \neq \emptyset$ (to apply this statement to our case take $Y^* \subseteq X$ the pre-image of Y under the quotient map and take g to be the quotient map).

We use induction on $\ell = \dim Y^*$. If $\ell = 0$ then Y^* is finite so every element of Y^* is in $\mathrm{dcl}(A)$ (see the earlier property DEQ) and therefore $g(Y^*) \subseteq \mathrm{dcl}(A)$.

Assume now that $\dim Y^* = \ell > 0$. If $\ell = 1$ then Y^* is a finite union of A-definable open intervals and the restriction of g to one of these gives an A-definable function $g : (a, b) \to G$. Its image is either finite, so again in $\mathrm{dcl}(A)$ (see [3]), or infinite in which case, by definable compactness, the limit point of $g(y)$ as y tends to b, exists in $\mathrm{Cl}(g(Y^*))$ and is A-definable.

Assume then that $\ell > 1$. We find a projection, $\pi^* : Y^* \to M^{\ell-1}$ whose image has dimension $\ell - 1$. For every $t \in \pi^*(Y^*)$, let $Y^*_t \subseteq Y^*$ be the pre-image of t under π^*. By dimension considerations, we can find an A-definable set $T \subseteq \pi^*(Y^*)$ such that for every $t \in T$, $\dim(Y^*_t) = 1$. Because $\dim Y^*_t = 1 < \ell$, we have, by induction, $\mathrm{dcl}(At) \cap \mathrm{Cl}(g(Y^*_t)) \neq \emptyset$. Using logical compactness, we get an A-definable function $\sigma : T \to G$ with $\sigma(t) \in \mathrm{Cl}(g(Y^*_t))$ for every $t \in T$. Because $\dim T < \ell$, we can apply induction and obtain

$$\mathrm{dcl}(A) \cap \mathrm{Cl}(\sigma(T)) \neq \emptyset.$$

But $\sigma(T) \subseteq \mathrm{Cl}(g(Y^*)$, so we are done. □

Lemma 8.22. *There exists a finite set B and a B-definable neighborhood $U_0 \ni 1$ in G such that G has Definable Choice over B, for definable subsets of U_0.*

Proof. Start with a fixed neighborhood U_0 of $1 \in G$, which we may assume is a subset of M^n. The group G induces on U_0 the structure of a local group, so just like in [20, Lemma 1.28], we may assume, by further shrinking U_0, that U_0 is a product of intervals, each endowed with the structure of a bounded group-interval (this might require the parameter set B). By Fact 4.4, U_0 has definable choice. □

We can now complete the proof of the theorem. Take an a-definable $Y \subseteq G$. By Lemma 8.21, there exists $h \in \mathrm{dcl}(A) \cap \mathrm{Cl}(Y)$. We can now replace Y by $Y_1 =$

$h^{-1}Y \cap U_0$. The set Y_1 is AB-definable and because $h \in \mathrm{Cl}(Y)$, the set Y_1 is also non-empty. By Lemma 8.22, we have $\mathrm{dcl}(ABh) \cap Y_1 \neq \emptyset$. But h is in $\mathrm{dcl}(A)$ so we have $\mathrm{dcl}(AB) \cap Y \neq \emptyset$.

8.4. *Interpretable groups are definable*

Theorem 8.23.

1. *If G is an interpretable group then it is definably isomorphic, over parameters, to a definable group.*
2. *If G is an interpretable group then there are generalized group-intervals $I_1, \dots, I_k$ and a definable injection $\sigma : G \to I_1 \times \cdots \times I_k$. Namely, G is definably isomorphic, over parameters, to a definable group which is a subset of a cartesian product of generalized group-intervals. We can also replace each group-interval I_j with a one-dimensional definable group H_j.*

Proof. We are going to prove the following statement, which incorporates both (1) and (2): *Every interpretable group G is definably isomorphic to a definable group which is gp-short.* By Fact 4.1, this implies the theorem.

We prove the results through several lemmas.

Lemma 8.24. *The result holds for G definably compact.*

Proof. By Proposition 7.11, there are intervals $I_i \subseteq M$, $i = 1 \dots, k$, each the image of G^{n_i} under a definable map $f_i : G^{n_i} \to I_i$ and a definable set $X' \subseteq \Pi_i I_i$ with a definable equivalence relation E' on X', such that G is definably bijective to X'/E'.

By Theorem 8.20, G (and hence each G^{n_i}) has Definable Choice. Therefore there are definable 1-dimensional subsets $J_i \subseteq G^{n_i}$, $i = 1, \dots, k$, such that $f_i | J_i : J_i \to I_i$ is a bijection. By Theorem 8.2, every J_i is gp-short and therefore each I_i is group-short. It follows that $\Pi_i I_i$ has definable choice (Fact 4.4), so Y/E' is in definable bijection with a definable subset of $\Pi_i I_i$ and each I_i is gp-short. □

Lemma 8.25. *Assume that $H_1 \subseteq G$ is a definable normal subgroup, and assume that H_1 and G/H_1 are each definably isomorphic to a definable, gp-short group. Then so is G.*

