DEFINABILITY OF RESTRICTED THETA FUNCTIONS AND FAMILIES OF ABELIAN VARIETIES

YA'ACOV PETERZIL AND SERGEI STARCHENKO

Abstract. We consider some classical maps from the theory of abelian varieties and their moduli spaces, and prove their definability on restricted domains in the o-minimal structure $\mathbb{R}_{an,exp}$. In particular, we prove that the projective embedding of the moduli space of the principally polarized abelian variety $Sp(2g,\mathbb{Z})\backslash\mathbb{H}_g$ is definable in $\mathbb{R}_{an,exp}$ when restricted to Siegel's fundamental set $\mathfrak{F}_g$. We also prove the definability on appropriate domains of embeddings of families of abelian varieties into projective spaces.

1. Introduction

The goal of this paper is to establish a link between the classical analytic theory of complex abelian varieties and the theory of o-minimal structures. In [18] we established a similar link in the one-dimensional case by showing that the analytic j-invariant and more generally the Weierstrass function $\wp(z,\tau)$, as a function of two variables, are definable in the o-minimal structure $\mathbb{R}_{an,exp}$ when restricted to appropriate domain ($\mathbb{R}_{an,exp}$ is the expansion of the real field by the real exponential function as well as all restrictions of real analytic functions to compact rectangular boxes $B \subseteq \mathbb{R}^n$).

In [21], and then in [20] Pila makes use of this o-minimal link in order to settle some open cases of the André-Oort Conjecture by applying his powerful theorem with Wilkie [22] about rational and algebraic points on definable sets in o-minimal structures. Although this is not part of our paper, we briefly recall the statement of the conjecture and its status:

The Andre-Oort Conjecture *Let X be a Shimura variety and let $Y \subseteq X$ be an irreducible algebraic subvariety. If Y contains a Zariski dense set of special points then Y must be a special subvariety.*

The conjecture has been proven in full generality under the assumption of the Generalized Riemann Hypothesis, see [14]. In [20] Pila proves the conjecture unconditionally for the Shimura variety $\mathbb{C}^n$. The results in our present paper (see Theorem 1.1) opens the way for a possible application of o-minimality towards a proof the same conjecture for Siegel modular varieties $Sp(2g,\mathbb{Z})\backslash\mathbb{H}_g$, $g \geq 1$. In addition, Theorem 1.2 could be helpful in handling the conjecture for some mixed Shimura varieties, namely families of abelian varieties over their moduli space. It

2010 *Mathematics Subject Classification.* Primary 03C64, 03C98, 14K25, 11F46; Secondary 14G35, 11G18.

Key words and phrases. O-minimality, abelian varieties, theta function, modular forms, André-Oort Conjecture.

The second author was partially supported by NSF.

should be emphasized though that o-minimality is just one of several ingredients in the above proof of Pila. Another one is the so-called lower bound on the number of special points, given through Galois theory is still missing in the general case of Andre-Oort, even for Siegel modular varieties. Furthermore, in order to apply o-minimality to the general case of Andre-Oort, one needs to prove a similar result to ours for general locally symmetric spaces. This requires analyzing automorphic forms in this setting, on their so-called Siegel fundamental sets, instead of just theta functions and Siegel modular forms which we handle here.

Definability of classical functions in o-minimal structures can be useful in various other ways. At the end of the paper we give an example of one such application to the theory of compactifications in complex analytic spaces.

Let us review the basic setting in which we are working. We consider, for every $g \geq 1$, the family of complex g-dimensional tori and in particular the sub-family of abelian varieties, namely those tori which are isomorphic to projective varieties. Every abelian variety of dimension g is isomorphic to $\mathcal{E}_\tau^D = \mathbb{C}^g/(\tau\mathbb{Z}^g + D\mathbb{Z}^g)$, where τ is a symmetric $g \times g$ complex matrix with a positive definite imaginary part and $D = diag(d_1, \ldots, d_g)$ with $d_1|d_2 \cdots |d_g$ non-negative integers (called a polarization type of $\mathcal{E}_\tau^D$). Thus, for every such matrix D, the family of all corresponding abelian varieties is parameterized by the *Siegel half space*

$$\mathbb{H}_g = \{\tau \in Sym(g, \mathbb{C}) : Im(\tau) > 0\}.$$

For each $\tau \in \mathbb{H}_g$ there is a holomorphic map $h_\tau : \mathbb{C}^g \to \mathbb{P}^k(\mathbb{C})$ (for some k depending only on D), which is invariant under the lattice $\tau\mathbb{Z}^g + D\mathbb{Z}^g$ and induces an embedding of $\mathcal{E}_\tau^D$ into $\mathbb{P}^k(\mathbb{C})$. While h_τ is periodic and therefore cannot be definable in an o-minimal structure, its restriction to a fundamental parallelogram $E_\tau^D \subseteq \mathbb{C}^g$ is immediately seen to be definable in the o-minimal structure $\mathbb{R}_{an}$ (since the real and imaginary parts of this map are real analytic).

