

ALGEBRAIC AND O-MINIMAL FLOWS ON COMPLEX AND REAL TORI

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ABSTRACT. We consider the covering map $\pi : \mathbb{C}^n \rightarrow \mathbb{T}$ of a compact complex torus. Given an algebraic variety $X \subseteq \mathbb{C}^n$ we describe the topological closure of $\pi(X)$ in \mathbb{T} . We obtain a similar description when \mathbb{T} is a real torus and $X \subseteq \mathbb{R}^n$ is a set definable in an o-minimal structure over the reals.

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1. INTRODUCTION

Let A be a complex abelian variety of dimension n , and let $\pi : \mathbb{C}^n \rightarrow A$ be its covering map. It follows from a theorem of Ax (see [1, Theorem 3]), that if $X \subseteq \mathbb{C}^n$

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is an algebraic variety then the Zariski closure of $\pi(X)$ is a union of finitely many cosets of abelian subvarieties of A .

In [6, 7], Ullmo and Yafaev attempt to characterize the *topological* closure of $\pi(X)$ in the above setting and also when X is a set definable in an o-minimal expansion of the real field.

They prove a similar result to Ax's for algebraic curves (see [6, Theorem 2.4]: If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic curve then the topological closure of $\pi(X)$ in A is

$$\text{cl}(\pi(X)) = \pi(X) \cup \bigcup_{k=1}^m Z_k,$$

where each Z_k is a real weakly special subvariety of A , namely a coset of a real Lie subgroup of A . They conjecture that the same is true for algebraic subvarieties $X \subseteq \mathbb{C}^n$ of arbitrary dimension.

In this article we give a full description of $\text{cl}(\pi(X))$ when X is an algebraic subvariety of \mathbb{C}^n of arbitrary dimension and also when $X \subseteq \mathbb{R}^n$ is definable in an o-minimal structure over the reals and $\pi : \mathbb{R}^n \rightarrow \mathbb{T}$ is the covering map of a compact real torus.

As we show, the conjecture from [6] fails as stated (see Section 8) and we prove a modified version by showing that the frontier of $\pi(X)$ consists of finitely many families of real weakly special subvarieties. Our theorem holds for arbitrary compact complex tori and not only for abelian varieties.

Theorem 1.1. *Let $\pi : \mathbb{C}^n \rightarrow \mathbb{T}$ be the covering map of a compact complex torus and let X be an algebraic subvariety of \mathbb{C}^n . Then there are finitely many algebraic subvarieties $C_1, \dots, C_m \subseteq \mathbb{C}^n$ and finitely many real subtori (i.e. real Lie subgroups) $\mathbb{T}_1, \dots, \mathbb{T}_m \subseteq \mathbb{T}$ of positive dimension such that*

$$\text{cl}(\pi(X)) = \pi(X) \cup \bigcup_{i=1}^m (\pi(C_i) + \mathbb{T}_i).$$

In addition,

- (i) *For every $i = 1, \dots, m$, we have $\dim_{\mathbb{C}} C_i < \dim_{\mathbb{C}} X$.*
- (ii) *If \mathbb{T}_i is maximal with respect to inclusion among the subtori then C_i is finite.*

Notice that in general the sets $\pi(X)$ and $\pi(C_i)$ need neither be closed nor definable in any o-minimal structure. Note also that when $\dim_{\mathbb{C}} X = 1$ then $\dim_{\mathbb{C}} C_i = 0$ hence Theorem 1.1 implies the result of Ullmo and Yafaev mentioned above.

In fact, as we show, the choice of C_i depends only on X (and not on \mathbb{T}) and furthermore, each subtorus $\mathbb{T}_i \subseteq \mathbb{T}$ in the above description is of the form $\text{cl}(\pi(V_i))$ with $V_i \subseteq \mathbb{C}^n$ a complex linear subspace which also depends only on X .

In order to prove the above result, we find it more convenient to work in \mathbb{C}^n rather than in \mathbb{T} . Let $\Lambda = \ker \pi$ be the corresponding lattice in \mathbb{C}^n and let $\text{cl}(X + \Lambda)$ denote the topological closure in \mathbb{C}^n . It is easy to see that $\text{cl}(\pi(X)) = \pi(\text{cl}(X + \Lambda))$ hence the above theorem can be deduced from our analysis of $\text{cl}(X + \Lambda)$ in \mathbb{C}^n , which we now describe.

For V a complex or real linear subspace of \mathbb{C}^n and Λ a lattice in \mathbb{C}^n we denote by V^Λ the smallest \mathbb{R} -linear subspace of \mathbb{C}^n containing Λ with a basis in Λ (equivalently, this is the connected component of 0 in the real Lie group $\text{cl}(V + \Lambda)$).

The following is the main result of the first part of this article.

Main Theorem (algebraic case)[see Theorem 6.3]. *Let $X \subseteq \mathbb{C}^n$ be an algebraic subvariety. There are linear \mathbb{C} -subspaces $V_1, \dots, V_m \subseteq \mathbb{C}^n$ of positive dimension and algebraic subvarieties $C_1, \dots, C_m \subseteq \mathbb{C}^n$ such that for any lattice $\Lambda < \mathbb{C}^n$ we have*

$$\text{cl}(X + \Lambda) = (x + \Lambda) \cup \bigcup_{i=1}^m (C_i + V_i^\Lambda + \Lambda).$$

In addition,

- (i) For each $i = 1, \dots, m$, we have $\dim_{\mathbb{C}} C_i < \dim_{\mathbb{C}} X$.
- (ii) For each V_i that is maximal among V_1, \dots, V_m , the set $C_i \subseteq \mathbb{C}^n$ is finite.

Notice that Theorem 1.1 is an immediate corollary of the above.

In the second part of the article we obtain a similar result in the o-minimal setting, and disprove the analogous o-minimal conjecture from [7]. In order to formulate the theorem, we fix an o-minimal expansion \mathbb{R}_{om} of the real field.

Theorem 1.2. *Let $\pi : \mathbb{R}^n \rightarrow \mathbb{T}$ be the covering map of a compact real torus and let $X \subseteq \mathbb{R}^n$ be a closed set definable in \mathbb{R}_{om} . There are finitely many definable closed sets $C_1, \dots, C_m \subseteq \mathbb{R}^n$ and finitely many real subtori $\mathbb{T}_1, \dots, \mathbb{T}_m \subseteq \mathbb{T}$ of positive dimension such that*

$$\text{cl}(\pi(X)) = \pi(X) \cup \bigcup_{i=1}^m (\pi(C_i) + \mathbb{T}_i).$$

In addition,

- (i) For every $i = 1, \dots, m$, we have $\dim_{\mathbb{R}} C_i < \dim_{\mathbb{R}} X$ (where $\dim_{\mathbb{R}}$ is the o-minimal dimension).
- (ii) If \mathbb{T}_i is maximal with respect to inclusion among $\mathbb{T}_1, \dots, \mathbb{T}_m$ then C_i is bounded in \mathbb{C}^n and in particular $\pi(C_i)$ is closed.

As in the algebraic case, the above result follows from a theorem on the closure of $X + \Lambda$ in \mathbb{R}^n , for $\Lambda = \ker \pi$.

Main Theorem (o-minimal case) [see Theorem 7.8]. *Let $X \subseteq \mathbb{R}^n$ be a closed set definable in an o-minimal expansion \mathbb{R}_{om} of the real field. There are linear \mathbb{R} -subspaces $V_1, \dots, V_m \subseteq \mathbb{R}^n$ of positive dimension, and for each $i = 1, \dots, m$ definable closed $C_i \subseteq \mathbb{R}^n$, such that for any lattice $\Lambda < \mathbb{R}^n$ we have*

$$\text{cl}(X + \Lambda) = (X + \Lambda) \cup \bigcup_{i=1}^m (C_i + V_i^\Lambda + \Lambda).$$

In addition,

- (i) For each $i = 1, \dots, m$, we have $\dim_{\mathbb{R}} C_i < \dim_{\mathbb{R}} X$.
- (ii) For each V_i that is maximal among V_1, \dots, V_m , the set $C_i \subseteq \mathbb{R}^n$ is bounded, and in particular $C_i + V_i^\Lambda + \Lambda$ is a closed set.

The proofs of the above theorems are carried out in two main steps. In the first step we describe the closure of $X + \Lambda$ in \mathbb{C}^n and \mathbb{R}^n , as a union of closures of sets of the form $\Lambda + A$, where A is an affine subspace of \mathbb{C}^n or \mathbb{R}^n . We call these affine spaces “affine asymptotes” to X (see Section 4). The analysis of the

closure in terms of affine asymptotes uses the model theoretic notion of types. We also apply at this step the theory of stabilizers of μ -types as was developed in [4] (see Section 3.3 below). Furthermore, using model theory of valued fields and of o-minimal structures we show that the family of affine asymptotes to X is itself constructible (in the algebraic case) and definable (in the o-minimal case). We introduce these model theoretic preliminaries in Section 2 and Section 3.

In the second step we use Baire Category Theorem to replace the infinitely many affine spaces by finitely many (each possibly infinite) families of translates of fixed linear spaces thus yielding Theorem 6.3 and Theorem 7.8.

We note that in the same articles, Ullmo and Yafaev formulate two measure theoretic conjectures about $\pi(X)$, in the algebraic and o-minimal settings, and we do not touch on them here.

Finally, although the article is not formulated in that language, our approach is influenced by van den Dries work on various notions of limits of definable families and their connection to model theory (see [9]).

2. PRELIMINARIES

2.1. Model theoretic preliminaries. We first introduce the model theoretic settings in which we will be working. We refer to [3] for basics on model theory and to [8] and [10] for basics on o-minimality.

We denote by \mathcal{L}_a the “algebraic” language, i.e. the language of rings $\mathcal{L}_a = \langle +, -, \cdot, 0, 1 \rangle$, and view the field \mathbb{C} as an \mathcal{L}_a -structure.

Working with the field \mathbb{R} and semialgebraic sets we use the “semialgebraic” language $\mathcal{L}_{sa} = \langle +, -, \cdot, <, 0, 1 \rangle$.

For the algebraic case we also need a language for valued field. We use the language $\mathcal{L}_{val} = \langle +, -, \cdot, \mathcal{O}, 0, 1 \rangle$, where \mathcal{O} is a unary predicate with an intended use for the valuation ring. Notice that $\mathcal{L}_a \subseteq \mathcal{L}_{val}$.

For the o-minimal case we fix an o-minimal expansion \mathbb{R}_{om} of the field \mathbb{R} and denote its language by \mathcal{L}_{om} . Notice that $\mathcal{L}_a \subseteq \mathcal{L}_{sa} \subseteq \mathcal{L}_{om}$.

We also work in expansions of \mathbb{R} and \mathbb{R}_{om} by various additive subgroups $\Lambda \leq \mathbb{R}^n$. To avoid different expansions it is convenient to treat them all at once. Thus we consider the expansion of \mathbb{R} by predicates for **all** subsets of \mathbb{R}^n , for all n . The language for this structure is denoted by \mathcal{L}_{full} . We let \mathbb{R}_{full} be the associated \mathcal{L}_{full} -structure on \mathbb{R} .

We choose a cardinal $\kappa > 2^\omega$ and fix a κ -saturated elementary extension \mathfrak{R}_{full} of \mathbb{R}_{full} . We denote by \mathfrak{R} the underlying real closed field and by \mathfrak{R}_{om} the o-minimal reduct $\mathfrak{R}_{full} \upharpoonright \mathcal{L}_{om}$. Notice that since both the real closed field \mathfrak{R} and the o-minimal structure \mathfrak{R}_{om} are reducts of \mathfrak{R}_{full} they are both κ -saturated.

To distinguish between subsets of \mathfrak{R} and \mathbb{R} we use the following convention: we let roman letters X, Y, Z etc. denote subsets of \mathbb{R}^n and script letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ etc. subsets of \mathfrak{R}^n .

Also if $X \subseteq \mathbb{R}^n$, then we can view X as an \mathcal{L}_{full} -definable set and denote by $X^\#$ the set $X(\mathfrak{R})$ of realizations of X in \mathfrak{R}_{full} .

Our model theoretic terminology is standard. By definable we mean definable with parameters. We use “ \mathcal{L}_{sa} -definable” and “semialgebraic” interchangeably.

If \mathcal{L}_\bullet is one of our languages and x a finite tuple of variables then an \mathcal{L}_\bullet -type $p(x)$ (over a set $A \subseteq \mathfrak{R}$) is a collection of \mathcal{L}_\bullet -formulas (over A) with free variables x . We identify an \mathcal{L}_\bullet -type $p(x)$ with the collection of subsets of $\mathfrak{R}^{|x|}$ defined by formulas in $p(x)$. We do not assume that a type is complete, but always assume it is consistent. Thus an \mathcal{L}_\bullet -type $p(x)$ over A is a collection of subsets of $\mathfrak{R}^{|x|}$ such that each subset is \mathcal{L}_\bullet -definable over A and $p(x)$ satisfies the finite intersection property (namely the intersection of a every finite subcollection of p is nonempty). Given a type $p(x)$ we denote by $p(\mathfrak{R})$ the set $\bigcap_{\mathcal{X} \in p} \mathcal{X}$. Two \mathcal{L}_\bullet -types $p(x)$ and $q(x)$ are called equivalent if for every finite $p_0(x) \subseteq p(x)$ there is finite $q_0(x) \subseteq q(x)$ with $q_0(\mathfrak{R}) \subseteq p_0(\mathfrak{R})$, and vice versa. It follows that $p(\mathfrak{R}) = q(\mathfrak{R})$.

Definition 2.1. A subset $\mathcal{X} \subseteq \mathfrak{R}^n$ is called *pro-semialgebraic* (over $A \subseteq \mathfrak{R}$) if there is $A_0 \subseteq \mathfrak{R}$ ($A_0 \subseteq A$) with $|A_0| < \kappa$ and a semialgebraic type $p(x)$ over A_0 such that $\mathcal{X} = p(\mathfrak{R})$.

Notice that if \mathcal{X} and $p(x)$ are as in the above definition and $\mathcal{X} = q(\mathfrak{R})$ for another semialgebraic type $q(x)$ over A_0 then by κ -saturation of \mathfrak{R} the types $p(x)$ and $q(x)$ are equivalent.