Proof. As in the proof of Lemma 8.24, it is sufficient to prove, for every definable map $f : G \to M$, that $f(G)$ is gp-short. Let $\pi : G \to G/H_1$ be the quotient map. For each $y \in G/H_1$, $G_y = \pi^{-1}(y)$ is in definable bijection with H_1 and therefore it is in definable bijection with a gp-short definable set. We now write $f(G)$ as a definable union $\bigcup_{y \in G/H_1} f(G_y)$. Each set $f(G_y)$ is gp-short and the parameter set G/H_1 is gp-short, so by Lemma 4.10, the union $f(G)$ is gp-short. □

Lemma 8.26. *If G is abelian then G is definably isomorphic to a definable group, which is gp-short.*

Proof. By Lemma 8.19, we can find a chain of definable groups $A_1 \leqslant \cdots \leqslant A_k \leqslant G$, such that $\dim(A_i/A_{i-1}) = 1$ and G/A_k is definably compact. By Corollary 7.8, each one-dimensional group is definably isomorphic to a definable group, and by Theorem 8.2, each such group is gp-short. So, using Lemma 8.25, we see that A_k is definably isomorphic to a definable, gp-short group. By Lemma 8.24, G/A_k is definably isomorphic to a definable (gp-short) group, so again by 8.25, the group G is definably isomorphic to a definable gp-short group. □

Lemma 8.27. *If G is definably simple (namely, G is non-abelian and has no definable non-trivial normal subgroup) and definably connected then G is definably isomorphic to a definable group which is gp-short.*

Proof. We fix $U_0 \ni 1$ a definable neighborhood which we may assume to be an open subset of M^n. The rest of the argument is identical to the proof in [20], because all that was used there was the basic facts about definable groups (whose analogues we proved here for interpretable groups) together with the existence of an M^n-neighborhood of the identity in G. To recall, the fact that G is centerless implies that we can write U_0 as a cartesian product of open rectangular boxes, pairwise orthogonal, $R_1 \times \cdots \times R_s$, where each R_j is itself a cartesian product of intervals which are non-orthogonal to each other (see Theorem 3.1 in [20]). Since G is definably simple we can show that there is only one such box, so we may write U_0 as a single cartesian product of pairwise non-orthogonal group-intervals. Moreover, each interval supports the structure of a definable real closed field and all these real closed fields must be definably isomorphic to each other (see [20, Theorem 3.2]). We now have a neighborhood U_0 of $1 \in G$ which we may assume to be a neighborhood of $0 \in R^n$ for a definable real closed field R. We repeat the construction of the Lie algebra $L(G)$ in R (which only requires working in a neighborhood of 1), and finally embed G into $GL(n, R)$ using the adjoint embedding. Clearly, the group $GL(n, R)$ is gp-short. □

We can now prove Theorem 8.23: We use induction on $\dim G$. By Lemma 8.25, we may assume that G is definably connected. If G has a definable infinite proper normal subgroup H_1 then, by induction, both H_1 and G/H_1 satisfy the result so again by 8.25 we are done. So, we may assume that no such infinite definable proper normal subgroup exists.

In this case, by DCC and the connectedness of G, every (finite) normal subgroup must be contained in $Z(G)$, which by assumption must be finite. Again, using Lemma 8.25, we can replace G by $G/Z(G)$, which now has no definable non-trivial normal subgroup. We are left with two possibilities: either G is abelian or definably simple, so we are done by 8.26 and 8.27.

To replace each I_j with a definable one-dimensional group, use Lemma 3.5.

9. Appendix: A uniform cell decomposition

Lemma 9.1. *let $\{X_t : t \in T\}$ be a $\emptyset$-definable family of subsets of M^k. Then there are finitely many $\emptyset$-definable collections $\{X_t^i : t \in T\}$, $i = 1, \dots, m$, such that: (i) For each $i = 1, \dots, m$ and each $t \in T$, $X_t^i \subseteq M^k$ is a cell, or an empty set. (ii) For each $t \in T$, X_t is the disjoint union of $X_t^1, \dots X_t^m$. (iii) For each $t, s \in T$, and $i = 1, \dots, m$, if $X_t = X_s$ then $X_t^i = X_s^i$.*

Proof. It is sufficient to prove: Assume that $X \subseteq M^n$ is definable over a parameter set $A \subseteq \mathcal{M}^{eq}$. Then there is a cell decomposition of X that is definable over A. Indeed, if we do that then we can define on the above T the equivalence relation $t \sim s$ iff $X_t = X_s$. We replace the original family with $\{X_{[t]} : t \in T/\sim\}$, with $X_{[t]} = X_t$. If each $X_{[t]}$ has a $[t]$-definable cell decomposition then, by compactness, there is a uniform cell decomposition of the X_t's parameterized by $T/\sim$. This easily gives us the required result.

We now fix $X \subseteq M^n$ and consider the o-minimal structure $\mathcal{M}_X = \langle M, <, X \rangle$ with a new predicate for X. Since the standard cell decomposition theorem holds in this structure there are 0-definable, pairwise disjoint cells $C_1, \dots, C_m$ whose union is X. Each C_i is clearly invariant under every automorphism of $\mathcal{M}_X$. Each C_i is given by a formula $\xi_i(x)$ in the structure $\mathcal{M}_X$. If we now return to $\mathcal{M}$, each $\xi_i(x)$ can be transformed into an $\mathcal{M}$-formula, possibly with parameters, which we call $\xi_i(x, a_i)$.

Each set $\xi_i(M^k, a_i)$ is invariant under any automorphism of $\mathcal{M}$ which fixes X set-wise, so in particular under any automorphism which fixes A point-wise. □

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