LOCALLY DEFINABLE SUBGROUPS OF SEMIALGEBRAIC GROUPS

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Abstract. We prove the following instance of a conjecture stated in [10]. Let G be an abelian semialgebraic group over a real closed field R and let X be a semialgebraic subset of G. Then the group generated by X contains a generic set and, if connected, it is divisible.

More generally, the same result holds when X is definable in any o-minimal expansion of R which is elementarily equivalent to $\mathbb{R}_{an,exp}$.

We observe that the above statement is equivalent to saying: there exists an m such that $\Sigma_{i=1}^{m}(X-X)$ is an approximate subgroup of G.

1. Introduction

Locally definable groups arise naturally in the study of definable groups in o-minimal structures. In this paper we are mostly interested in *definably generated* groups, namely locally definable groups which are generated by definable sets (see Section 2 for basic definitions). An important example of such groups is the universal cover of a definable group. Indeed, a definable group in an o-minimal structure can be endowed with a definable manifold structure making the group into a topological group and then, similarly to the Lie context, one can construct its universal covering group, in the category of locally definable groups, see [9]. This universal covering is generated by a definable set.

The universal covering is an example of a locally definable group $\mathcal{U}$ with a definable *(left) generic* set X; that is, a definable set such that $AX = \mathcal{U}$ for some countable subset $A \subseteq \mathcal{U}$ (see [11, Lemma 1.7]). In [10], it was conjectured that every definably generated abelian group is of this form:

Conjecture 1.1. *Let $\mathcal{U}$ be an abelian, connected, definably generated group. Then $\mathcal{U}$ contains a definable generic set.*

Note that by [10, Claim 3.11], we may assume in the above conjecture that $\mathcal{U}$ is generated by a definably compact set.

Date: November 18, 2019.

2010 *Mathematics Subject Classification.* 03C64,03C68, 22B99, 20N99.

Key words and phrases. semialgebraic groups, locally definable groups, approximate groups, generic sets, lattices.

The first author was supported by the program MTM2017-82105-P. The second author was supported by an Independent Research Grant from the German Research Foundation (DFG) and a Zukunftskolleg Research Fellowship.

It has been shown in recent papers that the above conjecture can be restated in several ways (see for example [4]). We will be using the equivalences below, for which we first need a definition.

Definition 1.2. Given an abelian, connected, definably generated group $\mathcal{U}$, we say that a locally definable normal subgroup $\Gamma < \mathcal{U}$ is a *lattice* if $\dim(\Gamma) = 0$, and $\mathcal{U}/\Gamma$ is definable; that is, there exist a definable group G and a locally definable surjective homomorphism from $\mathcal{U}$ onto G, whose kernel is Γ.

Fact 1.3 ([10, Proposition 3.5] and [11, Theorem 2.1])**.** *Let $\mathcal{U}$ be an abelian, connected, definably generated group. Then the following are equivalent:*

(1) *$\mathcal{U}$ contains a definable generic set.*
(2) *$\mathcal{U}$ admits a lattice.*
(3) *$\mathcal{U}$ admits a lattice isomorphic to $\mathbb{Z}^k$, for some k.*

Moreover, each of the above clauses implies that $\mathcal{U}$ is divisible.

In this note, we study Conjecture 1.1 for definably generated subgroups of definable groups. To that aim, we introduce the following notion.

Definition 1.4. Let $\mathcal{M}$ be an o-minimal structure. We say that an abelian locally definable group $\mathcal{G}$ has the *generic property with respect to $\mathcal{M}$* if every definably generated subgroup of $\mathcal{G}$ contains a definable generic set. We omit the reference to $\mathcal{M}$ if it is clear from the context (see Remark 2.3).

The main result of [11] can be stated as follows.

Fact 1.5. *Let $\mathcal{R}$ be an $\aleph_1$-saturated o-minimal expansion of a real closed field R. Then $\langle R^n, +\rangle$ has the generic property with respect to $\mathcal{R}$.*

Our first result, in Section 3, is that the generic property can be lifted under the presence of an exact sequence (Theorem 3.4).

Theorem. *Let $\mathcal{M}$ be an o-minimal structure. Assume that we are given an exact sequence of abelian locally definable groups and maps:*

$$0 \longrightarrow \mathcal{H} \xrightarrow{\ i\ } \mathcal{G} \xrightarrow{\ \pi\ } \mathcal{V} \longrightarrow 0$$

If $\mathcal{V}$ and $\mathcal{H}$ have the generic property, then so does $\mathcal{G}$.

This is a useful criterion that can be applied inductively in certain situations. As a corollary, we prove (Subsection 3.1) the following theorem.

Theorem. *Let $\mathcal{M}$ be an $\aleph_1$-saturated o-minimal structure.*

(1) If G is a definable abelian torsion-free group, then G has the generic property.
(2) If $\mathcal{M}$ expands a real closed field R and $G \subseteq Gl(n, R)$ is a definable abelian linear group, then G has the generic property.

In Section 4, we apply the above lifting result to study definably generated subgroups of semialgebraic groups. In order to formulate the next result, recall that $\mathbb{R}_{an,exp}$ is the expansion of the real field by the real exponential map and all restrictions of real analytic functions to the closed unit box in $\mathbb{R}^n$, for all $n \in \mathbb{N}$. By [7], it is o-minimal. The following theorem is the main result of the paper (Theorem 4.5), which generalizes Fact 1.5 above.

Theorem. *Let $\mathcal{R}$ be an $\aleph_1$-saturated o-minimal expansion of a real closed field R such that the theory $Th(\mathcal{R}) \cup Th(\mathbb{R}_{an,exp})$ is consistent and has an o-minimal completion. Then any abelian R-semialgebraic group G has the generic property with respect to $\mathcal{R}$. In particular, any semialgebraically generated subgroup of G contains a semialgebraic generic set.*

A special case of the above result is when $\mathcal{R}$ is elementarily equivalent to $\mathbb{R}_{an,exp}$.

A crucial key case of the above theorem is when G is an abelian variety. In [19] the authors prove the definability in $\mathbb{R}_{\mathrm{an,exp}}$, on appropriate domains, of embeddings of families of abelian varieties into projective spaces. From those results it is possible to extract the following non-standard property of abelian varieties.