Each $\mathcal{E}_\tau^D$ is naturally endowed with a polarization of type D. For each such D there is a corresponding discrete group $G_D \subseteq Sp(2g, \mathbb{Q})$ acting on $\mathbb{H}_g$ and two polarized abelian varieties in the family are isomorphic if and only if their corresponding τ's are in the same G_D-orbit.

Let us consider the family of principally polarized varieties, namely the case $D = I$, where $G_D = Sp(2g, \mathbb{Z})$. By [2], there is a holomorphic map $F : \mathbb{H}_g \to \mathbb{P}^m(\mathbb{C})$, for some m, inducing an embedding of $Sp(2g, \mathbb{Z})\backslash\mathbb{H}_g$ into $\mathbb{P}^m(\mathbb{C})$, whose image is Zariski open in some projective variety. Because of the periodicity of this map, one cannot hope that the whole map F will be definable in an o-minimal structure. Instead, we consider a restriction of F to the semialgebraic set $\mathfrak{F}_g \subseteq \mathbb{H}_g$, called *the Siegel fundamental set*, which contains a single representative for every $Sp(2g, \mathbb{Z})$-orbit. We prove here (see Theorem 8.3):

Theorem 1.1. *There is an open set $U \subseteq \mathbb{H}_g$ containing $\mathfrak{F}_g$, such that the restriction of the above $F : \mathbb{H}_g \to \mathbb{P}^m(\mathbb{C})$ to U is definable in the o-minimal structure $\mathbb{R}_{an,exp}$.*

Note that since F is $Sp(2g, \mathbb{Z})$-invariant, and $\mathfrak{F}_g$ is a fundamental set for $Sp(2g, \mathbb{Z})$, we have $F(U) = F(\mathbb{H}_g)$ and in this sense the above theorem says that the quotient $Sp(2g, \mathbb{Z})\backslash\mathbb{H}_g$ can be definably embedded into projective space in the structure $\mathbb{R}_{an,exp}$.

While the map $\tau \mapsto F(\tau)$ is a function on $\mathbb{H}_g$, the theorem above follows from the definability of a map in two sets of variables (z, τ). Namely, for every fixed polarization type D we consider the family of abelian varieties of polarization type D, $\mathcal{F}^D = \{\mathcal{E}^D_\tau : \tau \in \mathbb{H}_g\}$. As was noted in [18] the family $\mathcal{F}^D$ can be viewed as a semialgebraic family of complex tori, by identifying the underlying set of each $\mathcal{E}^D_\tau$ with its fundamental parallelogram E^D_τ. We let $h^D_\tau : E^D_\tau \to \mathbb{P}^k(\mathbb{C})$ be the embedding of the complex torus $\mathcal{E}^D_\tau$ into projective space and consider the family $\{h^D_\tau : \tau \in \mathbb{H}_g\}$. Again, for reasons of periodicity we cannot hope for this whole family to be definable in an o-minimal structure. However, using Siegel's reduction theory we can find a fundamental set $\mathfrak{F}^D_g \subseteq \mathbb{H}_g$, such that every $\mathcal{E}^D_\tau$ is isomorphic, as a polarized abelian variety of type D, to one of the form $\mathcal{E}^D_{\tau'}$, for $\tau' \in \mathfrak{F}^D_g$. We prove (see Theorem 8.5):

Theorem 1.2. *For every polarization type D, there is in $\mathbb{R}_{an,exp}$ a definable open set $U \subseteq \mathbb{H}_g$ containing $\mathfrak{F}^D_g$ and a definable family of maps $\{h^D_\tau : \tau \in U\}$, such that each $h^D_\tau : E^D_\tau \to \mathbb{P}^k(\mathbb{C})$ is an embedding of the abelian variety $\mathcal{E}^D_\tau$ into projective space.*

The map F and the family $\{h^D_\tau\}$ are defined in coordinates via Riemann theta functions, of the form $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)$, for $a, b \in \mathbb{R}^g$. Most of our work in the paper goes towards proving:

Theorem 1.3. *Fix a polarization type $D = diag(d_1, \dots, d_g)$. For each $a, b \in \mathbb{R}^g$, the restriction of $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}$ to the set*

$$\{(z,\tau) \in \mathbb{C}^g \times \mathbb{H}_g : \tau \in \mathfrak{F}_g \ , \ z \in E^D_\tau\}$$

is definable in $\mathbb{R}_{an,exp}$.