2.2. Basics on additive subgroups. Let W be a finite dimensional \mathbb{R} -vector space and Λ be an additive subgroup of W whose \mathbb{R} -span is the whole W . We do not assume that Λ is a lattice or even finitely generated.

We say that an \mathbb{R} -subspace $V \subseteq W$ is *defined over* Λ if it has a basis consisting of elements of Λ .

It is not hard to see that the family of subspaces defined over Λ is closed under arbitrary intersections and finite sums.

For a subspace $H \subseteq W$ we denote by H^Λ the smallest \mathbb{R} -subspace of W defined over Λ containing H .

We will need the following well-known fact.

Fact 2.2. *Let W be a finite dimensional \mathbb{R} -vector space and $\Lambda \leq W$ an additive subgroup whose \mathbb{R} -span is the whole W . If $H \subseteq W$ is an \mathbb{R} -subspace then $H^\Lambda + \Lambda \subseteq \text{cl}(H + \Lambda)$ (with equality when Λ is a lattice in W).*

Remark 2.3. For a \mathbb{C} -subspace $H \subseteq \mathbb{C}^n$ and an additive subgroup $\Lambda \subseteq \mathbb{C}^n$ whose \mathbb{R} -span is the whole \mathbb{C}^n , we still denote by H^Λ the smallest \mathbb{R} -subspace of \mathbb{C}^n containing H and having an \mathbb{R} -basis in Λ . Thus H^Λ need not be a \mathbb{C} -linear subspace of \mathbb{C}^n .

3. VALUED FIELD STRUCTURES ON \mathfrak{R}

We denote by $\mathcal{O}_\mathfrak{R}$ the convex hull of \mathbb{R} in \mathfrak{R} . It is a valuation ring of \mathfrak{R} , and we let $\mu_\mathfrak{R}$ be its maximal ideal. The set $\mu_\mathfrak{R}$ is the intersection of all open intervals $(-1/n, 1/n)$ for $n \in \mathbb{N}^{>0}$, hence it is pro-semialgebraic over \emptyset .

As an additive group $\mathcal{O}_\mathfrak{R}$ is the direct sum $\mathcal{O}_\mathfrak{R} = \mathbb{R} \oplus \mu_\mathfrak{R}$, hence for $\alpha \in \mathcal{O}_\mathfrak{R}$ there is unique $r_\alpha \in \mathbb{R}$ with $\alpha \in r_\alpha + \mu_\mathfrak{R}$. This r_α is called *the standard part of* α and we denote it by $\text{st}(\alpha)$. Thus we have *the standard part map* $\text{st}: \mathcal{O}_\mathfrak{R} \rightarrow \mathbb{R}$. Slightly abusing notations, we use $\text{st}(x)$ also to denote the map from $\mathcal{O}_\mathfrak{R}^n$ to \mathbb{R}^n defined by $\text{st}(x_1, \dots, x_n) = (\text{st}(x_1), \dots, \text{st}(x_n))$, and for a subset $\mathcal{X} \subseteq \mathfrak{R}^n$ we write $\text{st}(\mathcal{X})$ for the set $\text{st}(\mathcal{X} \cap \mathcal{O}_\mathfrak{R}^n)$.

3.1. Closure and the standard part map. We need the following claim that relates the topological closure and the standard part map. It follows from the saturation assumption on $\mathfrak{R}_{\text{full}}$. As usual for $\mathcal{X}, \mathcal{Y} \subseteq \mathfrak{R}^n$ we write $\mathcal{X} + \mathcal{Y}$ for the set $\{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\}$. Recall that for a subset $X \subseteq \mathbb{R}^n$ we denote by X^\sharp the subset of \mathfrak{R}^n defined in the structure $\mathfrak{R}_{\text{full}}$ by the predicate corresponding to X .

Claim 3.1. (1) For $X \subseteq \mathbb{R}^n$ we have $\text{cl}(X) = \text{st}(X^\sharp)$. In particular, for $X, Y \subseteq \mathbb{R}^n$ we have $\text{cl}(X + Y) = \text{st}(X^\sharp + Y^\sharp)$.

(2) Let Σ be a collection of subsets of \mathbb{R}^n closed under finite intersections. Then

$$\text{st} \left(\bigcap_{X \in \Sigma} X^\sharp \right) = \bigcap_{X \in \Sigma} \text{st}(X^\sharp).$$

In particular the set $\text{st} \left(\bigcap_{X \in \Sigma} X^\sharp \right)$ is closed.

3.2. The algebraic closure \mathfrak{C} of \mathfrak{R} as an ACVF structure.

Let $\mathfrak{C} = \mathfrak{R} + i\mathfrak{R}$, where $i = \sqrt{-1}$. It is an algebraically closed field containing \mathbb{C} . We identify the underlying set of \mathfrak{C} with \mathbb{R}^2 and the underlying set of \mathfrak{C} with \mathfrak{R}^2 . We also view \mathbb{R} and \mathfrak{R} as subfields of \mathfrak{C} and \mathfrak{C} , respectively, in an obvious way.

Let $\mathcal{O}_{\mathfrak{C}} \subseteq \mathfrak{C}$ be the set $\mathcal{O}_{\mathfrak{C}} = \mathcal{O}_{\mathfrak{R}} + i\mathcal{O}_{\mathfrak{R}}$. It is a valuation ring of \mathfrak{C} with the maximal ideal $\mu_{\mathfrak{C}} = \mu_{\mathfrak{R}} + i\mu_{\mathfrak{R}}$. Again we have that $\mathcal{O}_{\mathfrak{C}} = \mathbb{C} \oplus \mu_{\mathfrak{C}}$, and we let $\text{st} : \mathcal{O}_{\mathfrak{C}} \rightarrow \mathbb{C}$ be the standard part map.

Remark 3.2. We can also identify \mathbb{C} with the residue field $k = \mathcal{O}_{\mathfrak{C}}/\mu_{\mathfrak{C}}$ so that the residue map $\text{res} : \mathcal{O}_{\mathfrak{C}} \rightarrow k$ is the same as the standard part map $\text{st} : \mathcal{O}_{\mathfrak{C}} \rightarrow \mathbb{C}$.

We denote by $\mathfrak{C}_{\text{val}}$ the \mathcal{L}_{val} -structure $(\mathfrak{C}; +, -, \cdot, \mathcal{O}_{\mathfrak{C}}, 0, 1)$.

Proposition 3.3. If $\mathcal{X} \subseteq \mathfrak{C}^n$ is an \mathcal{L}_{val} -definable set then $\mathcal{X} \cap \mathbb{C}^n$ is \mathcal{L}_{a} -definable in the field \mathbb{C} , i.e. it is a constructible subset of \mathbb{C}^n .

If \mathcal{F} is an \mathcal{L}_{val} -definable family of subsets of \mathfrak{C}^n then the family $\{\mathfrak{F} \cap \mathbb{C}^n : \mathfrak{F} \in \mathcal{F}\}$ is constructible.

Proof. By Remark 3.2, the field \mathfrak{C} together with a predicate for \mathbb{C} and the map $\text{st}(x)$ is an algebraically closed field with an embedded residue field. By [2, Lemma 6.3] the embedded residue field \mathbb{C} is stably embedded, i.e. for every \mathcal{L}_{val} -definable subset $\mathcal{X} \subseteq \mathfrak{C}^n$ the set $\mathcal{X} \cap \mathbb{C}^n$ is \mathcal{L}_{a} -definable in the field of complex numbers.

The second part of the proposition is not stated in [2, Lemma 6.3], but it easily follows: By quantifier elimination from [2, Lemma 6.3], the theory of algebraically closed valued fields of characteristic zero with an embedded residue field is complete, and we can use a standard compactness argument. \square

Remark 3.4. Notice that the structure $\mathfrak{C}_{\text{val}}$ is not a reduct of $\mathfrak{R}_{\text{full}}$ (e.g. $\mathcal{O}_{\mathfrak{C}}$ is not definable in $\mathfrak{R}_{\text{full}}$), so $\mathfrak{C}_{\text{val}}$ need not be κ -saturated, and in fact it is not. For example the set $\{x \in \mathcal{O}_{\mathfrak{C}} \wedge x \notin (c + \mu_{\mathfrak{C}}) : c \in \mathbb{C}\}$ is an \mathcal{L}_{val} -type over \mathbb{C} but has no realization in \mathfrak{C} .

However, as we will see below (see Corollary 3.6), \mathcal{L}_{val} -types over \mathbb{C} that are realized in \mathfrak{C} can be viewed as pro-semialgebraic objects, and working with them we will make use of the saturation of the field \mathfrak{R} .

Using the identification of \mathfrak{C}^n with \mathfrak{R}^{2n} , we say that a subset $\mathcal{X} \subseteq \mathfrak{C}^n$ is semialgebraic (over $A \subseteq \mathfrak{R}$) if it is semialgebraic (over A) as a subset of \mathfrak{R}^{2n} . Similarly we say that a set $\mathcal{X} \subseteq \mathfrak{C}^n$ is pro-semialgebraic if it is pro-semialgebraic as a subset of \mathfrak{R}^{2n} .

For example, every constructible subset of \mathfrak{C}^n is also semialgebraic, and for every $n \in \mathbb{N}$ the set $\mu_{\mathfrak{C}}^n$ is pro-semialgebraic over \emptyset . However the set $\mathcal{O}_{\mathfrak{C}}$ is not pro-semialgebraic.

Given an element $\alpha \in \mathfrak{C}^n$ we will consider its \mathcal{L}_{val} -type over \mathbb{C} and also its semialgebraic type over \mathbb{R} . To distinguish these types we denote by $\text{tp}_{\text{val}}(\alpha/\mathbb{C})$ the complete \mathcal{L}_{val} -type of α over \mathbb{C} , i.e.

$$\text{tp}_{\text{val}}(\alpha/\mathbb{C}) = \{ \mathcal{X} \subseteq \mathfrak{C}^n : \alpha \in \mathcal{X}, \mathcal{X} \text{ is } \mathcal{L}_{\text{val}}\text{-definable over } \mathbb{C} \},$$

and by $\text{tp}_{\text{sa}}(\alpha/\mathbb{R})$ the semialgebraic type of α over \mathbb{R} , i.e.

$$\text{tp}_{\text{sa}}(\alpha/\mathbb{R}) = \{ \mathcal{X} \subseteq \mathfrak{C}^n : \alpha \in \mathcal{X}, \mathcal{X} \text{ is semialgebraic over } \mathbb{R} \}.$$

Notice that $\text{tp}_{\text{sa}}(\alpha/\mathbb{R})$ can be also written as

$$\{ X^{\sharp} \subseteq \mathfrak{C}^n : \alpha \in X^{\sharp}, X \subseteq \mathbb{C}^n \text{ is semialgebraic} \}.$$

The following theorem plays an essential role in this paper.

Theorem 3.5. *Let $\alpha \in \mathfrak{C}$ and $p(x) = \text{tp}_{\text{val}}(\alpha/\mathbb{C})$. There is an \mathcal{L}_{val} -type $s(x)$ over \mathbb{C} that is equivalent to $p(x)$ and such that every $\mathcal{X} \in s(x)$ is pro-semialgebraic over \mathbb{R} .*

Proof. By Robinson's quantifier elimination (see [5]), if $\mathcal{X} \subseteq \mathfrak{C}^n$ is \mathcal{L}_{val} -definable over \mathbb{C} then it is a finite Boolean combination of sets of the form X^{\sharp} where $X \subseteq \mathbb{C}^n$ is a constructible set, and sets of the form $\{z \in \mathfrak{C}^n : h(z) \in q(z)\mathcal{O}_{\mathfrak{C}}\}$ where $h, q \in \mathbb{C}[x]$.

Since the complement of a constructible set is constructible, and, for $h, q \in \mathbb{C}[x]$, the complement of the set $\{z \in \mathfrak{C}^n : h(z) \in q(z)\mathcal{O}_{\mathfrak{C}}\}$ is $\{z \in \mathfrak{C}^n : h(z) \neq 0, q(z)/h(z) \in \mu_{\mathfrak{C}}\}$, every \mathcal{L}_{val} -definable over \mathbb{C} set $\mathcal{X} \subseteq \mathfrak{C}^n$ is a finite **positive** Boolean combination of sets of the following three kinds:

1. X^{\sharp} , where $X \subseteq \mathbb{C}^n$ is a constructible set.
2. $\{z \in \mathfrak{C}^n : h(z) \neq 0, q(z)/h(z) \in \mu_{\mathfrak{C}}\}$, where $h, q \in \mathbb{C}[x]$.
3. $\{z \in \mathfrak{C}^n : h(z) \in q(z)\mathcal{O}_{\mathfrak{C}}\}$, where $h, q \in \mathbb{C}[x]$.

Notice that sets of all three kinds are \mathcal{L}_{val} -definable over \mathbb{C} . Every set of the first kind is also a semialgebraic set defined over \mathbb{R} , and every set of the second kind is pro-semialgebraic over \mathbb{R} (since $\mu_{\mathfrak{C}}$ is pro-semialgebraic).

Let $\mathcal{X} \in p(x)$ be of the third kind, i.e.

$$\mathcal{X} = \{z \in \mathfrak{C}^n : h(z) \in q(z)\mathcal{O}_{\mathfrak{C}}\}.$$

Since $\alpha \in \mathcal{X}$, we have $h(\alpha) \in q(\alpha)\mathcal{O}_{\mathfrak{C}}$. Then either $q(\alpha) = 0$ (and hence $h(\alpha) = 0$) or, for $c = \text{st}(h(\alpha)/q(\alpha))$, we have $h(\alpha)/q(\alpha) - c \in \mu_{\mathfrak{C}}$.

In either case we get a set \mathcal{Y} of the first or second kind with $\alpha \in \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$.

Thus if we take $s(x)$ to be the set of all $\mathcal{X} \in p$ of first and second kinds, then $s(x)$ is equivalent to $p(x)$ and consists of sets that are pro-semialgebraic over \mathbb{R} . \square

Corollary 3.6. *Let $\alpha \in \mathfrak{C}^n$ and $p(x) = \mathcal{L}_{\text{val}}(\alpha/\mathbb{C})$. There is an \mathcal{L}_{sa} -type $r_{\alpha}(x)$ over \mathbb{R} such that*

- (1) $r_{\alpha}(\mathfrak{C}) = p(\mathfrak{C})$.
- (2) For every finite $r'(x) \subseteq r_{\alpha}(x)$ there is $\mathcal{X} \in p(x)$ with $\mathcal{X} \subseteq r'(x)$.