Fact 1.6. [19] *Let $\mathcal{R} = \langle R, \dots \rangle$ be a model of $Th(\mathbb{R}_{an,exp})$ and let $A \subseteq \mathbb{P}^N(K)$, $K = R(i)$, be an embedded abelian variety of dimension g. Then there exist a locally definable subgroup $\mathcal{G}$ of $\langle R^g, + \rangle$ and a locally definable covering homomorphism $p : \mathcal{G} \to A$.*

For the sake of completeness, we provide a proof of the above fact in Appendix 5. Another important ingredient is the work of E. Barriga on semialgebraic groups, [3], which we recall in Fact 4.4.

1.1. **The connection to approximate subgroups.** Approximate subgroups have been studied extensively in various fields including model theory, see for example [6] and [12].

Definition 1.7. Given a group G, and $k \in \mathbb{N}$, a set $X \subseteq G$ is called a k-*approximate group* if $X = X^{-1}$ and there is a finite set $A \subseteq G$ of cardinality k such that $X \cdot X \subseteq A \cdot X$. We say that X is an *approximate group* if it is k-approximate for some $k \in \mathbb{N}$.

As we observe in Remark 2.3 below, the existence of a generic set inside a definably generated group $\langle X \rangle \subseteq G$ is equivalent to saying that there exists an m such that the set $X(m)$ (the addition of $X - X$ to itself m times) is an approximate group. Thus our various results and conjectures can be re-formulated in the language of approximate subgroups. For example, Conjecture 1.1 can be re-formulated as follows.

Conjecture 1.8. *Let $\mathcal{U}$ be a locally definable abelian group in an o-minimal structure and $X \subseteq \mathcal{U}$ a definable set. Then there exists $m \in \mathbb{N}$ such that $X(m)$ is an approximate group.*

Our main result above (Theorem 4.5) easily implies the following uniformity statement:

Theorem 1.9. *Let $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot, \dots \rangle$ be an o-minimal expansion of $\mathbb{R}_{an,exp}$. Let $\{G_t : t \in T\}$ an $\mathbb{R}_{an,exp}$-definable family of semialgebraic abelian groups, and $\{X_t : t \in T\}$ an $\mathcal{R}$-definable family, with each $X_t \subseteq G_t$. Then there is $k \in \mathbb{N}$, such that for every $t \in T$, the set $X_t(k)$ is a k-approximate subgroup of G.*

In Conjecture 1.8 we restricted our discussion to definable sets in o-minimal structures, but the same problem could be formulated for arbitrary smooth curves in $\mathbb{R}^n$.

Question 1.10. Let $X \subseteq \mathbb{R}^n$ be a connected smooth curve. Is there $m \in \mathbb{N}$ such that $X(m)$ is an approximate subgroup of $\langle \mathbb{R}^n, + \rangle$?

Let us see that when X is compact the answer to the above question is positive: Indeed, without loss of generality, $0 \in X$ and X is given by $\gamma : \mathbb{R} \to \mathbb{R}^n$. Moreover, we can assume that $\mathbb{R}^n$ is the minimal linear space containing X. Thus, there are $t_1, \dots, t_n$ such that $\gamma(\dot{t}_1), \dots, \gamma(\dot{t}_n)$ form a basis for $\mathbb{R}^n$ (otherwise, $\mathbb{R}^n$ would not be minimal). It follows that the map

$$(x_1, \dots, x_n) \mapsto \gamma(x_1) + \cdots + \gamma(x_n) : \mathbb{R}^n \to \mathbb{R}^n$$

is a submersion at the point $(\gamma(t_1), \dots, \gamma(t_n))$ and hence the point $\gamma(t_1) + \cdots + \gamma(t_n)$ is an internal point of $X(n)$ inside $\mathbb{R}^n$. Since $X(n) + X(n)$ is compact it can be covered by finitely many translates of $X(n)$, so that $X(n)$ is an approximate subgroup.

Note that even if the answer to Question 1.10 is positive, one does not expect any uniformity statement such as that of Theorem 1.9 to hold at this level of generality.

We finish this part of the introduction by pointing out that one cannot expect a positive answer to the above question without the model theoretic (o-minimality) or the topological (smoothness) assumptions. The example was suggested to us by P. Simon. A similar example was also proposed by E. Breuillard.

Example 1.11. Let $G = \mathbb{R}^{\mathbb{N}}$ with coordinate-wise addition and let $X \subseteq \mathbb{R}^{\mathbb{N}}$ be the set of all elements with at most one nonzero coordinate. We claim that for no n is the set $X(n)$ an approximate subgroup. Indeed, assume that the set $X(n+1)$ is covered by finitely many translates of $X(n)$. Let $p : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{2(n+1)}$ be the projection onto the first $2(n+1)$ coordinates. The set $p(X(n))$ consists of the tuples with at most $2n$ coordinates different than 0, so for any finite subset A of $\mathbb{R}^{\mathbb{N}}$ we have that $p(A + X(n)) = p(A) + p(X(n))$ has dimension $2n$. On the other hand, $p(X(n+1)) = \mathbb{R}^{2(n+1)}$, a contradiction.

Because $\langle \mathbb{R}^{\mathbb{N}}, + \rangle$ is isomorphic as a group to $\langle \mathbb{R}, + \rangle$, we can also find a set $X \subseteq \mathbb{R}$ such that for no n is the set $X(n)$ an approximate subgroup.

1.2. **The non-abelian case.** It has been shown in [4] that Fact 1.3 fails for non-abelian groups. More precisely, it was shown that every definable centerless group, in a sufficiently saturated o-minimal structure, contains a definably generated subgroup with a definable generic set, which is not the cover of any definable group. However, as far as we know the following question is still open.

Question 1.12. Let $\mathcal{U}$ be a definably generated group in an o-minimal structure. Does $\mathcal{U}$ contain a definable generic set?

In [14, Section 7] there is a discussion of locally definable (called Ind-definable) groups and it is shown (see Proposition 7.8 there) that every locally definable group contains a definably generated subgroup $\mathcal{U}$ of the same dimension which contains a definable generic set (using also [11, Theorem 2.1]).

Acknowledgements. We thank Eliana Barriga for reading and commenting on an early version of this paper.