As a corollary we conclude (see Theorem 8.2):

Corollary 1.4. *Let $\phi : \mathbb{H}_g \to \mathbb{C}$ be a modular form with respect to some congruence subgroup of $Sp(2g,\mathbb{Z})$. Then the restriction of ϕ to some open set containing $\mathfrak{F}_g$ is definable in $\mathbb{R}_{an,exp}$.*

Much attention has been given classically to the possible compactifications of the quotient $G_D \backslash \mathbb{H}_g$. At the end of the paper we show how removal of singularities in o-minimal structures can be used to show that the closure in projective space of the image of some of the above maps, is an algebraic variety.

1.1. **O-minimality.** Since the subject of o-minimality may be not very familiar to some readers we give a brief introduction to the topics and refer to [5, 6] for more detailed expositions. As usual $\mathcal{P}(X)$ denotes the collection of all subsets of a set X.

For the purpose of this paper, by *a structure* (over $\mathbb{R}$) we mean a collection $\mathcal{R} = \left(\mathcal{R}^{(k)}\right)_{k \geq 1}$ with $\mathcal{R}^{(k)} \subseteq \mathcal{P}(\mathbb{R}^k)$ satisfying the following properties:

(1) Every $\mathcal{R}^{(k)}$ is a Boolean subalgebra of $\mathcal{P}(\mathbb{R}^k)$.
(2) If $A \in \mathcal{R}^{(k)}$ and $B \in \mathbb{R}^{(l()}$ then $A \times B \in \mathcal{R}^{(k+l)}$.
(3) If $A \in \mathcal{R}^{(k)}$ and $P \colon \mathbb{R}^k \to \mathbb{R}^l$ is a polynomial map then the image $P(A)$ is in $\mathcal{R}^{(l)}$.

We say that a subset $X \subseteq \mathbb{R}^n$ is $\mathcal{R}$-definable, or definable in $\mathcal{R}$, if $X \in \mathcal{R}^{(k)}$. A function is $\mathcal{R}$-definable if its graph is an $\mathcal{R}$-definable set.

It is easy to see that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are structures then their intersection $\mathcal{R}_1 \cap \mathcal{R}_2 = \left(\mathcal{R}_1^{(k)} \cap \mathcal{R}_2^{(k)}\right)_{k\geq 1}$ is a structure again, hence for every collection $\mathcal{R} = \left(\mathcal{R}^{(k)}\right)_{k\geq 1}$ with $\mathcal{R}^{(k)} \subseteq \mathcal{P}(\mathbb{R}^k)$ there is a minimal structure containing $\mathcal{R}$ that is called the structure generated by $\mathcal{R}$.

Example 1.5. *The collection $\mathcal{R}_a$ of all algebraic sets, i.e. sets that are finite Boolean combinations of zero sets of polynomials, is not a structure: the projection of the algebraic set $x = y^2$ to the first component is the set $\{x \in \mathbb{R} \colon x \geq 0\}$ that is not algebraic.*

By the Tarski-Seidenberg principle, the collection $\mathcal{R}_{sa}$ of all semi-algebraic sets, i.e. sets defined by finitely many polynomial equations and inequalities ($<$,$\leq$), is a structure.

Definition 1.6. A structure $\mathcal{R}$ is called *o-minimal* (order-minimal) if every nonempty $X \in \mathcal{R}^{(1)}$ is a finite union of points and intervals (bounded or unbounded).

Example 1.7. *1. The semi-algebraic structure $\mathcal{R}_{sa}$ is o-minimal.*

2. A restricted analytic function *is a function $f\colon K \to R$, whose domain K is a compact semi-algebraic subset of $\mathbb{R}^n$, such that f has an analytic extension to some open $U \supseteq K$. The structure $\mathbb{R}_{an}$ generated by the graphs of all restricted analytic functions is o-minimal (this follows from the main result of* [9]*).*

3. The structure $\mathbb{R}_{an,exp}$ generated by $\mathbb{R}_{an}$ and the graph of the real exponential function $y = e^x$ is o-minimal (see [7] *and* [8]*). This is the main o-minimal structure used in the current paper.*

We present few facts about o-minimal structures to demonstrate their remarkable tameness properties.