Remark 3.7. Notice that unless $\alpha \in \mathbb{C}^n$ the semialgebraic type $r_{\alpha}(x)$ is different from the semialgebraic type $p_{\text{sa}}(x) = \text{tp}_{\text{sa}}(\alpha/\mathbb{R})$ and we only have strict inclusion $p_{\text{sa}}(\mathfrak{C}) \subset r_{\alpha}(\mathfrak{C}) = p(\mathfrak{C})$.

Using the κ -saturation of the field \mathfrak{R} and Corollary 3.6 we obtain the following.

Corollary 3.8. *Let $\alpha \in \mathfrak{C}^n$, $p(x) = \text{tp}_{\text{val}}(\alpha/\mathbb{C})$ and $\mathcal{Z} \subseteq \mathfrak{C}^n$ a pro-semialgebraic set. If $\mathcal{Z} \cap \mathcal{X} \neq \emptyset$ for every $\mathcal{X} \in p(x)$ then $\mathcal{Z} \cap p(\mathfrak{C}) \neq \emptyset$.*

Corollary 3.9. *Let $h(x)$ be a polynomial over \mathbb{C} , $\alpha \in \mathfrak{C}^n$, $\alpha_1 = h(\alpha)$, $p(x) = \text{tp}_{\text{val}}(\alpha/\mathbb{C})$ and $p_1(x) = \text{tp}_{\text{val}}(\alpha_1/\mathbb{C})$. Then $h(x)$ maps $p(\mathfrak{C})$ onto $p_1(\mathfrak{C})$.*

Proof. Let $s(x)$ be an \mathcal{L}_{val} -type over \mathbb{C} equivalent to $p(x)$ consisting of pro-semialgebraic sets, as in Theorem 3.5.

First notice that $p_1(x)$ is equivalent to the \mathcal{L}_{val} type

$$\{h(p'(\mathfrak{C})): p'(x) \subseteq p(x) \text{ is finite}\}.$$

Let $s_1(x) = \{h(s'(\mathfrak{C})): s'(x) \subseteq s(x) \text{ is finite}\}$. It is easy to see that $s_1(x)$ consists of pro-semialgebraic sets and since $s(x)$ and $p(x)$ are equivalent, $s_1(x)$ and $p_1(x)$ are equivalent as well.

As $h(x)$ is a polynomial over \mathbb{C} it is also a semialgebraic map, and by the κ -saturation of \mathfrak{A} we have $h(s(\mathfrak{C})) = s_1(\mathfrak{C})$. Since $s(\mathfrak{C}) = p(\mathfrak{C})$ and $s_1(\mathfrak{C}) = p_1(\mathfrak{C})$ the result follows. \square

3.3. On μ -stabilizers of types.

3.3.1. The o-minimal case.

We review briefly μ -stabilizers of \mathcal{L}_{om} -types over \mathbb{R} and refer to [4] for more details.

Since the structure \mathfrak{A}_{om} is κ -saturated the following definition is equivalent to [4, Definition 2.10].

Definition 3.10. For $\alpha \in \mathfrak{A}^n$ and $p(x) = \text{tp}_{\text{om}}(\alpha/\mathbb{R})$ we define *the μ -stabilizer of p* as

$$\text{Stab}_{\text{om}}^\mu(p) = \{v \in \mathbb{R}^n : v + (p(\mathfrak{A}) + \mu_{\mathfrak{A}}^n) = (p(\mathfrak{A}) + \mu_{\mathfrak{A}}^n)\},$$

and we also denote it by $\text{Stab}_{\text{om}}^\mu(\alpha/\mathbb{R})$.

The fact below follows from the main results of [4] (see Proposition 2.17 and Theorem 2.10 there).

Fact 3.11. *Let $\alpha \in \mathfrak{A}^n$.*

- (1) $\text{Stab}_{\text{om}}^\mu(\alpha/\mathbb{R})$ is an \mathcal{L}_{om} -definable subgroup of $(\mathbb{R}^n, +)$.
- (2) If α is unbounded, i.e. $\alpha \notin \mathcal{O}_{\mathfrak{A}}^n$, then $\text{Stab}_{\text{om}}^\mu(\alpha/\mathbb{R})$ is infinite.

3.3.2. The algebraic case.

Similarly to the o-minimal case we now define μ -stabilizers for \mathcal{L}_{val} -types over \mathbb{C} realized in \mathfrak{C} .

Definition 3.12. For $\alpha \in \mathfrak{C}^n$ and $p(x) = \text{tp}_{\text{val}}(\alpha/\mathbb{C})$ we define *the μ -stabilizer of p* as

$$\text{Stab}_{\text{val}}^\mu(p) = \{v \in \mathbb{C}^n : v + (p(\mathfrak{C}) + \mu_{\mathfrak{C}}^n) = p(\mathfrak{C}) + \mu_{\mathfrak{C}}^n\},$$

and we also denote it by $\text{Stab}_{\text{val}}^\mu(\alpha/\mathbb{C})$.

We denote by \mathcal{P}_α^μ the set $\mathcal{P}_\alpha^\mu = p(\mathfrak{C}) + \mu_n^{\mathfrak{C}}$. Thus $\text{Stab}_{\text{val}}^\mu(\alpha/\mathbb{C}) = \{v \in \mathbb{C}^n : v + \mathcal{P}_\alpha^\mu = \mathcal{P}_\alpha^\mu\}$.

Since the structure $\mathfrak{C}_{\text{val}}$ is not κ -saturated we need some preliminaries before we can prove an analogue of Fact 3.11.

Lemma 3.13. *Let $\alpha \in \mathfrak{C}^n$. For $v \in \mathbb{C}^n$ we have $v + \mathcal{P}_\alpha^\mu = \mathcal{P}_{v+\alpha}^\mu$.*

Proof. Let $p(x) = \text{tp}_{\text{val}}(\alpha/\mathbb{C})$, $\beta = v + \alpha$ and $q(x) = \text{tp}(\beta/\mathbb{C})$.

It is easy to see that $v + p(\mathfrak{C}) = q(\mathfrak{C})$, hence

$$v + \mathcal{P}_\alpha^\mu = v + p(\mathfrak{C}) + \mu_n^{\mathfrak{C}} = q(\mathfrak{C}) + \mu_n^{\mathfrak{C}} = P_\beta^\mu = P_{v+\alpha}^\mu.$$

□

Proposition 3.14. *For $\alpha \in \mathfrak{C}^n$ we have*

$$\mathcal{P}_\alpha^\mu = \cap \{ \mathcal{X} + \mu_n^{\mathfrak{C}} : \mathcal{X} \subseteq \mathfrak{C}^n \text{ is } \mathcal{L}_{\text{val}}\text{-definable over } \mathbb{C} \text{ with } \alpha \in \mathcal{X} + \mu_n^{\mathfrak{C}} \}.$$

Proof. Let $p(x) = \text{tp}_{\text{val}}(\alpha/\mathbb{C})$.

The inclusion \subseteq is easy. Indeed, let $\mathcal{X} \subseteq \mathfrak{C}^n$ be \mathcal{L}_{val} -definable over \mathbb{C} with $\alpha \in \mathcal{X} + \mu_n^{\mathfrak{C}}$. Then $\mathcal{X} + \mu_n^{\mathfrak{C}}$ is \mathcal{L}_{val} -definable over \mathbb{C} as well, hence $p(\mathfrak{C}) \subseteq \mathcal{X} + \mu_n^{\mathfrak{C}}$ and

$$\mathcal{P}_\alpha^\mu = p(\mathfrak{C}) + \mu_n^{\mathfrak{C}} \subseteq (\mathcal{X} + \mu_n^{\mathfrak{C}}) + \mu_n^{\mathfrak{C}} = \mathcal{X} + \mu_n^{\mathfrak{C}}.$$

For the inclusion \supseteq , let

$$\beta \in \cap \{ \mathcal{X} + \mu_n^{\mathfrak{C}} : \mathcal{X} \subseteq \mathfrak{C}^n \text{ is } \mathcal{L}_{\text{val}}\text{-definable over } \mathbb{C} \text{ with } \alpha \in \mathcal{X} + \mu_n^{\mathfrak{C}} \}.$$

We need to show $\beta \in p(\mathfrak{C}) + \mu_n^{\mathfrak{C}}$, or equivalently $(\beta + \mu_n^{\mathfrak{C}}) \cap p(\mathfrak{C}) \neq \emptyset$.

Let $\mathcal{X} \in p(x)$. Then $(\mathcal{X} + \mu_n^{\mathfrak{C}}) \in p(x)$ hence $\beta \in \mathcal{X} + \mu_n^{\mathfrak{C}}$ and $(\beta + \mu_n^{\mathfrak{C}}) \cap \mathcal{X} \neq \emptyset$. Since the set $\beta + \mu_n^{\mathfrak{C}}$ is pro-semialgebraic, by Corollary 3.8, we obtain $(\beta + \mu_n^{\mathfrak{C}}) \cap p(\mathfrak{C}) \neq \emptyset$. □

Corollary 3.15. *For $\alpha, \beta \in \mathfrak{C}^n$ the following conditions are equivalent.*

- (1) $\mathcal{P}_\alpha^\mu = \mathcal{P}_\beta^\mu$.
- (2) $\mathcal{P}_\alpha^\mu \cap \mathcal{P}_\beta^\mu \neq \emptyset$.
- (3) $\alpha \in \mathcal{X} + \mu_n^{\mathfrak{C}} \Leftrightarrow \beta \in \mathcal{X} + \mu_n^{\mathfrak{C}}$, for every $\mathcal{X} \subseteq \mathfrak{C}^n$ that is \mathcal{L}_{val} -definable over \mathbb{C} .

Proof. Obviously (1) \Rightarrow (2).

(3) \Rightarrow (1) follows from Proposition 3.14.

We are left to show that (2) implies (3). We will assume (3) fails and show that $\mathcal{P}_\alpha^\mu \cap \mathcal{P}_\beta^\mu = \emptyset$.

Assume (3) fails, and say $\alpha \in \mathcal{X} + \mu_n^{\mathfrak{C}}$, but $\beta \notin \mathcal{X} + \mu_n^{\mathfrak{C}}$ for some $\mathcal{X} \subseteq \mathfrak{C}^n$ that is \mathcal{L}_{val} -definable over \mathbb{C} .

Let $\mathcal{Y} = \mathfrak{C}^n \setminus (\mathcal{X} + \mu_n^{\mathfrak{C}})$. We have $\beta \in \mathcal{Y}$, the set \mathcal{Y} is \mathcal{L}_{val} -definable over \mathbb{C} and $\mathcal{Y} + \mu_n^{\mathfrak{C}} = \mathcal{Y}$.

By Proposition 3.14 we have $\mathcal{P}_\alpha^\mu \subseteq \mathcal{X} + \mu_n^{\mathfrak{C}}$ and $\mathcal{P}_\beta^\mu \subseteq \mathcal{Y}$. Since $(\mathcal{X} + \mu_n^{\mathfrak{C}}) \cap \mathcal{Y} = \emptyset$ we get $\mathcal{P}_\alpha^\mu \cap \mathcal{P}_\beta^\mu = \emptyset$. □

Thus the family of sets $\{\mathcal{P}_\alpha^\mu : \alpha \in \mathfrak{C}^n\}$ partitions \mathfrak{C}^n , and, by Lemma 3.13, translations by elements of \mathfrak{C}^n respect this partition.

We are ready to show that the μ -stabilizers $\text{Stab}_{\text{val}}^\mu(\alpha/\mathbb{C})$ have properties analogous to μ -stabilizers in o-minimal theories.

Theorem 3.16. *Let $\alpha \in \mathfrak{C}^n$.*

- (1) *The μ -stabilizer $\text{Stab}_{\text{val}}^\mu(\alpha/\mathbb{C})$ is an algebraic subgroup of $(\mathfrak{C}^n, +)$, i.e. a \mathbb{C} -subspace of \mathfrak{C}^n .*
- (2) *If $\alpha \in \mathfrak{C}^n$ is unbounded, i.e. $\alpha \notin \mathcal{O}_{\mathfrak{C}}^n$, then $\text{Stab}_{\text{val}}^\mu(\alpha/\mathbb{C})$ is infinite.*

Proof. (1). By Lemma 3.13 we have

$$\text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C}) = \{v \in \mathbb{C}^n : \mathcal{P}_{\alpha}^{\mu} = \mathcal{P}_{v+\alpha}^{\mu}\}.$$

Let $v \in \mathbb{C}^n$. Applying Corollary 3.15 we obtain that $v \in \text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$ if and only if $\alpha \in \mathcal{X} + \mu_{\mathfrak{C}}^{\mathfrak{C}} \Leftrightarrow v + \alpha \in \mathcal{X} + \mu_{\mathfrak{C}}^{\mathfrak{C}}$, for every $\mathcal{X} \subseteq \mathfrak{C}^n$ that is \mathcal{L}_{val} -definable over \mathbb{C} .

If $\mathcal{X} \subseteq \mathfrak{C}^n$ is \mathcal{L}_{val} -definable over \mathbb{C} and $u \in \mathbb{C}^n$ then the set $u + \mathcal{X}$ is \mathcal{L}_{val} -definable over \mathbb{C} as well. Thus for $v \in \mathbb{C}^n$ we have that $v \in \text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$ if and only if for every \mathcal{L}_{val} -definable over \mathbb{C} set $\mathcal{X} \subseteq \mathfrak{C}^n$ and every $u \in \mathbb{C}^n$ we have $\alpha \in u + \mathcal{X} + \mu_{\mathfrak{C}}^{\mathfrak{C}} \Leftrightarrow v + \alpha \in u + \mathcal{X} + \mu_{\mathfrak{C}}^{\mathfrak{C}}$.