2. Preliminaries

Let $\mathcal{M}$ be an arbitrary κ-saturated o-minimal structure for κ sufficiently large. By *bounded* cardinality, we mean cardinality smaller than κ. We refer the reader to [1] and [8] for the basics concerning locally definable groups. A *locally definable group* is a group $\langle \mathcal{U}, \cdot \rangle$ whose universe is a directed union $\mathcal{U} = \bigcup_{k \in \mathbb{N}} X_k$ of definable subsets of M^n for some fixed n, and for every $i, j \in \mathbb{N}$, the restriction of group multiplication to $X_i \times X_j$ is a definable function (by saturation, its image is contained in some X_k). The dimension of $\mathcal{U}$ is by definition $\dim(\mathcal{U}) = \max\{\dim(X_k) : k \in \mathbb{N}\}$.

A map $\phi : \mathcal{U} \to \mathcal{H}$ between locally definable groups is called *locally definable* if for every definable $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{H}$, the set $graph(\phi) \cap (X \times Y)$ is definable. Equivalently, the restriction of ϕ to any definable set is a definable map. If ϕ is surjective, then there exists a locally definable section $s : \mathcal{H} \to \mathcal{U}$ of ϕ.

For a locally definable group $\mathcal{U}$, we say that $\mathcal{V} \subseteq \mathcal{U}$ is *a compatible subset of* $\mathcal{U}$ if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap \mathcal{V}$ is a definable set (note that in this case $\mathcal{V}$ itself is a countable union of definable sets). We say that $\mathcal{U}$ is *connected* if there is no proper compatible subgroup of bounded index. By [8], every locally definable group $\mathcal{U}$ has a connected component $\mathcal{U}^0$, that is, a connected compatible subgroup of $\mathcal{U}$ of the same dimension. Moreover, $\mathcal{U}$ admits a locally definable topological structure that makes the group operations continuous. Note that we still use the term "definably connected" when referring to definable sets. Note also that if $\phi : \mathcal{U} \to \mathcal{V}$ is a locally definable homomorphism between locally definable groups, then $\ker(\phi)$ is a compatible locally definable normal subgroup of $\mathcal{U}$. In fact, the following holds.

Fact 2.1. [8, Theorem 4.2] *If $\mathcal{U}$ is a locally definable group and $\mathcal{H} \subseteq \mathcal{U}$ is a locally definable normal subgroup then $\mathcal{H}$ is a compatible subgroup of $\mathcal{U}$ if and only if there exists a locally definable surjective homomorphism of locally definable groups $\phi : \mathcal{U} \to \mathcal{V}$ whose kernel is $\mathcal{H}$.*

In Definition 1.4 we introduced the notion of an abelian locally definable group having the generic property. Now, we stress some easy properties regarding that notion. For that, we need the following notation that will be used throughout the paper.

Notation 2.2. Let G be an abelian group and X a subset. The set $X(m)$ denotes the addition of $X-X$ to itself m times. We say that X is symmetric if $X=-X$.

Remark 2.3. (1) An abelian locally definable group $\mathcal{G}$ has the generic property if and only if for every definable subset $Y\subseteq \mathcal{G}$, there are $m,k\in\mathbb{N}$ and $0\in A\subseteq Y(3m)$, $|A|\le k$, such that $Y(m)+Y(m)\subseteq A+Y(m)$. In particular, $Y(m)$ is a k-approximate group.

(2) If $\mathcal{G}$ has the generic property and $\mathcal{H}$ is a locally definable subgroup of $\mathcal{G}$, then $\mathcal{H}$ has also the generic property.

(3) Let $\mathcal{G}$ and $\mathcal{V}$ be abelian locally definable groups, and $\pi:\mathcal{G}\to\mathcal{V}$ a surjective locally definable homomorphism. If $\mathcal{G}$ has the generic property, then so does $\mathcal{V}$. Indeed, for $X\subseteq\mathcal{V}$ definable, let $Y\subseteq\mathcal{G}$ be any definable set with $\pi(Y)=X$ (such Y exists by saturation). Since $\mathcal{G}$ has the generic property, there are $m,k\in\mathbb{N}$ and a set $0\in A\subseteq Y(3m)$, $|A|\le k$ such that $Y(m)+Y(m)\subseteq A+Y(m)$. In particular, we get that $0\in\pi(A)\subset X(3m)$ and

$$X(m)+X(m)=\pi(Y(m)+Y(m))\subseteq\pi(A+Y(m))=\pi(A)+X(m),$$

as required.

(4) The generic property is preserved under taking reducts. Namely, let $\mathcal{M}'$ be an o-minimal expansion of $\mathcal{M}$. By (1) above, if $\mathcal{G}$ is a locally definable group in $\mathcal{M}$ with the generic property with respect to $\mathcal{M}'$, then $\mathcal{G}$ has the generic property with respect to $\mathcal{M}$. It is also clear, again using (1), that the generic property is preserved under taking elementary substructures. That is, let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. Let $\mathcal{G}=\bigcup_{\ell\in\mathbb{N}}X_\ell$ be a locally definable group in $\mathcal{M}$, and denote by $\mathcal{G}(N)$ its realization in $\mathcal{N}$. Then $\mathcal{G}$ has the generic property with respect to $\mathcal{M}$ if and only if $\mathcal{G}(N)$ has the generic property with respect to $\mathcal{N}$. For, let $Y\subset X_\ell(N)$ be a subset of $\mathcal{G}(N)$ definable over a finite tuple $d\in N$. Replace the parameters d by variables t, and take the definable set $T=\{t:Y_t\subset X_\ell(N)\}$. Since T is definable over $\mathcal{M}$, we can consider the definably family in $\mathcal{M}$ of definable subsets $\{Y_t:t\in T(M)\}$ of G. By (1) and saturation of $\mathcal{M}$ there are $m,k\in\mathbb{N}$ such that for all $t\in T(M)$ there is $0\in A_t\subseteq Y_t(3m)$, $|A_t|\le k$, such that $Y_t(m)+Y_t(m)\subseteq A_t+Y_t(m)$, as required.