Fact 1.8. *Let $\mathcal{R}$ be an o-minimal structure.*

1. *Every $\mathcal{R}$-definable set has finitely many connected component, and each connected component is $\mathcal{R}$-definable.*
2. *For an $\mathcal{R}$-definable set $A \subseteq \mathbb{R}^n$*
$$\dim(cl(A) \setminus A) < \dim(A),$$
where the dimension of A is the maximal d such that A contains a C^1-submanifold of $\mathbb{R}^n$ of dimension d.
3. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be an $\mathcal{R}$-definable function. For every $p \geq 0$ there is a C^p Verdier stratification of $\mathbb{R}^n$ such that each strata is $\mathcal{R}$-definable and f is of the class C^p on each of them.*

Identifying as usual $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we also have a notion of $\mathcal{R}$-definable functions from subsets $\mathbb{C}^n$ to $\mathbb{C}^m$.

Example 1.9. *The restriction of the complex exponential function e^z to the strip $\{z \in \mathbb{C} \colon 0 \leq Im(z) \leq 2\pi\}$ is definable in $\mathbb{R}_{an,exp}$, since the restrictions of $\sin(x)$ and $\cos(x)$ to $[0, 2\pi]$ are $\mathbb{R}_{an}$-definable. However, full complex exponentiation cannot be definable in any o-minimal structure because its zero set will be a definable set with infinitely many components.*

We refer to [17] for more details on tameness of complex analytic functions definable in o-minimal structures.

2. Conventions and Notations

Because of the need for heavy notation we begin with a list of symbols to be used in the paper.

2.1. **Conventions.** For a ring R we will denote by $M_g(R)$ the set of of all $g \times g$ matrices over R, and by $Sym(g, R)$ the set of all symmetric $g \times g$ matrices over R.

We always consider $a \in R^g$ as a column vector. For $a \in R^g$ and $M \in M_g(R)$, we denote by ${}^t a$ and ${}^t M$ the transpose of a and M, respectively.
For a matrix $y \in M_g(R)$ and $a \in R^g$ we denote ${}^t aya$ by $y[a]$.

We denote complex exponentiation by exp.

2.2. **Notations used in the paper.**

$\mathbb{H}_g$	Siegel upper half-space of degree g	Page 6
Λ_τ^D	The lattice $\tau\mathbb{Z}^g + D\mathbb{Z}^g$	Page 6
$\mathcal{E}_\tau^D$	The torus $\mathbb{C}^g/\Lambda_\tau^D$	Page 6
$Sp(2g, K)$	The symplectic group	Page 6
G_D	A special subgroup of $Sp(2g, \mathbb{Q})$	Page 7
$\mathfrak{M}_g$	The set of Minkowski reduced matrices	Page 7
$\mathfrak{F}_g(G)$	A fundamental set for the action of G	Page 9
$\vartheta(z, \tau), \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)$	Riemann Theta functions	Page 9
φ^D	A holomorphic map from $\mathbb{C}^g \times \mathbb{H}_g$ into $\mathbb{P}_n(\mathbb{C})$ whose components are theta functions	Page 10
φ_τ^D	The map $z \mapsto \varphi^D(z, \tau)$	Page 10
$\mathbb{R}_n$	The standard polyhedral cone $(\mathbb{R}^{\geq 0})^n$	Page 10
$\mathcal{T}(S)$	A tube in $\mathbb{C}^n$ corresponding to a set $S \subseteq \mathbb{R}^n$	Page 11
$\mathbf{e}(z)$	The complex function $\mathbf{e}(z) = e^{2\pi i z}$	Page 11
$\overline{\mathfrak{M}}_g$	An integral polyhedral cone containing $\mathfrak{M}_g$	Page 14
C_m	An integral polyhedral cone in $\mathbb{R}^g \times Sym(g, \mathbb{R})$	Page 14
E_τ^D	The fundamental parallelogram of Λ_τ^D	Page 18
$\bar{E}_\tau^D$	The topological closure of $\bar{E}_\tau^D$	Page 18
$\mathcal{X}^D(V), \mathcal{X}^D_{<K}(V)$	Subsets of $\mathbb{C}^g \times \mathbb{H}_g$	Page 18
$G_D(D)_0$	A special subgroup of G_D	Page 23
$\mathfrak{F}_g^D$	The fundamental set $\mathfrak{F}_g(G_D(D)_0)$	Page 24
$\Psi^D(\tau)$	The map $\Psi^D(\tau) = \varphi^D(0, \tau)$	Page 23
$\Phi^D(z, \tau)$	The map $\Phi^D(z, \tau) = (\varphi_D(z, \tau), \Psi^D(\tau))$	Page 23

3. Polarized abelian varieties and the Siegel fundamental domain

In this section we review briefly some known facts about tori and polarized abelian varieties. We refer to [3] for more details.