For a set $\mathcal{X} \subseteq \mathfrak{C}^n$ that is \mathcal{L}_{val} -definable over \mathbb{C} let

$$F_{\mathcal{X}} = \{u \in \mathbb{C}^n : \alpha \in u + \mathcal{X} + \mu_{\mathfrak{C}}^{\mathfrak{C}}\}.$$

Let $v \in \mathbb{C}^n$. It follows from the above discussion that $v \in \text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$ if and only if $-v + F_{\mathcal{X}} = F_{\mathcal{X}}$, equivalently $\mathcal{X}_F = \mathcal{X}_F + v$ for every $\mathcal{X} \subseteq \mathfrak{C}^n$ that is \mathcal{L}_{val} -definable over \mathbb{C} .

For a set $\mathcal{X} \subseteq \mathfrak{C}^n$ that is \mathcal{L}_{val} -definable over \mathbb{C} let

$$G_{\mathcal{X}} = \{v \in \mathbb{C}^n : v + F_{\mathcal{X}} = F_{\mathcal{X}}\}$$

be the stabilizer of the set $F_{\mathcal{X}}$ in $(\mathbb{C}^n, +)$.

Obviously each $G_{\mathcal{X}}$ is a subgroup of $(\mathbb{C}^n, +)$. By Proposition 3.3 every $F_{\mathcal{X}}$ is a constructible subset of \mathbb{C}^n . Hence each $G_{\mathcal{X}}$ is an algebraic subgroup of $(\mathbb{C}^n, +)$.

As we observe above

$$\text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C}) = \bigcap \{G_{\mathcal{X}} : \mathcal{X} \subseteq \mathfrak{C}^n \text{ is } \mathcal{L}_{\text{val}}\text{-definable over } \mathbb{C}\}.$$

Thus $\text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$ is an intersection of algebraic subgroups of $(\mathbb{C}^n; +)$. Since the field \mathbb{C} satisfies the Decreasing Chan Condition on algebraic subgroups, $\text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$ is an intersection of finitely many $G_{\mathcal{X}}$, hence is algebraic.

(2). Assume $\alpha \in \mathfrak{C}^n$ is unbounded.

Let $p_{\text{sa}}(x) = \text{tp}_{\text{sa}}(\alpha/\mathbb{R})$ be the semialgebraic type of α over \mathbb{R} . Since by Fact 3.11(2) $\text{Stab}_{\text{sa}}^{\mu}(p)$ is infinite, it is sufficient to show that $\text{Stab}_{\text{sa}}^{\mu}(p) \subseteq \text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$.

Let $v \in \text{Stab}_{\text{sa}}^{\mu}(p)$. We have $v + \alpha \in p_{\text{sa}}(\mathfrak{C}) + \mu_{\mathfrak{C}}^{\mathfrak{C}}$. By Theorem 3.5 (see also Remark 3.7), $p_{\text{sa}}(\mathfrak{C}) \subseteq \text{tp}_{\text{val}}(\alpha/\mathbb{C})$. Thus $v + \alpha \in p_{\text{sa}}(\mathfrak{C}) + \mu_{\mathfrak{C}}^{\mathfrak{C}} \subseteq \mathcal{P}_{\alpha}^{\mu}$. By Corollary 3.15, $\mathcal{P}_{\alpha}^{\mu} = \mathcal{P}_{v+\alpha}^{\mu}$, and, by Lemma 3.13, $v \in \text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$. \square

4. AFFINE ASYMPTOTES

Using an idea of Ullmo and Yafaev from [6, Section 2] we introduce the notion of affine asymptotes.

As usual if V is a vector space over \mathbb{C} , then a translate of a \mathbb{C} -linear subspace of V is called an *affine \mathbb{C} -subspace of V* or a *\mathbb{C} -flat subset of V* .

Definition 4.1. Let $\alpha \in \mathfrak{C}^n$. The smallest \mathbb{C} -flat subset $A \subseteq \mathbb{C}^n$ with $\alpha \in A^{\sharp} + \mu_{\mathfrak{C}}^{\mathfrak{C}}$ is called the *asymptotic \mathbb{C} -flat of α* or just the *\mathbb{C} -flat of α* and is denoted by $A_{\alpha}^{\mathbb{C}}$.

To justify the above definition we need to show that such smallest \mathbb{C} -flat exists. It follows from the following proposition.

Proposition 4.2. *Let $\alpha \in \mathfrak{C}^n$. If $A_1, A_2 \subseteq \mathbb{C}^n$ are \mathbb{C} -flat subsets with $\alpha \in A_i^{\sharp} + \mu_{\mathfrak{C}}^{\mathfrak{C}}$, $i = 1, 2$, then $\alpha \in (A_1 \cap A_2)^{\sharp} + \mu_{\mathfrak{C}}^{\mathfrak{C}}$.*

Proof. By an elementary linear algebra, \mathbb{C} -flat subsets of \mathbb{C}^n are exactly solution sets of the linear systems $Mx = r$, for an $m \times n$ -matrix M over \mathbb{C} and $r \in \mathbb{C}^m$ (m is arbitrary).

We need a claim.

Claim 4.3. *Let A be a \mathbb{C} -flat subset of \mathbb{C}^n given as the solution set of $Mx = r$, where M is an $m \times n$ matrix over \mathbb{C} . Then $\alpha \in A^\sharp + \mu_{\mathbb{C}}^n$ if and only if $M\alpha \in r + \mu_{\mathbb{C}}^m$.*

Proof of the claim. If $\alpha \in A^\sharp + \mu_{\mathbb{C}}^n$ then $\alpha \in \beta + \mu_{\mathbb{C}}^n$ for some $\beta \in A^\sharp$, and $M\alpha \in M\beta + M\mu_{\mathbb{C}}^n \subseteq r + \mu_{\mathbb{C}}^m$.

For the right to left direction, assume $M\alpha \in r + \mu_{\mathbb{C}}^m$. Replacing \mathbb{C}^m by the range of M if needed we will assume that the range of M is the whole \mathbb{C}^m .

Let $V_0 \subseteq \mathbb{C}^n$ be the kernel of M . We choose $V_1 \subseteq \mathbb{C}^n$ a \mathbb{C} -subspace complementary to V_0 , so $\mathbb{C}^n = V_0 \oplus V_1$. We write α as $\alpha = \alpha_0 + \alpha_1$ with $\alpha_0 \in V_0^\sharp$, $\alpha_1 \in V_1^\sharp$. Since the restriction of M to V_1 is an invertible \mathbb{C} -linear map from V_1 onto \mathbb{C}^m and $M\alpha_1 \in r + \mu_{\mathbb{C}}^m$, there is $\beta_1 \in V_1^\sharp$ with $\beta_1 \in \alpha_1 + \mu_{\mathbb{C}}^n$ and $M\beta_1 = r$. Obviously $\alpha_0 + \beta_1 \in A^\sharp + \mu_{\mathbb{C}}^n$, hence $\alpha \in A^\sharp + \mu_{\mathbb{C}}^n$.

This finishes the proof of the claim. \square

We now proceed with the proof of the proposition.

Let $A = A_1 \cap A_2$. For $i = 1, 2$ we choose $m_i \times n$ -matrices M_i over \mathbb{C} and $r_i \in \mathbb{C}^{m_i}$, such that A_i is the solution set of $M_i x = r_i$. Then A is the solution set of $Mx = r$, where M is the $m \times n$ matrix $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ and $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$.

Using Claim 4.3 for α and A_1 and A_2 , we see that $M\alpha = r + \epsilon$ for some $\epsilon \in \mu_{\mathbb{C}}^{m_1+m_2}$. Using Claim 4.3 again we see that $\alpha \in A^\sharp + \mu_{\mathbb{C}}^n$. \square

Definition 4.4. For a constructible set $X \subseteq \mathbb{C}^n$ we will denote by $\mathcal{A}^{\mathbb{C}}(X)$ the set of all \mathbb{C} -flats of elements of X^\sharp , namely

$$\mathcal{A}^{\mathbb{C}}(X) = \{A_\alpha^{\mathbb{C}} : \alpha \in X^\sharp\}.$$

We say that a family \mathcal{F} of subsets of \mathbb{C}^n is a *constructible family* if there is a constructible set $T \subseteq \mathbb{C}^k$ and a constructible set $Y \subseteq \mathbb{C}^n \times T$ such that $\mathcal{F} = \{Y_t : t \in T\}$, where Y_t is the fiber of Y above t .

The next theorem follows easily from Proposition 3.3.

Theorem 4.5. *If $X \subseteq \mathbb{C}^n$ is a constructible set then the family $\mathcal{A}^{\mathbb{C}}(X)$ is also constructible.*

Example 4.6. (1). Consider the curve $X \subseteq \mathbb{C}^2$ given by $xy = 1$.

Let $\alpha = (\alpha_1, \alpha_2) \in X^\sharp$. If α is bounded, i.e. $\alpha \in \mathcal{O}_{\mathbb{C}}^2$, then $A_\alpha^{\mathbb{C}}$ is just the point $\text{st}(\alpha)$. Also notice that since X is closed we have $\text{st}(\alpha) \in X$.

If α is unbounded then either $\alpha_1 \notin \mathcal{O}_{\mathbb{C}}$ and $\alpha_2 \in \mu_{\mathbb{C}}$ or $\alpha_1 \in \mu_{\mathbb{C}}$ and $\alpha_2 \notin \mathcal{O}_{\mathbb{C}}$.

In the first case we get $A_\alpha^{\mathbb{C}} = \mathbb{C} \times 0$, and in the second $A_\alpha^{\mathbb{C}} = 0 \times \mathbb{C}$.

Thus $\mathcal{A}^{\mathbb{C}}(X)$ consists of all points in X together with two lines $\mathbb{C} \times 0$ and $0 \times \mathbb{C}$.

(2). Consider the curve $X \subseteq \mathbb{C}^2$ given by $y = x^2$.

Let $\alpha \in X^\sharp$. Again if α is bounded then $A_\alpha^{\mathbb{C}}$ is $\text{st}(\alpha)$.

If α is unbounded then $\alpha = (\xi, \xi^2)$ with $\xi \notin \mathcal{O}_{\mathbb{C}}$. It is easy to see that ξ and ξ^2 are \mathbb{C} -independent modulo $\mathcal{O}_{\mathbb{C}}$, i.e. for $c_1, c_2 \in \mathbb{C}$, if $c_1\xi + c_2\xi^2 \in \mathcal{O}_{\mathbb{C}}$ then $c_1 = c_2 = 0$. It follows then that $A_\alpha^{\mathbb{C}} = \mathbb{C}^2$.

Thus in this case $\mathcal{A}^{\mathbb{C}}(X)$ consists of all points in X together with the plane \mathbb{C}^2 .

(3). If we take $X = \mathbb{C}^2$ then $\mathcal{A}^{\mathbb{C}}(X)$ will be the set of all \mathbb{C} -flat subsets of \mathbb{C}^2 .

Remark 4.7. Since the theory of algebraically closed fields of characteristic zero with an embedded residue field is complete we can define $\mathcal{A}^{\mathbb{C}}(X)$ using the field of convergent Puiseux series instead of \mathfrak{C} , and get an analytic interpretation of $\mathcal{A}^{\mathbb{C}}(X)$ as follows.

We denote by $\mathbb{C}(\{z\})$ the field of germs meromorphic functions at $0 \in \mathbb{C}$.

For constructible set $X \subseteq \mathbb{C}^n$ we denote by X_{an} the set of $\mathbb{C}(\{z\})$ -points on X , in other words

$$X_{\text{an}} = \{(f_1, \dots, f_n) \in \mathbb{C}(\{x\})^n : (f_1(z), \dots, f_n(z)) \in X \text{ for all } z \text{ near } 0\}.$$

For $f \in X_{\text{an}}$ let $A_f \subseteq \mathbb{C}^n$ be the smallest \mathbb{C} -flat subset of \mathbb{C}^n such that the distance from $f(z)$ to A_z tends to 0 as z approaches 0. Then $\mathcal{A}^{\mathbb{C}}(X) = \{A_f : f \in X_{\text{an}}\}$.

We need some basic properties of $A_{\alpha}^{\mathbb{C}}$.

Lemma 4.8. *Let $\alpha \in \mathfrak{C}^n$. Then α is bounded (i.e. $\alpha \in \mathcal{O}_{\mathfrak{C}}^n$) if and only if $\dim_{\mathbb{C}}(A_{\alpha}^{\mathbb{C}}) = 0$. Also if $\dim_{\mathbb{C}}(A_{\alpha}^{\mathbb{C}}) = 0$ then $A_{\alpha}^{\mathbb{C}} = \text{st}(\alpha)$.*

Proof. It follows from the definition of $A_{\alpha}^{\mathbb{C}}$ that $\dim_{\mathbb{C}}(A_{\alpha}^{\mathbb{C}}) = 0$ if and only if $\alpha \in c + \mu_{\mathfrak{C}}^n$ for some $c \in \mathbb{C}^n$, i.e. α is bounded and $A_{\alpha}^{\mathbb{C}} = c$. \square

Lemma 4.9. *If $\alpha \in \mathfrak{C}^n$ then $A_{\alpha}^{\mathbb{C}}$ is invariant under translations by elements of $\text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$.*

Proof. Notice that if α and β realize the same \mathcal{L}_{val} -type then $A_{\alpha}^{\mathbb{C}} = A_{\beta}^{\mathbb{C}}$. Moreover, the same is true if $\mathcal{P}_{\alpha}^{\mu} = \mathcal{P}_{\beta}^{\mu}$, where $\mathcal{P}_{\alpha}^{\mu}$ as in Definition 3.12

Now, if $v \in \text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$ then $\mathcal{P}_{\alpha}^{\mu} = v + \mathcal{P}_{\alpha}^{\mu} = \mathcal{P}_{v+\alpha}^{\mu}$. By what we just noted,

$$A_{\alpha}^{\mathbb{C}} = A_{v+\alpha}^{\mathbb{C}} = v + A_{\alpha}^{\mathbb{C}},$$

as claimed. \square

The lemma below follows easily from Lemma 4.9 and we leave its proof to the reader.

Lemma 4.10. *Let $\alpha \in \mathfrak{C}^n$, $H_{\alpha} = \text{Stab}_{\text{val}}^{\mu}(\alpha/\mathbb{C})$ and $V_1 \subseteq \mathbb{C}^n$ a subspace complementary to H_{α} . Let $\pi: \mathbb{C}^n \rightarrow V_1$ be the projection along H_{α} and $\alpha_1 = \pi(\alpha)$. Then $A_{\alpha}^{\mathbb{C}} = H_{\alpha} \oplus A_{\alpha_1}^{\mathbb{C}}$.*

5. DESCRIBING $\text{cl}(X + \Lambda)$ USING ASYMPTOTIC FLATS

The main goal of this section is to describe $\text{cl}(X + \Lambda)$ as the union:

$$\text{cl}(X + \Lambda) = \bigcup_{A \in \mathcal{A}^{\mathbb{C}}(X)} \text{cl}(A + \Lambda).$$

The next proposition is the key ingredient.