We can now formulate:

Proposition 2.4. *Let $\mathcal{G}$ be an abelian locally definable group in $\mathcal{M}$ (which is still sufficiently saturated). Then the following are equivalent:*

(1) $\mathcal{G}$ has the generic property.

(2) For every definable family $\{X_t : t \in T\}$ of subsets of $\mathcal{G}$ there exist $m, k \in \mathbb{N}$ such that for every $t \in T$, there exists a subset $0 \in A \subseteq X_t(3m)$, of size at most k such that

$$X_t(m) + X_t(m) \subseteq A + X_t(m).$$

In particular, $\mathcal{G}$ has the generic property in $\mathcal{M}$ if and only if it has the generic property in any/some elementary extension of $\mathcal{M}$.

Proof. This follows immediately from Remark 2.3 (1) and saturation. □

Remark 2.5. While we focus here on o-minimal structures, the notions we defined make sense in any sufficiently saturated structure, in which case Remark 2.3 and Proposition 2.4 are also true.

3. Group extensions

In this section we study the existence of definable generic sets when dealing with abelian group extensions, in an arbitrary o-minimal structure $\mathcal{M}$. As a corollary, we prove that definably generated subgroups of abelian torsion-free definable groups contain definable generic sets. When $\mathcal{M}$ expands a real closed field R, we deduce a similar result for definable linear groups over R.

Proposition 3.1. *Assume that we are given an exact sequence of locally definable abelian groups and maps,*

$$0 \longrightarrow \mathcal{H} \xrightarrow{\ i\ } \mathcal{G} \xrightarrow{\ \pi\ } \mathcal{V} \longrightarrow 0\ ,$$

where $\mathcal{V}$ is connected and admits a lattice. Let $Y \subseteq \mathcal{V}$ be a definable generic set and $s : Y \to \mathcal{G}$ a definable section. Then the intersection $\langle s(Y)\rangle \cap i(\mathcal{H})$ is definably generated.

Proof. By [11, Lemma 1.7], $\mathcal{V} = \langle Y\rangle$. In particular, π sends the group $\langle s(Y)\rangle$ onto $\mathcal{V}$. Without loss of generality, we may assume that i is the identity map.

Henceforth we will use that given a definable set $Z \subseteq \mathcal{V}$, we can assume that $Z \subseteq Y$. Indeed, by saturation there is n such that $Z \subseteq Y(n)$ and by definable choice there is a section $r : Z \to s(Y)(n) \subseteq \langle s(Y)\rangle$. Thus we can extend the section $s : Y \to \mathcal{G}$ to a section $\widetilde{s} : Y \cup (Z \setminus Y) \to \mathcal{G}$ via r in such a way that $\langle s(Y)\rangle = \langle \widetilde{s}(Y \cup Z)\rangle$. Therefore we can work with the generic set $Y \cup Z$ instead of Y, as required. For example, we can assume that $Y \ni 0$ is symmetric (extend the section s to the set $-Y \cup \{0\}$). Moreover, we can set $s(0) = 0$. For, let $y_0 := s(0) \in \mathcal{H}$ and consider the definable section $\widetilde{s} : Y \to \mathcal{G}$ such that $\widetilde{s} = s$ in $Y \setminus \{0\}$ and $\widetilde{s}(0) = 0$. If $\widetilde{D}$ is a definable set which generates $\langle \widetilde{s}(Y)\rangle \cap \mathcal{H}$, then $D := \widetilde{D} \cup \{y_0\}$ generates $\langle s(Y)\rangle \cap \mathcal{H}$, as required.

By Fact 1.3 and since Y is generic, the locally definable group $\mathcal{V}$ admits a lattice $\Gamma \simeq \mathbb{Z}^k$. Since $\mathcal{V}/\Gamma$ is definable and Y generic in $\mathcal{V}$, there is a finite set $A \subseteq \mathcal{V}$ such that $Y + A + \Gamma = \mathcal{V}$. Indeed, to see that note that the image of Y in $\mathcal{V}/\Gamma$ is a generic so finitely many translates of it cover the group. Now, without loss of generality, we can assume that A contains a fixed set of generators $\gamma_1,\dots,\gamma_k$ of Γ. Therefore we can assume that $Y+\Gamma=\mathcal{V}$ and $\gamma_1,\dots,\gamma_k\in Y$ (extending the section s to $Y+A$).

Let $\Delta=\langle s(\gamma_1),\dots,s(\gamma_k)\rangle$ and note that $\pi|_\Delta:\Delta\to\Gamma$ is an isomorphism. Consider the symmetric finite set (notice that $Y(2)\cap\Gamma$ is finite)

$$\Delta_0:=\{\delta\in\Delta:\pi(\delta)\in Y(2)\}$$

and the definable set

$$D:=(\Delta_0+s(Y)(2))\cap\mathcal{H}\ \subseteq\ \langle\Delta+s(Y)\rangle\cap\mathcal{H}\ =\ \langle s(Y)\rangle\cap\mathcal{H}.$$

Note that $0\in D$, and we now claim that D generates $\langle s(Y)\rangle\cap\mathcal{H}$. To prove that, it is sufficient to show the following:

Claim. For all n and for every $\delta_1,\dots,\delta_{2^n}\in\Delta$ and $y_1,\dots,y_{2^n}\in Y$, if $\Sigma_{i=1}^{2^n}\delta_i+s(y_i)\in\mathcal{H}$ then $\Sigma_{i=1}^{2^n}\delta_i+s(y_i)\in\langle D\rangle$.

Indeed, granted the claim, pick $\Sigma_{i=1}^m s(y_i)\in\langle s(Y)\rangle\cap\mathcal{H}$. Define $\delta_1=\dots=\delta_{2^m}=0$ and $y_{m+1}=\dots=y_{2^m}=0$. Since $\Sigma_{i=1}^{2^n}\delta_i+s(y_i)=\Sigma_{i=1}^m s(y_i)\in\mathcal{H}$ we deduce $\Sigma_{i=1}^m s(y_i)\in\langle D\rangle$, as required.

Proof of the claim. By induction on n. The case $n=0$ gives $\pi(\delta_1)+y_1=0$, hence $\pi(\delta_1)\in Y\subseteq Y(2)$, so $\delta_1\in\Delta_0$. Therefore $\delta_1+s(y_1)\in D$.