Proposition 5.1. *Let $\alpha \in \mathfrak{C}^n$ and let $\Lambda \leq \mathbb{C}^n$ be an additive subgroup whose \mathbb{R} -span is \mathbb{C}^n . Let $p(x) = \text{tp}_{\text{val}}(\alpha/\mathbb{C})$. Then $\text{st}(p(\mathfrak{C}) + \Lambda^{\sharp})$ is a closed subset of \mathbb{C}^n and*

$$\text{cl}(A_{\alpha}^{\mathbb{C}} + \Lambda) = \text{st}(p(\mathfrak{C}) + \Lambda^{\sharp}).$$

Proof. Let r_α be a semialgebraic type as in Corollary 3.6, namely $\rho_\alpha(\mathfrak{C}) = p(\mathfrak{C})$. By Claim 3.1, $\text{st}(r_\alpha(\mathfrak{C}) + \Lambda^\sharp)$ is closed. Hence $\text{st}(p(\mathfrak{C}) + \Lambda^\sharp)$ is closed as well.

For the inclusion $\text{cl}(A_\alpha^\mathbb{C} + \Lambda) \supseteq \text{st}(p(\mathfrak{C}) + \Lambda^\sharp)$, let $\beta \in p(\mathfrak{C})$. We need to show that $\text{st}(\beta + \Lambda^\sharp) \subseteq \text{cl}(A_\alpha^\mathbb{C} + \Lambda)$. Notice that since β and α have the same \mathcal{L}_{val} -type over \mathbb{C} we have $A_\beta^\mathbb{C} = A_\alpha^\mathbb{C}$.

By the definition of $A_\beta^\mathbb{C}$, there is $\gamma \in (A_\beta^\mathbb{C})^\sharp$ with $\beta \in \gamma + \mu_\mathfrak{C}^n$. Hence we have $\text{st}(\beta + \Lambda^\sharp) \subseteq \text{st}((A_\beta^\mathbb{C})^\sharp + \Lambda^\sharp)$, and by Claim 3.1,

$$\text{st}(\beta + \Lambda^\sharp) \subseteq \text{cl}(A_\beta^\mathbb{C} + \Lambda) = \text{cl}(A_\alpha^\mathbb{C} + \Lambda).$$

We are left to show the inclusion

$$\text{cl}(A_\alpha^\mathbb{C} + \Lambda) \subseteq \text{st}(p(\mathfrak{C}) + \Lambda^\sharp).$$

Since the right side is invariant under translations by elements of Λ and is closed it is sufficient to prove

$$A_\alpha^\mathbb{C} \subseteq \text{st}(p(\mathfrak{C}) + \Lambda^\sharp).$$

We proceed by induction on $\dim_{\mathbb{C}}(A_\alpha^\mathbb{C})$, and to simplify notation we denote \mathbb{C}^n by V .

Base case: $\dim_{\mathbb{C}}(A_\alpha^\mathbb{C}) = 0$. Then, by Lemma 4.8, α is bounded with $A_\alpha^\mathbb{C} = \text{st}(\alpha)$, and the proposition is trivial in this case.

Inductive step. Assume $\dim_{\mathbb{C}}(A_\alpha^\mathbb{C}) > 0$, hence α is unbounded.

Let $H_\alpha = \text{Stab}_{\text{val}}^\mu(\alpha/\mathbb{C})$. By Theorem 3.16(2), $\dim_{\mathbb{C}}(H_\alpha) > 0$.

Choose a \mathbb{C} -subspace $V_1 \subset V$, complementary to H_α , i.e

$$V = H_\alpha \oplus V_1,$$

and let $\pi: V \rightarrow V_1$ be the projection of V onto V_1 along H_α .

We can write α as

$$\alpha = \alpha_0 + \alpha_1 \text{ with } \alpha_0 \in H_\alpha^\sharp \text{ and } \alpha_1 = \pi(\alpha) \in V_1^\sharp.$$

By Lemma 4.10 we have $A_\alpha^\mathbb{C} = H_\alpha \oplus A_{\alpha_1}^\mathbb{C}$, so $\dim_{\mathbb{C}}(A_{\alpha_1}^\mathbb{C}) < \dim_{\mathbb{C}}(A_\alpha^\mathbb{C})$.

To prove the proposition, it is sufficient to show that for any $a \in A_{\alpha_1}^\mathbb{C}$ we have

$$(5.1) \quad a + H_\alpha \subseteq \text{st}(p(\mathfrak{C}) + \Lambda^\sharp).$$

We fix $a \in A_{\alpha_1}^\mathbb{C}$.

We now apply the inductive hypothesis to $A_{\alpha_1}^\mathbb{C}$ and $\Lambda_1 = H_\alpha + \Lambda$ (which clearly still \mathbb{R} -spans V). We obtain

$$a \in \text{st}(p_1(\mathfrak{C}) + H_\alpha^\sharp + \Lambda^\sharp),$$

where $p_1 = \text{tp}_{\text{val}}(\alpha_1/\mathbb{C})$.

Thus we have

$$a = \text{st}(\beta_1 + h + \lambda) \text{ for some } \beta_1 \in p_1(\mathfrak{C}), h \in H_\alpha^\sharp, \lambda \in \Lambda^\sharp.$$

By Corollary 3.8, π maps $p(\mathfrak{C})$ onto $p_1(\mathfrak{C})$, hence there is $\beta \in p(\mathfrak{C})$ with $\pi(\beta) = \beta_1$. Thus $\beta_1 = \beta + h'$ for some $h' \in H_\alpha^\sharp$ and

$$a = \text{st}(\beta + h'' + \lambda) \text{ with } \beta \in p(\mathfrak{C}), h'' \in H_\alpha^\sharp, \lambda \in \Lambda^\sharp.$$

Let $H = H_\alpha^\Lambda$. Since H has an \mathbb{R} -basis in Λ , there is a compact subset $F \subseteq H$ with $H \subseteq F + \Lambda$.

We now use the fact that $\mathfrak{R}_{\text{full}}$ is an elementary extension of \mathbb{R}_{full} . Since $h'' \in H^\sharp$, there is $\lambda_1 \in \Lambda^\sharp$ with $\lambda_1 - h'' \in F^\sharp$. Because F is compact, there is $h^* \in F^\sharp \subseteq H^\sharp$ with $h^* = \text{st}(\lambda_1 - h'')$. Thus

$$a + h^* = \text{st}(\beta + \lambda_1 + \lambda) \in \text{st}(\beta + \Lambda^\sharp),$$

and

$$a + h^* + H_\alpha \subseteq \text{st}(\beta + H_\alpha + \Lambda^\sharp).$$

By the definition of μ -stabilizers, $\beta + H_\alpha \subseteq p(\mathfrak{C}) + \mu_{\mathfrak{C}}^n$, hence

$$a + h^* + H_\alpha \subseteq \text{st}(p(\mathfrak{C}) + \Lambda^\sharp).$$

Since the right side is closed and invariant under translations by elements of Λ , we conclude that

$$a + h^* + \text{cl}(H_\alpha + \Lambda) \subseteq \text{st}(p(\mathfrak{C}) + \Lambda^\sharp).$$

By Fact 2.2, $H \subseteq \text{cl}(H_\alpha + \Lambda)$, and since $h^* \in H$, we have

$$a + H \subseteq \text{st}(p(\mathfrak{C}) + \Lambda^\sharp),$$

hence

$$a + H_\alpha \subseteq \text{st}(p(\mathfrak{C}) + \Lambda^\sharp).$$

This proves (5.1) and therefore the proposition. \square

We can now deduce the main theorem of this section.

Theorem 5.2. *Let $X \subseteq \mathbb{C}^n$ be a constructible set and $\Lambda \leq \mathbb{C}^n$ be an additive subgroup whose \mathbb{R} -span is the whole \mathbb{C}^n . Then*

$$\text{cl}(X + \Lambda) = \bigcup_{A \in \mathcal{A}^{\mathbb{C}}(X)} \text{cl}(A + \Lambda),$$

and the family of \mathbb{C} -flats $\mathcal{A}^{\mathbb{C}}(X)$ is constructible.

Proof. By Claim 3.1 we have

$$\text{cl}(X + \Lambda) = \text{st}(X^\sharp + \Lambda^\sharp) = \bigcup_{\alpha \in X^\sharp} \text{st}(\alpha + \Lambda^\sharp) = \bigcup_{p(x) \in S_{\text{val}}^{\mathbb{C}}(X)} \text{st}(p(\mathfrak{C}) + \Lambda^\sharp),$$

where $S_{\text{val}}^{\mathbb{C}}(X) = \{\text{tp}_{\text{val}}(\alpha/\mathbb{C}) : \alpha \in X^\sharp\}$.

By Proposition 5.1, the union on the right equals

$$\bigcup_{A \in \mathcal{A}^{\mathbb{C}}(X)} \text{cl}(A + \Lambda),$$

thus proving the required equality.

By Theorem 4.5, the family $\mathcal{A}^{\mathbb{C}}(X)$ is constructible. \square

6. COMPLETING THE PROOF IN THE ALGEBRAIC CASE

Our next goal is to show that in Theorem 5.2 one can replace the union $\bigcup_{A \in \mathcal{A}^{\mathbb{C}}(X)} \text{cl}(A + \Lambda)$ by a finite union of sets of the form $C_i + V_i^\Lambda + \Lambda$, where each C_i is constructible and each V_i a \mathbb{C} -linear subspace of \mathbb{C}^n .

6.1. On families of affine subspaces. By $\text{Grass}_k(\mathbb{C}^n)$ we denote the Grassmanian variety of all affine k -dimensional \mathbb{C} -subspaces of \mathbb{C}^n . Each $\text{Grass}_k(\mathbb{C}^n)$ is a quasi-projective subvariety of some $\mathbb{P}^{l_k}(\mathbb{C})$, and we often identify $A \in \text{Grass}_k(\mathbb{C}^n)$ with the corresponding \mathbb{C} -flat $A \subseteq \mathbb{C}^n$.

The Euclidean topology on $\text{Grass}_k(\mathbb{C}^n)$ induced by $\mathbb{P}^{l_k}(\mathbb{C})$ coincides with the topology induced by the distance function on the set of all \mathbb{C} -flats in \mathbb{C}^n that is defined as follows: Let $A_1, A_2 \subseteq \mathbb{C}^n$ be two \mathbb{C} -flat subsets. Let $\xi_1 \in A_1$ be the point of A_1 closest to the origin with respect to the Euclidean norm on \mathbb{C}^n , and similarly we choose $\xi_2 \in A_2$. Let $S_1 = \{v \in A_1: \|v - \xi_1\| = 1\}$ and $S_2 = \{v \in A_2: \|v - \xi_2\| = 1\}$. The distance between A_1 and A_2 is now defined to be the Hausdorff distance between S_1 and S_2 .

For a \mathbb{C} -flat $A \subseteq \mathbb{C}^n$ we denote by $L(A)$ the linear part of A , i.e. the linear subspace of \mathbb{C}^n such that A is a translate of L .

For a subset $T \subseteq \text{Grass}_k(\mathbb{C}^n)$ we denote by $\text{CL}(T)$ the \mathbb{C} -linear span of $\bigcup_{A \in T} L(A)$ in \mathbb{C}^n . Slightly abusing notation, for a \mathbb{C} -subspace $W \subseteq \mathbb{C}^n$ of arbitrary dimension we let

$$L^{-1}[W] = \{A \in \text{Grass}_k(\mathbb{C}^n): L(A) \text{ is a subspace of } W\}.$$

It is not hard to see that $L^{-1}[W]$ is a Zariski closed subset of $\text{Grass}_k(\mathbb{C}^n)$.

For a \mathbb{C} -flat $A \subseteq \mathbb{C}^n$ and an additive subgroup $\Lambda \leq \mathbb{C}^n$ whose \mathbb{R} -span is the whole \mathbb{C}^n we denote by A^Λ the \mathbb{C} -flat $a + L(A)^\Lambda$, where $a \in A$. Obviously the definition of A^Λ does not depend on the choice of a . Notice that if Λ is a lattice in \mathbb{C}^n then $a + L(A)^\Lambda$ is the connected component of $\text{cl}(A + \Lambda)$ containing a .

Remark 6.1. Let $A \subseteq \mathbb{C}^n$ be a \mathbb{C} -flat and $V \subseteq \mathbb{C}^n$ a \mathbb{C} -subspace with $L(A) \subseteq V$. Then $A + V$ is also a \mathbb{C} -flat with $L(A + V) = V$. Also if $A = a + L(A)$ then $A + V = a + V$.

Recall that a constructible subset $T \subseteq \text{Grass}_k(\mathbb{C}^n)$ is called *irreducible* if its Zariski closure is an irreducible subvariety of $\text{Grass}_k(\mathbb{C}^n)$

Proposition 6.2. *Let $T \subseteq \text{Grass}_k(\mathbb{C}^n)$ be an irreducible constructible set and $V = \text{CL}(T)$. Let $\Lambda < \mathbb{C}^n$ be a countable additive subgroup whose \mathbb{R} -span is the whole \mathbb{C}^n .*

- (1) *The set $\{A \in T: L(A)^\Lambda = V^\Lambda\}$ is topologically dense in T with respect to the Euclidean topology on $\text{Grass}_k(\mathbb{C}^n)$.*
- (2) *The set $\bigcup_{A \in T}(A + \Lambda)$ is topologically dense in $\bigcup_{A \in T}(A + V + \Lambda)$ with respect to the Euclidean topology on \mathbb{C}^n .*

Proof. (1). Since Λ is countable, there are at most countably many \mathbb{R} -subspaces of \mathbb{C}^n defined over Λ , hence by the Baire category theorem it is sufficient to show that for any proper \mathbb{R} -subspace $W \leq V^\Lambda$ defined over Λ the set $T_W = \{A \in T: L(A) \leq W\}$ is nowhere dense in T . Let $W \leq V^\Lambda$ be a proper \mathbb{R} -subspace of V^Λ defined over Λ . Let $W' = W \cap iW$ be the largest \mathbb{C} -subspace of \mathbb{C}^n contained in W . For $A \in T$ the space $L(A)$ is a \mathbb{C} -subspace of \mathbb{C}^n , hence $L(A) \leq W$ if and only if $L(A) \leq W'$. Thus $T_W = L^{-1}[W']$.

Since T is irreducible and $L^{-1}[W']$ is Zariski closed in $\text{Grass}_k(\mathbb{C}^n)$, the set T_W is nowhere dense in T if and only if $T \not\subseteq L^{-1}[W']$. Assume $T \subseteq L^{-1}[W']$. Then $V = \text{CL}(T) \subseteq W' \subseteq W$, and since W is defined over Λ we would have $V^\Lambda \subseteq W$, contradicting the assumption on W .

(2). For $A \in T$, let $\xi_A \in A$ be the point on A closest to the origin with respect to the Euclidean metric on \mathbb{C}^n . It is not hard to see that the map $A \mapsto \xi_A$, as a map from T to \mathbb{C}^n , is continuous with respect to Euclidean topologies.

By Fact 2.2, for any \mathbb{C} -subspace $W \subseteq \mathbb{C}^n$ and $\xi \in \mathbb{C}^n$ we have $\xi + W^\Lambda + \Lambda \subseteq \text{cl}(\xi + W + \Lambda)$. Therefore, writing each $A \in T$ as $A = \xi_A + L(A)$, we have

$$\bigcup_{A \in T} (\xi_A + L(A)^\Lambda + \Lambda) \subseteq \text{cl} \left(\bigcup_{A \in T} (A + \Lambda) \right).$$

Thus it is sufficient to show that the set $\bigcup_{A \in T} (\xi_A + L(A)^\Lambda + \Lambda)$ is dense in $\bigcup_{A \in T} (\xi_A + V^\Lambda + \Lambda)$.

The latter follows from clause (1) and the continuity of the map $A \mapsto \xi_A$. \square

6.2. Proof of the main theorem in the algebraic case. We can now prove our main result in the algebraic case.

For a \mathbb{C} -subspace $W \subseteq \mathbb{C}^n$, we denote by W^\perp its orthogonal complement with respect to the standard inner product on \mathbb{C}^n .

Theorem 6.3. *Let $X \subseteq \mathbb{C}^n$ be an algebraic subvariety. There are \mathbb{C} -subspaces $V_1, \dots, V_m \subseteq \mathbb{C}^n$ of positive dimension and algebraic subvarieties $C_1 \subseteq V_1^\perp, \dots, C_m \subseteq V_m^\perp$ such that for any lattice $\Lambda < \mathbb{C}^n$ we have*

$$(6.1) \quad \text{cl}(X + \Lambda) = (x + \Lambda) \cup \bigcup_{i=1}^m (C_i + V_i^\Lambda + \Lambda).$$

In addition,

- (i) *For each $i = 1, \dots, m$, we have $\dim C_i < \dim X$.*
- (ii) *For each V_i that is maximal among V_1, \dots, V_m , the set $C_i \subseteq \mathbb{R}^n$ is finite.*

Proof. Notice that it is sufficient to find C_i 's as above that are constructible, and then replace each C_i with its topological closure if needed.

By Theorem 5.2 we have

$$\text{cl}(X + \Lambda) = \bigcup_{A \in \mathcal{A}^c(X)} \text{cl}(A + \Lambda),$$

and we view $\mathcal{A}^c(X)$ as a constructible subset of $\text{Graff}_0(\mathbb{C}^n) \cup \dots \cup \text{Graff}_n(\mathbb{C}^n)$.

Since X is a closed set, we have $\text{st}(X^\#) = X$, and using Lemma 4.8 we identify $\mathcal{A}^c(X) \cap \text{Graff}_0(\mathbb{C}^n)$ with X .

Since every constructible subset of $\text{Graff}_k(\mathbb{C}^n)$ is a finite union of irreducible constructible sets, we can write $\mathcal{A}^c(X)$ as

$$(6.2) \quad \mathcal{A}^c(X) = X \cup T_1 \cup \dots \cup T_m,$$

where each T_i is an irreducible constructible subset of some $\text{Graff}_{k_i}(\mathbb{C}^n)$ with $k_i > 0$.

Thus we have

$$\text{cl}(X + \Lambda) = (X + \Lambda) \cup \left(\bigcup_{i=1}^m \bigcup_{A \in T_i} \text{cl}(A + \Lambda) \right).$$

For each $i = 1, \dots, m$, let $V_i = \text{CL}(T_i)$. Obviously each V_i has positive dimension. By Proposition 6.2(2), every closed set containing $\bigcup_{A \in T_i} (A + \Lambda)$ must also contain

$\bigcup_{A \in T_i} (A + V_i + \Lambda)$, hence

$$\text{cl}(X + \Lambda) = (X + \Lambda) \cup \left(\bigcup_{i=1}^m \bigcup_{A \in T_i} \text{cl}(A + V_i + \Lambda) \right).$$

Fix $i = \{1, \dots, m\}$. Since $L(A) \subseteq V_i$ for $A \in T_i$, we have that for each $A \in T_i$ the intersection $(A + V_i) \cap V_i^\perp$ is a singleton that we denote by c_A . Obviously $A + V_i = c_A + V_i$. Let $C_i = \{c_A : A \in T_i\}$. It is not hard to see that C_i is a constructible subset of V_i^\perp .

We have

$$\begin{aligned} \text{cl}(X + \Lambda) &= (X + \Lambda) \cup \left(\bigcup_{i=1}^m \bigcup_{A \in T_i} \text{cl}(c_A + V_i + \Lambda) \right) = \\ &= (X + \Lambda) \cup \left(\bigcup_{i=1}^m \bigcup_{c \in C_i} (c + V_i^\perp + \Lambda) \right) = \\ &= (X + \Lambda) \cup \bigcup_{i=1}^m (C_i + V_i^\perp + \Lambda). \end{aligned}$$

This finishes the proof of the main part of Theorem 6.3.

Proof of Clause (i). Although, considering X as a semialgebraic set, Clause (i) can be derived from the corresponding clause in the o-minimal case, we also provide an algebraic argument.

Let V be one of V_i 's, $i = 1, \dots, m$, and $C \subseteq V^\perp$ the corresponding constructible subset. We need to show that $\dim_{\mathbb{C}}(C) < \dim_{\mathbb{C}}(X)$.

By our choice of V and C , for each $c \in C$ there is $\alpha \in X^\sharp \setminus \mathcal{O}_{\mathbb{C}}^n$ with $A_\alpha^{\mathbb{C}} \subseteq c + V$, equivalently $\alpha \in c + V^\sharp + \mu_{\mathbb{C}}^n$.

Changing coordinates, we may assume that $V = \mathbb{C}^k$, $V^\perp = \mathbb{C}^l$ with $k + l = n$ and we write \mathbb{C}^n as $\mathbb{C}^l \times \mathbb{C}^k$. Let \overline{X}^Z be the Zariski closure of X in $\mathbb{C}^l \times \mathbb{P}^k(\mathbb{C})$ under the embedding $\mathbb{C}^l \times \mathbb{C}^k \hookrightarrow \mathbb{C}^l \times \mathbb{P}^k(\mathbb{C})$.

Since $\dim_{\mathbb{C}}(\overline{X}^Z \setminus X) < \dim_{\mathbb{C}}(X)$, it is sufficient to show that C is contained in the projection of $\overline{X}^Z \setminus X$ into \mathbb{C}^l .

Let $\bar{c} = (c_1, \dots, c_l) \in C$. Choose $\bar{\alpha} \in X^\sharp \setminus \mathcal{O}_{\mathbb{C}}^n$ with $\bar{\alpha} \in \bar{c} + (\mathbb{C}^k)^\sharp + \mu_{\mathbb{C}}^n$.

We can write $\bar{\alpha}$ as $\bar{\alpha} = (\bar{c}', \bar{\alpha}')$, with $\bar{c}' = \bar{c} + \mu_{\mathbb{C}}^l$ and $\bar{\alpha}' \in \mathbb{C}^k + \mu_{\mathbb{C}}^k$. Notice that we must have $\bar{\alpha}' \notin \mathcal{O}_{\mathbb{C}}^k$.

Choose $\varepsilon \in \mathbb{C}$ so that $\varepsilon \bar{\alpha}' \in \mathcal{O}_{\mathbb{C}}^k \setminus \mu_{\mathbb{C}}^k$, (for example we can take $\varepsilon = \frac{1}{\alpha'_i}$ where α'_i is a component of $\bar{\alpha}'$ with smallest valuation). Since $\bar{\alpha}' \notin \mathcal{O}_{\mathbb{C}}^k$ we must have $\varepsilon \in \mu_{\mathbb{C}}$. Let $\bar{a} = (a_1, \dots, a_k) = \text{st}(\varepsilon \bar{\alpha}')$.

We use $\bar{x} = (x_1, \dots, x_l)$ for coordinates in \mathbb{C}^l and $[y_0 : y_1 : \dots : y_k]$ for homogeneous coordinates in $\mathbb{P}^k(\mathbb{C})$, hence \mathbb{C}^k is identified with points represented by homogeneous coordinates $[1 : y_1 : \dots : y_k]$.

We claim that $(c_1, \dots, c_l; 0 : a_1 : \dots : a_k) \in \overline{X}^Z$, and since it is not in $\mathbb{C}^l \times \mathbb{C}^k$ we would be done.

To show that $(c_1, \dots, c_l; 0 : a_1 : \dots : a_k) \in \overline{X}^Z$ we need to check that $p(\bar{c}, 0, \bar{a}) = 0$ for every polynomial $p(x_1, \dots, x_l, y_0, \dots, y_k) \in \mathbb{C}[\bar{x}, \bar{y}]$ that is homogeneous in \bar{y} and with $p(x_1, \dots, x_l, 1, y_1, \dots, y_k)$ vanishing on X .

Let $p(x_1, \dots, x_l, y_0, \dots, y_k)$ be such polynomial. Since $\bar{\alpha} \in X^\sharp$ we have $p(\bar{c}', 1, \bar{\alpha}') = 0$. Since p is homogeneous in \bar{y} , we have $p(\bar{c}', \varepsilon, \varepsilon \bar{\alpha}') = \varepsilon^s p(\bar{c}', 1, \bar{\alpha}')$ for some $s \in \mathbb{N}$, hence $p(\bar{c}', \varepsilon, \varepsilon \bar{\alpha}') = 0$.

Since p is a polynomial over \mathbb{C} , $\bar{c} = \text{st}(\bar{c}')$, $\varepsilon \in \mu_{\mathfrak{C}}$, and $\bar{a} = \text{st}(\varepsilon \bar{\alpha}')$, we have $p(\bar{c}, 0, \bar{a}) = 0$, i.e. what we need. That finishes the proof of clause (i). \square

Proof of Clause (ii). Let $T \subseteq \text{Graff}_{k_i}(\mathbb{C})$ be one of the T_i 's in (6.2), and let V and C be the corresponding V_i and C_i . Thus $T \subseteq \mathcal{A}^{\mathbb{C}}(X)$ is an irreducible constructible set, $V = \text{CL}(T)$ and

$$C = \{a \in V^\perp : A \subseteq a + V \text{ for some } A \in T\}.$$

Assume C is infinite. Since C is constructible it is unbounded with respect to the Euclidean metric on \mathbb{C}^n . We will show that there is $\alpha^* \in X^\sharp$ with $L(A_{\alpha^*}^{\mathbb{C}})$ properly containing V . This would imply that $A_{\alpha^*}^{\mathbb{C}} \in T_j$ for some $j = 1, \dots, m$, and $L(A_{\alpha^*}^{\mathbb{C}}) \subseteq V_j$. Therefore such V can not be maximal among the V_i 's

In order to find such an α^* we use the following observation which can be deduced from the definition of asymptotic \mathbb{C} -flats: for a \mathbb{C} -subspace $W \subseteq \mathbb{C}^n$ and $\alpha \in \mathfrak{C}^n$ we have

$$L(A_\alpha^{\mathbb{C}}) \subseteq W \text{ if and only if } \alpha \in W^\sharp + \mathcal{O}_{\mathfrak{C}}^n.$$

Let Σ be the collection of all \mathbb{C} -subspaces $W \subseteq \mathbb{C}^n$ with $V \not\subseteq W$.

By the above observation, to prove the proposition, we need to show that there is $\alpha^* \in X^\sharp$ with $\alpha^* \notin (V^\sharp + \mathcal{O}_{\mathfrak{C}}^n)$ (hence $L(A_{\alpha^*}^{\mathbb{C}}) \not\subseteq V$), and also $\alpha^* \notin (W^\sharp + \mathcal{O}_{\mathfrak{C}}^n)$, for any $W \in \Sigma$ (hence $V \subseteq L(A_{\alpha^*}^{\mathbb{C}})$).

Assume no such α^* as above exists. Then the intersection

$$X^\sharp \cap \{u \in \mathfrak{C}^n : u \notin (V^\sharp + \mathcal{O}_{\mathfrak{C}}^n)\} \cap \bigcap_{W \in \Sigma} \{u \in \mathfrak{C}^n : u \notin (W^\sharp + \mathcal{O}_{\mathfrak{C}}^n)\}$$

would be empty.

Notice that every set in the above intersection is pro-semialgebraic over \mathbb{R} .

By the saturation of \mathfrak{R} , it follows then that there are $W_1, \dots, W_l \in \Sigma$ and $R \in \mathbb{R}$ such that

$$X^\sharp \subseteq (V^\sharp + B^c(0, R)^\sharp) \cup \bigcup_{i=1}^l (W_i^\sharp + B^c(0, R)^\sharp),$$

where $B^c(0, R)$ is a closed ball in \mathbb{C}^n of radius R centered at the origin.

It follows then that for every $A \in \mathcal{A}^{\mathbb{C}}(X)$

- (1) either $A \subseteq V + B^c(0, R)$;
- (2) or $A \subseteq W_i + B^c(0, R)$ for some $i = 1, \dots, l$.

Notice that in the second case we have $L(A) \subseteq W_i$.