Assume now that $\Sigma_{i=1}^{2^n}\delta_i+s(y_i)\in\mathcal{H}$. We want to show that $\Sigma_{i=1}^{2^n}\delta_i+s(y_i)$ is in $\langle D\rangle$. We write the sum in pairs:

$$\Sigma_{i=1}^{2^n}(\delta_i+s(y_i))=\Sigma_{k=1}^{2^{n-1}}(s(y_{2k-1})+s(y_{2k})+\delta_{2k-1}+\delta_{2k}).$$

Now, because $Y+\Gamma=\mathcal{V}$, for each $k=1,\dots,2^{n-1}$ there is $w_k\in Y$ and $\beta_k\in\Gamma$ such that $y_{2k-1}+y_{2k}=\beta_k+w_k$. Let $\alpha_k\in\Delta$ be such that $\pi(\alpha_k)=\beta_k$. Note that $\beta_k\in Y(2)$, so that $\alpha_k\in\Delta_0$. Hence,

$$(s(y_{2k-1})+s(y_{2k})-\alpha_k-s(w_k))\in D.$$

Also because the image under π of $s(y_{2k-1})+s(y_{2k})-\alpha_k-s(w_k)$ is 0, it belongs to $\mathcal{H}$.

Thus the above sum also equals

$$\begin{aligned}\Sigma_{i=1}^{2^n}(\delta_i+s(y_i))=&\Sigma_{k=1}^{2^{n-1}}\big(s(y_{2k-1})+s(y_{2k})-\alpha_k-s(w_k)\big)+\\&\Sigma_{k=1}^{2^{n-1}}\big(\delta_{2k-1}+\delta_{2k}+\alpha_k+s(w_k)\big).\end{aligned}$$

We already showed that $\Sigma_{k=1}^{2^{n-1}}\big(s(y_{2k-1})+s(y_{2k})-\alpha_k-s(w_k)\big)\in\langle D\cap\mathcal{H}\rangle$, so if we denote $\widetilde{\delta}_k:=\delta_{2k-1}+\delta_{2k}+\alpha_k\in\Delta$ then

$$\Sigma_{k=1}^{2^{n-1}}\big(\widetilde{\delta}_k+s(w_k)\big)\in\mathcal{H},$$

and it remains to see that it belongs to $\langle D\rangle$. This follows by induction, so the claim is proved and with it Proposition 3.1. □

Proposition 3.2. *With $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{V}$ as in Proposition 3.1, assume that $X\subseteq\mathcal{G}$ is a definable set with $\langle\pi(X)\rangle=\mathcal{V}$. Then $\langle X\rangle\cap i(\mathcal{H})$ is definably generated.*

Proof. Again, we may assume that i is the identity map. Since $\mathcal{V}$ admits a lattice it contains a definable generic set Y. Without loss of generality, we may assume that $\pi(X)(1) \subseteq Y$. By saturation $Y \subseteq \pi(X)(\ell)$ for some $\ell \in \mathbb{N}$ and therefore by definable choice we can pick a section $s : Y \to \langle X \rangle$. Moreover, we can assume that $s(\pi(X)) \subseteq X$. Let $E := X(1) \cap \mathcal{H}$. By Proposition 3.1 we have that $\mathcal{H}_0 := \langle s(Y) \rangle \cap \mathcal{H}$ is definably generated. Thus, to prove that $\langle X \rangle \cap \mathcal{H}$ is definably generated it suffices to show that $\langle X \rangle \cap \mathcal{H} = \langle E \rangle + \mathcal{H}_0$.

To that aim, pick $x_1 \dots, x_n \in X$ such that $\Sigma_{i=1}^n x_i \in \mathcal{H}$. We can write

$$\Sigma_{i=1}^n x_i = \Sigma_{i=1}^n (x_i - s(\pi(x_i)) + \Sigma_{i=1}^n s(\pi(x_i)).$$

Note that $x_i - s(\pi(x_i)) \in E$ for each $i = 1, \dots, n$ and therefore $\Sigma_{i=1}^n (x_i - s(\pi(x_i)) \in \langle E \rangle$. Moreover, $\Sigma_{i=1}^n s(\pi(x_i)) = \Sigma_{i=1}^n x_i - \Sigma_{i=1}^n (x_i - s(\pi(x_i)) \in \mathcal{H}$. Since also $s(\pi(x_i)) \in s(\pi(X)) \subseteq s(Y)$ for each $i = 1, \dots, n$, we get $\Sigma_{i=1}^n s(\pi(x_i)) \in \mathcal{H}_0$ and so $\Sigma_{i=1}^n x_i \in \langle E \rangle + \mathcal{H}_0$, as required. □

Before the main corollary we need also the following lemma.

Lemma 3.3. *Let $\mathcal{G}$ be an abelian locally definable group which is definably generated. Then its connected component is definably generated by a definably connected set (with regard to the group topology). In particular, if every* connected *definably generated subgroup of $\mathcal{G}$ contains a definable generic set, then $\mathcal{G}$ has the generic property.*

Proof. Let $\mathcal{G}$ be a locally definable group and $X \subseteq \mathcal{G}$ be a definable set which generates $\mathcal{G}$. Let $X_1, \dots, X_k$ be its connected components. Fix an element a_i in each X_i, and let $\Gamma = \langle a_1, \dots, a_k \rangle$. Consider the connected set $\widetilde{X} = \bigcup X_i - a_i$, and notice that $\langle X \rangle = \langle \widetilde{X} \rangle + \Gamma$. Since $\langle \widetilde{X} \rangle$ is a locally definable subgroup of $\mathcal{G}$ of bounded index, it is compatible ([10, Fact 2.3(2)]), and since it is connected, it must be the connected component of $\mathcal{G}$.