For $j = 1, \dots, l$, let $Z_j = T \cap L^{-1}[W_j]$. Since $\text{CL}(T) = V$, $V \not\subseteq W_j$, and T is irreducible we have $\dim_{\mathbb{C}} Z_j < \dim_{\mathbb{C}} T$.

The function $h: T \rightarrow C$ that assigns to each A the unique $c_A \in C$ with $A \subseteq c_A + V$ is continuous with respect to the Euclidean topologies and surjective. Since C is unbounded, the set $h^{-1}(C \setminus B^c(0, R))$ is a non-empty open subset in the Euclidean topology of T . By (1) and (2) it is covered by subsets Z_1, \dots, Z_l of smaller dimension. A contradiction.

That finishes the proof of Clause (ii) and with it the proof of the main theorem in the algebraic case.

7. THE O-MINIMAL RESULT

In this section we prove the Main Theorem in the o-minimal case. The proof follows closely that of Theorem 6.3 so we shall be brief.

In this section “definable” means \mathcal{L}_{om} -definable in structures \mathbb{R}_{om} or \mathfrak{R}_{om} . We use $\dim_{\mathbb{R}}$ to denote the o-minimal dimension of definable sets.

7.1. Asymptotic \mathbb{R} -flats. Similarly to the algebraic case, for $\alpha \in \mathfrak{R}^n$ we let $A_{\alpha}^{\mathbb{R}}$ be the smallest affine \mathbb{R} -subspace $A \subseteq \mathbb{R}^n$ such that $\alpha \in A + \mu_{\mathfrak{R}}^n$. The proof of its existence is identical to the proof in the complex case. We call $A_{\alpha}^{\mathbb{R}}$ *the asymptotic \mathbb{R} -flat of α* or just *the \mathbb{R} -flat of α* . For $X \subseteq \mathbb{R}^n$ definable in \mathbb{R}_{om} , we define

$$\mathcal{A}^{\mathbb{R}}(X) = \{A_{\alpha}^{\mathbb{R}} : \alpha \in X^{\#}\}.$$

We can now prove the analogue of Theorem 4.5, using the theory of tame pairs.

Theorem 7.1. *Let $X \subseteq \mathbb{R}^n$ be a set definable in \mathbb{R}_{om} . Then the family of affine space $\mathcal{A}^{\mathbb{R}}(X)$ is also definable in \mathbb{R}_{om} .*

Proof. We consider the structure obtained by expanding the o-minimal structure \mathfrak{R}_{om} by a predicate symbol for the real field and a function symbol $\text{st}(x)$ for the standard part map from $\mathcal{O}_{\mathfrak{R}}$ into \mathbb{R} . Thus we are working with a pair of o-minimal structures $(\mathfrak{R}_{\text{om}}, \mathbb{R}_{\text{om}}, \text{st})$ in which the first structure is an elementary extension of the second one and in addition the latter is Dedekind complete in the first. Such an extension is called *tame* and if we let T be the theory of \mathfrak{R}_{om} then the theory of $(\mathfrak{R}_{\text{om}}, \mathbb{R}_{\text{om}}, \text{st})$ is denoted T_{tame} . The model theory of tame extensions was studied by van den Dries and others, e.g. see [9, Section 8]. The main result we need states:

Fact 7.2 ([9, Proposition 8.1]). *If $X \subseteq \mathbb{R}^n$ is definable in $(\mathfrak{R}_{\text{om}}, \mathbb{R}_{\text{om}}, \text{st})$ then X is definable in \mathbb{R}_{om} . Moreover, if \mathcal{F} is a family of subsets of \mathbb{R}^n that is definable in $(\mathfrak{R}_{\text{om}}, \mathbb{R}_{\text{om}}, \text{st})$, then it is also definable in \mathbb{R}_{om} .*

To be precise, it is only the first part of the above result which is proved in [9], but the second part follows immediately by working in an arbitrary model of T_{tame} .

To complete the proof of Theorem 7.1, we just need to observe that the family $\mathcal{A}^{\mathbb{R}}(X)$ is definable in $(\mathfrak{R}_{\text{om}}, \mathbb{R}_{\text{om}}, \text{st})$. \square

The next step towards proving the Main Theorem is the following analogue of Proposition 5.1.

Proposition 7.3. *For $\alpha \in \mathfrak{R}^n$ and an additive subgroup $\Lambda \leq \mathbb{R}^n$ whose \mathbb{R} -span is \mathbb{R}^n , let $p(x) = \text{tp}_{\text{om}}(\alpha/\mathbb{R})$. Then $\text{st}(p(\mathfrak{R}) + \Lambda^{\#})$ is a closed subset of \mathbb{R}^n and*

$$\text{cl}(A_{\alpha}^{\mathbb{R}} + \Lambda) = \text{st}(p(\mathfrak{R}) + \Lambda^{\#}).$$

Proof. The inclusion \supseteq is similar to the argument in Proposition 5.1: We need to see that every element of the form $a = \text{st}(\beta + \lambda)$ for $\beta \in p(\mathfrak{R})$ and $\lambda \in \Lambda^{\#}$ belongs to $\text{cl}(A_{\alpha}^{\mathbb{R}} + \Lambda)$. It is easy to see that $A_{\beta}^{\mathbb{R}} = A_{\alpha}^{\mathbb{R}}$ and hence there exists $\beta' \in (A_{\alpha}^{\mathbb{R}})^{\#}$ such that $\beta - \beta' \in \mu_{\mathfrak{R}}^n$. It follows that

$$a = \text{st}(\beta' + \lambda) \in \text{st}((A_{\alpha}^{\mathbb{R}})^{\#} + \Lambda^{\#}) = \text{cl}(A_{\alpha}^{\mathbb{R}} + \Lambda).$$

We need to prove the inclusion $\text{cl}(A_{\alpha}^{\mathbb{R}} + \Lambda) \subseteq \text{st}(p(\mathfrak{R}) + \Lambda^{\#})$. As in the algebraic case, we use induction on $\dim A_{\alpha}^{\mathbb{R}}$ and assume that $\dim_{\mathbb{R}} A_{\alpha}^{\mathbb{R}} > 0$, so $\alpha \notin \mathcal{O}_{\mathfrak{R}}^n$.

We denote by R_α the μ -stabilizer of the o-minimal type p , namely $R_\alpha = \text{Stab}_{\text{om}}^\mu(\alpha/\mathbb{R})$, as introduced in Section 3.3.1. By Fact 3.11, $\dim_{\mathbb{R}} R_\alpha > 0$. The group R_α is an \mathcal{L}_{om} -definable subgroup of $(\mathbb{R}^n, +)$ and hence it is an \mathbb{R} -subspace. We write $\mathbb{R}^n = R_\alpha \oplus V_1$ for some complementary \mathbb{R} -space V_1 (with $\dim_{\mathbb{R}} V_1 < n$) and let $\pi : \mathbb{R}^n \rightarrow V_1$ be the projection along R_α . We write accordingly $\alpha = \alpha_0 + \alpha_1$.

We let $p_1 = \text{tp}_{\text{om}}(\alpha_1/\mathbb{R})$. Using the saturation of \mathfrak{R} we have

$$\pi(p(\mathfrak{R})) = p_1(\mathfrak{R}).$$

(Note that the above equality is immediate, unlike the algebraic case where we had to use Corollary 3.9.)

As in Lemma 4.9, we have $R_\alpha \subseteq L(A_\alpha^{\mathbb{R}})$. It follows, as in Lemma 4.10, that $A_\alpha^{\mathbb{R}} = R_\alpha + A_{\alpha_1}^{\mathbb{R}}$. Thus, it is enough to show that for any $a \in A_{\alpha_1}^{\mathbb{R}}$, we have $a + R_\alpha \subseteq \text{st}(p(\mathfrak{R}) + \Lambda^\#)$.

The remaining argument is identical to the proof of Proposition 5.1 so we omit it. \square

We can now conclude the o-minimal analogue of Theorem 5.2.

Theorem 7.4. *If $X \subseteq \mathbb{R}^n$ is a definable set then*

$$\text{cl}(X + \Lambda) = \bigcup_{A \in \mathcal{A}^{\mathbb{R}}(X)} \text{cl}(A + \Lambda).$$

Proof. By Claim 3.1 (1),

$$\text{cl}(X + \Lambda) = \bigcup_{\alpha \in X^\#} \text{st}(\alpha + \Lambda^\#) = \bigcup_{p \in S_{\text{om}}^{\mathbb{R}}(X)} \text{st}(p(\mathfrak{R}) + \Lambda^\#),$$

where $S_{\text{om}}^{\mathbb{R}}(X)$ is the collection of all complete \mathcal{L}_{om} -types on X over \mathbb{R} . Using Proposition 7.3 we obtain the desired result. \square

7.2. Neat families in the o-minimal context. We proceed similarly to Section 6.1.

We denote by $\text{Graff}_k(\mathbb{R}^n)$ the Grassmannian variety of all affine k -dimensional \mathbb{R} -subspaces of \mathbb{R}^n . For an \mathbb{R} -flat $A \subseteq \mathbb{R}^n$ we denote by $L(A)$ the \mathbb{R} -subspace of \mathbb{R}^n whose translate is A .

For a subset $T \subseteq \text{Graff}_k(\mathbb{R}^n)$ we denote by $\mathbb{R}L(T)$ the \mathbb{R} -linear span of $\bigcup_{A \in T} L(A)$ in \mathbb{R}^n .

Given an \mathbb{R} -flat $A \subseteq \mathbb{R}^n$ and an additive subgroup $\Lambda \leq \mathbb{R}^n$ whose \mathbb{R} -span equals \mathbb{R}^n , we denote by A^Λ the \mathbb{R} -flat $a + L(A)^\Lambda$, where $a \in A$.

Definition 7.5. A definable $T \subseteq \text{Graff}_k(\mathbb{R})$ is called *neat* if

- (a) T is a connected \mathbb{R} -submanifold of $\text{Graff}_k(T)$.
- (b) For any nonempty open subset $U \subseteq T$, we have $\mathbb{R}L(U) = \mathbb{R}L(T)$.

The notion of neatness helps us replace the irreducibility assumption in Proposition 6.2. We have:

Proposition 7.6. *Let $T \subseteq \text{Graff}_k(\mathbb{R}^n)$ be a definable neat family and $V = \mathbb{R}L(T)$. Let $\Lambda < \mathbb{R}^n$ be a countable additive subgroup of \mathbb{R}^n whose \mathbb{R} -span is the whole \mathbb{R}^n . Then,*

- (1) *The set $\{A \in T : L(A)^\Lambda = V^\Lambda\}$ is topologically dense in T .*
- (2) *The set $\bigcup_{A \in T} (A + \Lambda)$ is topologically dense in $\bigcup_{A \in T} (A + V + \Lambda)$ with respect to the Euclidean topology on \mathbb{R}^n .*

Proof. (1) Since Λ is countable there are countably many spaces $L(A)^\Lambda$ as A varies in T . Using Baire Categoricality it is enough to prove that for any proper subspace $W \subsetneq V^\Lambda$ which is defined over Λ , the set $L^{-1}[W] = \{A \in T : L(A) \subseteq W\}$ is nowhere dense in T . Because this set is definable we just need to prove that it does not contain an open set. But since T is neat, for every open $U \subseteq T$ we have $\mathbb{R}L(U) = V$, and in particular, U is not contained in $L^{-1}[W]$.

(2) The proof is identical to that of Proposition 6.2(2). □

The next result will replace the decomposition of an algebraic variety into its irreducible components.

Theorem 7.7. *Let $T \subseteq \text{Graff}_k(\mathbb{R}^n)$ be a definable family of \mathbb{R} -flats in \mathbb{R}^n . Then T can be decomposed into a finite union of neat families.*

Proof. We use induction on $\dim_{\mathbb{R}}(T)$. If $\dim_{\mathbb{R}}(T) = 0$ then T is finite and the theorem is trivial.

Assume $\dim_{\mathbb{R}}(T) > 0$.

For $U_1 \subseteq U_2 \subseteq T$, we have $\mathbb{R}L(U_1) \subseteq \mathbb{R}L(U_2)$. Hence, for $A \in T$ there exists a neighborhood $U \subseteq T$ of A and a subspace $V_A \subseteq \mathbb{R}^n$ such that for every neighborhood $U' \subseteq U$ of A we have $\mathbb{R}L(U') = V_A$.

The map $A \mapsto V_A$ is definable hence we may partition T into finitely many connected submanifolds T_1, \dots, T_r , such that on each T_i the dimension of V_A is constant. By induction, it is enough to prove that those T_i of maximal dimension are neat. So we assume that $\dim T_i = \dim T$ and T_i is an open subset of T .

We need to show that for each nonempty open $W \subseteq T_i$, we have $\mathbb{R}L(W) = \mathbb{R}L(T_i)$.

We first claim that for each $A \in T_i$, there is a neighborhood U of A such that $V_A = V_B$ for all $B \in U$. Indeed, pick $U \subseteq T_i$ such that $\mathbb{R}L(U) = V_A$. By definition, for every $B \in U$ we have $V_B \subseteq V_A$, but because $\dim V_B = \dim V_A$ we must have $V_A = V_B$ for all $B \in U$.

Now, since T_i is connected, it easily follows that for all $A, B \in T_i$, we have $V_A = V_B$.

To finish we just note that for every $A \in T_i$, we have $L(A) \subseteq V_A$, so $\mathbb{R}L(T_i) = V_A = \mathbb{R}L(W)$, for every nonempty open $W \subseteq T_i$. □

7.3. Proof of the main theorem. We recall the Main Theorem in the o-minimal setting:

Theorem 7.8. *Let $X \subseteq \mathbb{R}^n$ be a closed set definable in an o-minimal expansion \mathbb{R}_{om} of the real field. There are linear \mathbb{R} -subspaces $V_1, \dots, V_m \subseteq \mathbb{R}^n$ of positive dimension, and for each $i = 1, \dots, m$ a definable closed $C_i \subseteq V_i^\perp$, such that for any lattice $\Lambda < \mathbb{R}^n$ we have*

$$\text{cl}(X + \Lambda) = (X + \Lambda) \cup \bigcup_{i=1}^m (C_i + V_i^\Lambda + \Lambda).$$

In addition:

- (i) For each $i = 1, \dots, m$ we have $\dim C_i < \dim X$
- (ii) For each V_i that is maximal among V_1, \dots, V_m the set $C_i \subseteq \mathbb{R}^n$ is bounded, and in particular $C_i + V_i^\Lambda + \Lambda$ is a closed set.

Proof. As in the algebraic case it is sufficient to find definable C_i 's as above that are that are not necessarily closed, and then replace each C_i with its topological closure if needed.

By Theorem 7.4 for any lattice $\Lambda \subseteq \mathbb{R}^n$ we have

$$\text{cl}(X + \Lambda) = \bigcup_{A \in \mathcal{A}^{\mathbb{R}}(X)} (\text{cl}(A) + \Lambda),$$

and the family $\mathcal{A}^{\mathbb{R}}(X)$ is definable.

By Theorem 7.7, applied to each $\mathcal{A}^{\mathbb{R}}(X) \cap \text{Graff}_k(\mathbb{R})$ for $k > 0$, we decompose $\mathcal{A}^{\mathbb{R}}(X) \setminus X$ into finitely many neat sets

$$(7.1) \quad \mathcal{A}^{\mathbb{R}}(X) \setminus X = T_1 \cup \dots \cup T_m.$$

We now finish the proof exactly as in Theorem 6.3, with Proposition 7.6 replacing Proposition 6.2, and conclude: There are linear subspaces $V_1, \dots, V_m \subseteq \mathbb{R}^n$ of positive dimension and definable sets $C_i \subseteq V_i^{\perp}$ such that for any lattice $\Lambda \subseteq \mathbb{R}^n$ we have:

$$\text{cl}(X + \Lambda) = (X + \Lambda) \cup \bigcup_{i=1}^r (C_i + V_i^{\perp} + \Lambda).$$

This ends the proof of the main statement of Theorem 7.8. We are left to prove the remaining two clauses in the theorem.

7.3.1. *Proof of Clause (i).* We need to prove that each C_i in the description of $\text{cl}(X + \Lambda)$ has dimension smaller than $\dim_{\mathbb{R}} X$. For that we recall first that for each $c \in C_i$ there exists $\alpha \in X^{\sharp} \setminus \mathcal{O}_{\mathfrak{R}}^n$ such that $A_{\alpha}^{\mathbb{R}} + V_i = c + V_i$.

We let $V = V_i$ and identify \mathbb{R}^n with $V^{\perp} \times V$. The idea of the proof is that C corresponds to the projection of the frontier of X in $V^{\perp} \times V^*$, where V^* is the one-point compactification of V . By o-minimality, this frontier will have dimension smaller than $\dim_{\mathbb{R}} X$. We now describe the details. Let

$$X' = \{(x_1, x_2, 1/|x_2|) \in V^{\perp} \times V \times \mathbb{R} : x_2 \neq 0 \& (x_1, x_2) \in X\}.$$

Consider the projection $(x_1, x_2, 1/|x_2|) \mapsto (x_1, 1/|x_2|)$ of X' into $V^{\perp} \times \mathbb{R}$, and the image of X' under this projection, call it Y' .

Claim. For every $c \in C$, $(c, 0)$ is in the closure of Y' .

Proof. Assume that $c + V = A_{\alpha}^{\mathbb{R}} + V \in \mathcal{F}_V$ for $\alpha \in X^{\sharp} \setminus \mathcal{O}_{\mathbb{R}}^n$. Because $L(A_{\alpha}^{\mathbb{R}}) \subseteq V$ it follows that $\alpha \in c + V^{\sharp} + \mu_{\mathfrak{R}}^n$ and hence α can be written as $\alpha_1 + \alpha_2$ with $\text{st}(\alpha_1) = c$ and $\alpha_2 \in V^{\sharp}$ necessarily unbounded. We thus have and $\text{st}(1/|\alpha|) = 0$ and therefore $(c, 0)$ is in $\text{cl}(Y')$. This ends the proof of the claim. \square

In order to finish we note that every element of the form $(c, 0) \in V^{\perp} \times \mathbb{R}$ is necessarily in the frontier $\text{Fr}(Y') = \text{cl}(Y') \setminus Y'$, thus

$$\dim_{\mathbb{R}} C \leq \dim_{\mathbb{R}} \text{Fr}(Y') < \dim_{\mathbb{R}} Y',$$

(the last equality follows from o-minimality).

But $\dim_{\mathbb{R}} Y' \leq \dim_{\mathbb{R}} X' \leq \dim_{\mathbb{R}} X$, so $\dim_{\mathbb{R}} C < \dim_{\mathbb{R}} X$.

This ends the proof of Clause (i).

7.3.2. *Proof of Clause (ii).* The proof below follows closely the proof of Clause (ii) in the algebraic case.

We start with $T = T_i$ as in (7.1), and $V = V_i$, $C = C_i \subseteq V_i^\perp$ the corresponding linear space and definable set, respectively. Namely, $T \subseteq \text{Graff}_{k_i}(\mathbb{R})$ is neat, $V = \mathbb{R}L(T)$, and

$$C = \{c \in V^\perp : A \subseteq c + V \text{ some } A \in T\}.$$

We assume that C is unbounded and we show that there exists $\alpha^* \in X^\sharp$ such that $L(A_\alpha^\mathbb{R})$ properly contains V . This implies that V cannot be maximal among the V_j 's.

Let Σ be the collection of all proper \mathbb{R} -subspaces of \mathbb{R}^n which do not contain V .

As in the algebraic case, if no such α^* exists then the intersection

$$X^\sharp \cap \{u \in \mathfrak{R}^n : u \notin (V^\sharp + \mathcal{O}_{\mathfrak{R}}^n)\} \cap \bigcap_{W \in \Sigma} \{u \in \mathfrak{R}^n : u \notin (W^\sharp + \mathcal{O}_{\mathfrak{R}}^n)\}$$

would be empty, and we would conclude that there is a closed ball $B \subseteq \mathbb{R}^n$ and $W_1, \dots, W_l \in \Sigma$ such that for every $A \in \mathcal{A}_\mathbb{R}(X)$,

- (1) either $A \subseteq V + B$;
- (2) or $L(A) \subseteq W_i$.

For every $A \in T$ we have $L(A) \subseteq V$, hence $L(A) \subseteq W_i$ implies that $L(A) \subseteq W_i \cap V$. Because T is neat and each $W_i \cap V$ is a proper subspace of V the dimension of the set $T' = \bigcup_{i=1}^m \{A \in T : L(A) \subseteq W_i\}$ is smaller than $\dim_{\mathbb{R}} T$. So, for all $A \in T$ outside a set T' of smaller dimension, we have $A \subseteq V + B$. Let us see that this is impossible.

For $A \in T$, we denote by $c(A) \in V^\perp$ the unique $c \in V^\perp$ so that $A \subseteq c + V$. The map $A \mapsto c(A)$ is continuous and surjective so the pre-image of $C \setminus B$ is a non-empty, open subset of T . Since T is a connected manifold the intersection of this pre-image with $T \setminus T'$ is non-empty, so there exists $A \in T \setminus T'$ with $A \not\subseteq V + B$. Contradiction. This ends the proof of Clause (ii) and with that we end the proof of Theorem 7.8. \square

8. AN EXAMPLE

In this section we provide a counter-example to conjectures [6, Conjecture 1.2] and [7, Conjecture 1.6].

Let X be the surface

$$X = \{(x, y, z) \in \mathbb{C}^3 : x(1 - yz) = 1\}$$

and $\Lambda = \mathbb{Z}^3 + i\mathbb{Z}^3$.

For $i = 1, 2, 3$ we denote by Π_i the corresponding coordinate plane in \mathbb{C}^3 , namely $\Pi_1 = 0 \times \mathbb{C} \times \mathbb{C}$, $\Pi_2 = \mathbb{C} \times 0 \times \mathbb{C}$ and $\Pi_3 = \mathbb{C} \times \mathbb{C} \times 0$. Notice that all Π_i are defined over Λ , hence each $\Pi_i + \Lambda$ is closed.

Lemma 8.1.

$$(8.1) \quad \text{cl}(X + \Lambda) = (X + \Lambda) \cup \bigcup_{i=1}^3 (\Pi_i + \Lambda) \cup \bigcup_{t \in \mathbb{C}^*} \left((0, t, \frac{1}{t}) + \mathbb{C} \times 0 \times 0 + \Lambda \right).$$

Proof. We use the formula

$$\text{cl}(X + \Lambda) = (X + \Lambda) \cup \bigcup_{A \in \mathcal{A}^*(X)} \text{cl}(A + \Lambda).$$

Inclusion \supseteq . We first show that $\text{cl}(X + \Lambda)$ contains every set in the union on the right side of (8.1).

Obviously $\text{cl}(X + \Lambda)$ contains $X + \Lambda$.

For $\Pi_1 + \Lambda$, consider $\alpha = (\frac{1}{1-\delta^3}, \delta, \delta^2)$, with nonzero $\varepsilon \in \mu_{\mathfrak{C}}$ and $\delta = 1/\varepsilon$. We show that $\Pi_1 = A_{\alpha}^{\mathfrak{C}}(X)$. Clearly $\delta \notin \mathcal{O}_{\mathfrak{C}}$ and $\alpha \in X^{\#} \setminus \mathcal{O}_{\mathfrak{C}}^3$. Since $\frac{1}{1-\delta^3} \in \mu_{\mathfrak{C}}$, we have $\alpha \in \Pi_1^{\#} + \mu_{\mathfrak{C}}^3$. Also, δ and δ^2 are \mathbb{C} -linearly independent modulo $\mathcal{O}_{\mathfrak{C}}$ (i.e. for $c_1, c_2 \in \mathbb{C}$, if $c_1\delta + c_2\delta^2 \in \mathcal{O}_{\mathfrak{C}}$ then $c_1 = c_2 = 0$), hence $\alpha \notin L^{\#} + \mathcal{O}_{\mathfrak{C}}^3$ for any proper \mathbb{C} -subspace $L \subsetneq \Pi_1$. Thus $A_{\alpha}^{\mathfrak{C}} = \Pi_1$ and, by Theorem 5.2,

$$\Pi_1 + \Lambda \subseteq \text{cl}(X + \Lambda).$$

For $\Pi_2 + \Lambda$, consider $\alpha = (\delta^2, \varepsilon - \varepsilon^3, \delta)$. We have that $\alpha \in X^{\#} \setminus \mathcal{O}_{\mathfrak{C}}^3$ and as above it is easy to see that $A_{\alpha}^{\mathfrak{C}} = \Pi_2$. Hence

$$\Pi_2 + \Lambda \subseteq \text{cl}(X + \Lambda).$$

Similarly, for $\alpha = (\delta^2, \delta, \varepsilon - \varepsilon^3)$, we have $A_{\alpha}^{\mathfrak{C}} = \Pi_3$, Hence

$$\Pi_3 + \Lambda \subseteq \text{cl}(X + \Lambda).$$

For $t \in \mathbb{C}^*$ and $\varepsilon \in \mu_{\mathfrak{C}}$, let $\alpha_t = (\frac{t}{\varepsilon}, t - \varepsilon, \frac{1}{t})$. It is easy to see that $\alpha_t \in X^{\#} \setminus \mathcal{O}_{\mathfrak{C}}^3$ and $A_{\alpha_t}^{\mathfrak{C}} = (0, t, \frac{1}{t}) + \mathbb{C} \times 0 \times 0$.

Thus the right side of (8.1) is contained in $\text{cl}(X + \Lambda)$.

Inclusion \subseteq . To show that $\text{cl}(X + \Lambda)$ is contained in the right side of (8.1), it is sufficient to show that for any $\alpha \in X^{\#} \setminus \mathcal{O}_{\mathfrak{C}}^3$ the set $\text{cl}(A_{\alpha}^{\mathfrak{C}} + \Lambda)$ is contained in the right side of (8.1).

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in X^{\#} \setminus \mathcal{O}_{\mathfrak{C}}^3$.

Observe that if some $\alpha_i \in \mu_{\mathfrak{C}}$ then $\alpha \in \Pi_i^{\#} + \mu_{\mathfrak{C}}$. In this case $A_{\alpha}^{\mathfrak{C}} \subseteq \Pi_i$ and $\text{cl}(A_{\alpha}^{\mathfrak{C}} + \Lambda) \subseteq \Pi_i + \Lambda$, hence it is contained in the right side.

The only remaining case is when $\alpha \in X^{\#}$ is unbounded and none of α_i is in $\mu_{\mathfrak{C}}$. It is easy to see that in this case α_1 must be unbounded, $\alpha_2 \in t + \mu_{\mathfrak{C}}$, and $\alpha_3 \in \frac{1}{t} + \mu_{\mathfrak{C}}$ for some $t \in \mathbb{C}^*$. Then $A_{\alpha} = (0, t, \frac{1}{t}) + \mathbb{C} \times 0 \times 0$, $\text{cl}(A_{\alpha} + \Lambda) = (0, t, \frac{1}{t}) + \mathbb{C} \times 0 \times 0 + \Lambda$, and it is contained in the right side of (8.1). \square

Thus in the notations of the main theorem we can write $\text{cl}(X + \Lambda)$ as

$$\text{cl}(X + \Lambda) = (X + \Lambda) \cup \bigcup_{i=1}^4 (C_i + V_i + \Lambda),$$

where $V_i = \Pi_i$ for $i = 1, \dots, 3$, $C_1 = C_2 = C_3 = 0 \times 0 \times 0$, $V_4 = \mathbb{C} \times 0 \times 0$, and $C_4 = \{(x, y, z) \in \mathbb{C}^3 : x = 0, yz = 1\}$.

Consider the projection map $\pi : \mathbb{C}^3 \rightarrow E^3$, where E is the elliptic curve $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. The set $\pi(C_4 + V_4)$ is just $\pi(C_4) + E \times 0 \times 0$ and it is not hard to see that this set is not contained in any proper real analytic subvariety of E^3 .

Thus $\text{cl}(\pi(X))$ cannot be written as the union of $\pi(X)$ with a real analytic subvariety of E^3 . Because X is semialgebraic, this shows the failure of the Conjecture 1.6 from [7].

For the same reason $\text{cl}(\pi(X))$ cannot be written as the union of $\pi(X)$ with finitely many real weakly special subvarieties of E^3 . This shows also the failure of Conjecture 1.2 from [6].

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