For the second part of the statement, let $\mathcal{H}$ be a definably generated subgroup of $\mathcal{G}$. Then, by what we just showed, its connected component $\mathcal{H}^0$ is definably generated and therefore by hypothesis it contains a definable generic set Y, that is, there is a bounded $A \subseteq \mathcal{H}^0$ such that $A + Y = \mathcal{H}^0$. Since $\mathcal{H}^0$ has bounded index in $\mathcal{H}$, there is a bounded $B \subseteq \mathcal{H}$ such that $B + \mathcal{H}^0 = \mathcal{H}$. In particular $A + B + Y = \mathcal{H}$, as required. □

Theorem 3.4. *Assume that we are given an exact sequence of abelian locally definable groups and maps*

$$0 \longrightarrow \mathcal{H} \xrightarrow{\ i\ } \mathcal{G} \xrightarrow{\ \pi\ } \mathcal{V} \longrightarrow 0.$$

If $\mathcal{V}$ and $\mathcal{H}$ have the generic property then $\mathcal{G}$ has the generic property.

Proof. By Lemma 3.3 it is sufficient to consider subgroups of $\mathcal{G}$ which are generated by definably connected sets. Let X be a definably connected set. Since $\pi(\langle X \rangle) = \langle \pi(X) \rangle$ is a definably generated connected group, we have the exact sequence of locally definable groups

$$0 \to \langle X \rangle \cap i(\mathcal{H}) \to \langle X \rangle \to \langle \pi(X) \rangle \to 0.$$

By hypothesis the connected group $\langle \pi(X) \rangle$ contains a definable generic set, that is, there exists a definable set $Z_1 \subseteq \langle X \rangle$ such that $\pi(Z_1)$ is generic in $\langle \pi(X) \rangle$. In particular the group $\langle \pi(X) \rangle$ admits a lattice (see Fact 1.3) and therefore by Proposition 3.2 the group $\langle X \rangle \cap i(\mathcal{H})$ is definably generated. Again by hypothesis, we have that $\langle X \rangle \cap i(\mathcal{H})$ contains a definable generic set Z_2. Finally, it is not hard to see that $Z_1 + Z_2$ is generic in $\langle X \rangle$. □

3.1. **Some applications of Theorem 3.4.** First, we study definably generated subgroups of abelian torsion-free definable groups (see basic facts on torsion-free groups definable in o-minimal structures in Section 2.1 in [18]).

Corollary 3.5. *Any abelian torsion-free definable group G in an o-minimal structure $\mathcal{M}$ has the generic property with respect to $\mathcal{M}$.*

Proof. We prove it by induction on $\dim(G)$. Assume first that $\dim(G) = 1$ and prove first a more general result.

Lemma 3.6. *If $\mathcal{U}$ is a 1-dimensional torsion-free locally definable group then it has the generic property.*

Proof. By [8, Corollary 8.3], the group $\mathcal{U}$ can be linearly ordered. By Lemma 3.3 it suffices to study a subgroup generated by a set of the form $(-b, b) := \{x \in G : -b < x < b\}$, that is,

$$\langle (-b, b) \rangle = \bigcup_{n \in \mathbb{N}} (-nb, nb).$$

It is easy to verify that the group $\Gamma = \mathbb{Z}b$ is a lattice in $\langle (-b, b) \rangle$ because $\langle (-b, b) \rangle / \Gamma$ is isomorphic to the definable group $\big([0, b), \mathrm{mod}\ b\big)$. □

Now, assume that $\dim(G) > 1$. Then, by [20], there exists a definable subgroup H of G of dimension 1. In particular, we have the exact sequence

$$0 \to H \to G \to G/H \to 0.$$

Since H and G/H are abelian torsion-free definable groups smaller dimension it follows by induction that they have the generic property. By Theorem 3.4, so does G. □

Next we prove:

Corollary 3.7. *Let $\mathcal{M}$ be an o-minimal expansion of a real closed field and $G \subseteq Gl(n, R)$ a definable abelian linear group. Then G has the generic property with respect to $\mathcal{M}$.*

Proof. We may assume that G is definably connected with regard its group topology. By [16, Proposition 3.10], G is definably isomorphic to a semialgebraic linear group, and hence it is the connected component of $H(R)$ for some abelian linear algebraic subgroup of $GL(n, K)$, defined over R (here K is the algebraic closure of R). By [16, Fact 3.1], G is semialgebraically isomorphic to a group of the form $T^m \times (R^*_{>0})^k \times (R^+)^n$, where $T = SO(2, R)$.

By Corollary 3.5, the group $(R^*_{>0})^k \times (R^+)^n$ has the generic property, so by Theorem 3.2 it is enough to show that $T = SO(2, R)$ has the generic property. The universal covering of T is a torsion-free 1-dimensional locally definable group, so by Lemma 3.6, it has the generic property. Thus G has the generic property. □

4. Semialgebraic groups

The main purpose of this section is to show, see Theorem 4.5 below, that every semialgebraic abelian group over a real closed field R has the generic property with respect to certain o-minimal expansions of R, which we now fix.

In the rest of the section, we fix $\mathcal{R}$ to be an $\aleph_1$-saturated o-minimal L-structure expanding a real closed field R such that the $(L \cup L_{\mathbf{an,exp}})$-theory $\mathrm{Th}(\mathcal{R}) \cup T_{\mathbf{an,exp}}$ is consistent and has an o-minimal completion, call it T_0. We denote by $K := R(i)$ its algebraic closure.

For example, any real closed field, or more generally an $\aleph_1$-saturated structure elementarily equivalent to $\mathbb{R}_{an,exp}$ clearly satisfies the above.

We start by analysing the case of abelian varieties.

Proposition 4.1. *Every embedded abelian variety $A \subseteq \mathbb{P}^N(K)$ over K has the generic property with respect to $\mathcal{R}$.*

Proof. First, note that by our assumptions there exists an elementary extension $\mathcal{R} \prec \mathcal{R}'$ such that $\mathcal{R}'$ can be expanded to a model of T_0. Furthermore, we may assume that this structure is $\aleph_1$-saturated. By Proposition 2.4 and the fact that the generic property is preserved under taking reducts and elementary substructures (Remark 2.3(4)), it is sufficient to prove the result in $\mathcal{R}'$. Thus, all in all, we can assume that $\mathcal{R}$ is an $\aleph_1$-saturated model of T_0.

By Fact 1.6, there exist a locally definable subgroup $\mathcal{G}$ of some $\langle R^g, +\rangle$ and a locally definable covering homomorphism $p : \mathcal{G} \to A$. Thus, by Fact 1.5 and Remark 2.3, the group A has the generic property. □

Proposition 4.2. *Let H be an irreducible abelian K-algebraic group. Then H has the generic property with respect to $\mathcal{R}$.*

Proof. As in the proof of Proposition 4.1, we can assume that $\mathcal{R}$ is an $\aleph_1$-saturated model of $T_0 := \mathrm{Th}(\mathcal{R}) \cup T_{\mathrm{an,exp}}$.

By Corollary 3.7, the result is true when H is linear (notice that every linear subgroup of $GL(n, K)$ can be viewed as a linear subgroup of $GL(m, R)$ for some R).

For the general case, by Chevalley's theorem, there are a linear group L and an abelian variety A such that the following is an exact sequence:

$$0 \to L \to H \to A \to 0.$$

Thus, by Theorem 3.4 and Proposition 4.1, the group H also has the generic property. □

Remark 4.3. If H is an abelian K-algebraic group defined over R, then by Remark 2.3 and Proposition 4.2, the group of R-rational points $H(R)^{\mathrm{o}}$ has the generic property.

Before reaching our main theorem we recall the following result of Barriga, [3, Theorem 10.2], which describes every semialgebraic group in terms of the R-points of an associated algebraic group over R.

Fact 4.4. *Let G be a definably compact and connected semialgebraic abelian group over R. Then there exists a K-algebraic group H defined over R, an open connected locally semialgebraic subgroup $\mathcal{W}$ of the o-minimal universal covering group $\widetilde{H(R)^{o}}$ of the connected component of $H(R)$, and a locally semialgebraic surjective covering homomorphism $\theta : \mathcal{W} \to G$, with 0-dimensional kernel.*

Theorem 4.5. *For $\mathcal{R}$ an o-minimal structure expanding a real closed field R, as before, let G be an abelian semialgebraic group over R. Then G has the generic property with respect to $\mathcal{R}$. In particular, any semialgebraically generated subgroup of G contains a generic semialgebraic subset.*

Proof. By Lemma 3.3 it is enough to show that every locally definable subgroup of G generated by a definably connected set contains a definable generic subset. Thus, we can assume that G is connected.

By [20, Theorem 1.2], applied finitely many times, G contains a torsion-free subgroup H such that the quotient G/H is definably compact. Thus, by Theorem 3.4 and Corollary 3.5, we may assume that G is definably compact.

Using the notation of Fact 4.4, we have a covering homomorphism $\theta : \mathcal{W} \to G$, with $\mathcal{W}$ a definably generated subgroup of the locally definable group $\widetilde{H(R)^{\mathrm{o}}}$.

Denote by $p : \widetilde{H(R)^{\mathrm{o}}} \to H(R)^{\mathrm{o}}$ the universal covering map. We have the exact sequence

$$0 \to \ker(p) \to \widetilde{H(R)^{\mathrm{o}}} \to H(R)^{\mathrm{o}} \to 0.$$

Note that $\ker(p)$ is discrete and therefore its only semialgebraically generated connected subgroup is the trivial one, so by Lemma 3.3 the group $\ker(p)$ has the generic property. Thus, by Theorem 3.4 and Remark 4.3, we deduce that $\widetilde{H(R)^{\mathrm{o}}}$ has the generic property. In particular, by Remark 2.3(2), the same is true for $\mathcal{W}$, and by (3), also for G, as required. □

We already pointed out in the Introduction that Theorem 1.9 follows from Theorem 4.5. The following is also a direct consequence of the above together with a theorem of Hrushovski and Pillay [13]. In that paper the authors prove (among other things) that any semialgebraic group over a real closed field R is locally semialgebraic isomorphic to the R-points of an algebraic group defined over R (i.e., there are semialgebraically isomorphic open neighborhoods of the identity of each group). We point out that almost with the same proof, the latter result is also valid for a locally semialgebraic group.

Corollary 4.6. *If R is an $\aleph_1$-saturated real closed field extension of $\mathbb{R}$ and $\mathcal{G}$ is an abelian locally semialgebraic group definable over $\mathbb{R}$ and semialgebraically generated, then $\mathcal{G}$ has the generic property.*

Proof. We can assume that $\mathcal{G}$ is connected by Lemma 3.3. Moreover, considering the universal covering [9] we can assume that $\mathcal{G}$ is simply-connected (in the locally semialgebraic category).

We prove it by induction on the dimension of $\mathcal{G}$, the initial case $\dim(\mathcal{G}) = 0$ being trivial. We can assume also that the semialgebraic set Y that generates $\mathcal{G}$ is definable over $\mathbb{R}$ and it is closed with respect to the group topology of $\mathcal{G}$.

Case 1. If Y is not definably compact, then by [8, Thm.5.2] there is a compatible definable 1-dimensional, definably connected torsion-free subgroup H of $\mathcal{G}$. Since H has the generic property by Corollary 3.5, and $\mathcal{G}/H$ has the generic property by induction, we are done by Theorem 3.4.

Case 2. If Y definably compact, then consider the realization of $\mathcal{G}(\mathbb{R})$ of $\mathcal{G}$ over $\mathbb{R}$, which is a Lie group. By [13, Prop.3.1 & Cor.4.9] there is a semialgebraic open neighborhood U of the identity of $\mathcal{G}(\mathbb{R})$ and an open semialgebraic neighborhood V of the identity of the connected component $H^0(\mathbb{R})$ of an algebraic group H defined over $\mathbb{R}$, for which there exist a local isomorphism

$$f : U \to V$$

defined over $\mathbb{R}$. Now, since $\mathcal{G}$ is simply-connected by [1, Cor.6.8] $\mathcal{G}(\mathbb{R})$ is simply-connected. On the other hand, denote by $\widetilde{H^0(R)}$ the universal covering of $H^0(R)$, which is a locally semialgebraic group defined over $\mathbb{R}$ [9]. Moreover, the realization of $\widetilde{H^0(R)}$ in the real field is naturally the universal covering $\widetilde{H^0(\mathbb{R})}$ of $H^0(\mathbb{R})$. By the monodromy principle of simply-connected Lie groups, there exists a Lie isomorphism

$$F : \mathcal{G}(\mathbb{R}) \to \widetilde{H^0(\mathbb{R})}$$

that extends f. Now, for any $n \in \mathbb{N}$ the set $Y(\mathbb{R})(n)$ is a compact subset of $\mathcal{G}(\mathbb{R})$ and therefore finitely many translates of U cover it. We deduce that $F|_{Y(\mathbb{R})(n)}$ is a semialgebraic map defined over $\mathbb{R}$. Thus, it makes sense to consider the realization of each $F|_{Y(\mathbb{R})(n)}$ over R, so that we obtain a locally semialgebraic monomorphism (not necessarily surjective)

$$\mathcal{G} \hookrightarrow \widetilde{H^0(R)}.$$

Note that by Lemma 4.2 the group $H^0(R)$ has the generic property, so that by Theorem 3.4 $\widetilde{H^0(R)}$ has the generic property. Thus, $\mathcal{G}$ has the generic property, as required. □

5. Appendix: Abelian Varieties

Fact 1.6 is a consequence of the results in [20]. Maybe not in this form, we believe it is well-known by the experts (e.g., a similar statement is used in

[21, §5.2.2 and §5.3]). For the sake of completeness, we sketch a proof in this appendix. As in [20], we quote several facts concerning abelian varieties, see [5] for details.

For a positive $g \in \mathbb{N}$, by a *complex g-torus* we mean the quotient group $\mathbb{C}^g/\Lambda$ where Λ is a lattice, i.e., a subgroup of $(\mathbb{C}^g, +)$ generated by $2g$ vectors which are $\mathbb{R}$-linearly independent. It is a compact complex Lie group of dimension g. A torus $\mathbb{C}^g/\Lambda$ is called an *abelian variety* if it is biholomorphic to a complex *embedded abelian variety*, namely a projective connected complex algebraic group.

By a theorem of Baily [2], for any $g \in \mathbb{N}$ there is a countable collection of constructible families of embedded abelian varieties $\{\mathcal{A}^D\}_{D\in\mathcal{D}}$, parameterized by certain polarizations $D \in \mathcal{D}$, each of the form

$$\mathcal{A}^D = \{A^D_t \subset \mathbb{P}^N_D(\mathbb{C}) : t \in S_D\},$$

such that every g-dimensional embedded abelian variety is isomorphic to a member of one of the $\mathcal{A}^D$'s (see also [19, Thm. 8.11]).

We now denote by $\mathbb{T}$ the semialgebraic compact group $([0,1), +\,\mathrm{mod}1)$, which is isomorphic to $\mathbb{R}/\mathbb{Z}$. A consequence of [19, Theorem 8.10] is the following:

Fact 5.1. *For any $g \in \mathbb{N}$ and for each $D \in \mathcal{D}$ there exists in $\mathbb{R}_{an,exp}$ a definable family of group isomorphisms between the members of $\mathcal{A}^D$ and $\mathbb{T}^{2g}$.*

Indeed, the family of maps which is given by Φ^D in [19, Theorem 8.10] yields group biholomorphisms between complex tori of the form $\mathcal{E}_t = \mathbb{C}^g/\Lambda_t$ and the members of $A_t \in \mathcal{A}^D$. Each $\mathcal{E}_t$, via its fundamental domain, is definably group-isomorphic to $\mathbb{T}^{2g}$.

Proposition 5.2. *Let $\mathcal{A} = \{A_t : t \in T\}$ be a constructible family without parameters of embedded g-dimensional abelian varieties of $\mathbb{P}^N(\mathbb{C})$.*

Then there are constructible $T_1, \ldots, T_k \subseteq T$ covering T, and there are finitely many $D_1, \ldots, D_k \in \mathcal{D}$ and $d \in \mathbb{N}$ such that for any $j = 1, \ldots, k$, and $t \in T_j$ there exists an algebraic isomorphism between A_t and an abelian variety in $\mathcal{A}^{D_j}$. Moreover, for each j, there is a constructible family of isomorphisms $\mathcal{H}^{D_j}$, between the members of $\{A_t : t \in T_j\}$ and of $\mathcal{A}^{D_j}$.

Proof. By Fact 5.1, each $A_t \in \mathcal{A}$ is bi-regularly isomorphic to some $A \in \mathcal{A}^D$, for $D \in \mathcal{D}$. Because the complex field is $\aleph_1$-saturated and there are countably many $D \in \mathcal{D}$ it follows that there exist $D_1, \ldots, D_k \in \mathcal{D}$ such that for any $t \in T$ we have that A_t is bi-regularly isomorphic to an abelian variety in $\mathcal{A}^{D_j}$ for some $j \in \{1, \ldots, k\}$. Again by saturation, the degree of the isomorphism, as t varies in T_j, is uniformly bounded by some d.

Now, for each $j \in \{1, \ldots, k\}$ the set T_j of $t \in T$ such that there exists a bi-regular isomorphism of degree at most d between A_t and an abelian variety in $\mathcal{A}^{D_j}$ is constructible without parameters. Because of the above bound on the degree, there exists a constructible family of isomorphisms $\mathcal{H}^{D_j}$ as required. □

Proof of Fact 1.6. We now return to the setting of 1.6 with R a real closed field and $K = R(i)$ its algebraic closure. Let $A \subseteq \mathbb{P}^N(K)$ be an embedded abelian variety. Let $c \in K^\ell$ be a tuple of coefficients defining algebraically the variety A.

We can replace the parameter c by a tuple t of free variables and therefore we obtain (without parameters) a constructible family $\mathcal{A} = \{A_t : t \in T\}$ of abelian subvarieties of $\mathbb{P}^N(K)$ of dimension $g := \dim(A)$.

Consider the realization $\mathcal{A}(\mathbb{C})$ of $\mathcal{A}$ in $\mathbb{C}$. By Proposition 5.2 we can assume that $\mathcal{A}$ is a sub-family of $\mathcal{A}^D$ for some $D \in \mathcal{D}$.

By Fact 5.1 there is a definable family in $\mathbb{R}_{an,exp}$ of group isomorphisms between the members of $\mathcal{A}^D$ and $\mathbb{T}^{2g}$. Going back to R and K, we can find a definable isomorphism between A and $\mathbb{T}^{2g}$ with its natural realization in R. Finally, there is a locally definable covering map from a locally definable subgroup of R^{2g} onto $\mathbb{T}^{2g}$, so also onto A. □

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