3.1. **Complex g-tori, abelian varieties and polarization.** For a positive $g \in \mathbb{N}$, by *a complex g-torus* we mean the quotient group $\mathbb{C}^g/\Lambda$, where $\Lambda \subseteq \mathbb{C}^g$ is a lattice, i.e. a subgroup of $(\mathbb{C}^g, +)$ generated by $2g$ vectors which are $\mathbb{R}$-linearly independent. Under the induced structure a complex g-torus $\mathbb{C}^g/\Lambda$ is a compact complex Lie group of dimension g, and vice versa, every g-dimensional compact complex Lie group is bi-holomorphic to a complex g-torus.

If $f\colon \mathbb{C}^g/\Lambda \to \mathbb{C}^g/\Lambda'$ is a bi-holomorphism between two tori then f can be lifted to a $\mathbb{C}$-linear map $F\colon \mathbb{C}^g \to \mathbb{C}^g$ with $F(\Lambda) = \Lambda'$, and vice-versa, every such F induces an isomorphism between corresponding tori.

A torus $\mathbb{C}^g/\Lambda$ is called *an abelian variety* if it is biholomorphic with a projective variety in $\mathbb{P}^k(\mathbb{C})$ for some k. If $g > 1$ then not every g-torus is an abelian variety. The following criterion is the well-known Riemann condition.

Theorem 3.1 (Riemann condition)**.** *A g-torus $\mathcal{E}$ is an abelian variety if and only if it is bi-holomorphic with a torus $\mathbb{C}^g/\Lambda$ with $\Lambda = \tau\mathbb{Z}^g + D\mathbb{Z}^g$, where D is a diagonal matrix $D = Diag(d_1, \ldots, d_g) = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_g \end{pmatrix}$ with positive integers $d_1|d_2|\cdots|d_g$ and τ is a complex $g \times g$ symmetric matrix with a positive definite imaginary part.*

Definition 3.2.

(1) The set of matrices
$$\mathbb{H}_g = \{\tau \in Sym(g, \mathbb{C})\colon Im(\tau) \text{ is positive definite } \}$$
is called *the Siegel upper half-space of degree g.*

(2) An integer diagonal matrix $D = Diag(d_1, \ldots, d_g)$ with positive $d_1, \ldots, d_g$ satisfying $d_1|d_2|\ldots|d_g$ is called *a polarization type.* When $D = I_g$ we say that the polarization is *principal.*

(3) We say that a g-torus $\mathcal{E}$ admits a polarization of the type $D = Diag(d_1, \ldots, d_g)$ if $\mathcal{E}$ is bi-holomorphic with a torus $\mathbb{C}^g/\Lambda$ with $\Lambda = \tau\mathbb{Z}^g + D\mathbb{Z}^g$ for some $\tau \in \mathbb{H}_g$.

Thus a torus $\mathcal{E}$ is an abelian variety if and only is it admits a polarization.

Remark 3.3. It is not hard to see that every abelian variety $\mathbb{C}^g/\Lambda$ admits infinitely many polarization types. For example, since for every positive integer k abelian varieties $\mathbb{C}^g/\big(\tau\mathbb{Z}^g + D\mathbb{Z}^g\big)$ and $\mathbb{C}^g/\big(k\tau\mathbb{Z}^g + kD\mathbb{Z}^g\big)$ are isomorphic (via the map $z \mapsto kz$), every abelian variety admitting a polarization of type D also admits a polarization of type kD for any positive $k \in \mathbb{N}$.

Notation 3.4. For a polarization type D and $\tau \in \mathbb{H}_g$ we will denote by Λ^D_τ the lattice
$$\Lambda^D_\tau = \tau\mathbb{Z}^g + D\mathbb{Z}^g,$$
and by $\mathcal{E}^D_\tau$ the abelian variety
$$\mathcal{E}^D_\tau = \mathbb{C}^g/(\tau\mathbb{Z}^g + D\mathbb{Z}^g).$$

3.2. **Action of $Sp(2g, \mathbb{R})$ and isomorphisms of polarized abelian varieties.** For $K = \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$ we will denote by $Sp(2g, K)$ the corresponding symplectic group
$$Sp(2g, K) = \left\{ M \in Gl(2g, K)\colon M \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} {}^tM = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \right\}.$$

It is a subgroup of $Gl(2g, K)$ closed under transposition.

The group $Sp(2g, \mathbb{R})$ acts (on the left) on the Siegel upper half-space $\mathbb{H}_g$ via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \tau = (\alpha\tau + \beta)(\gamma\tau + \delta)^{-1}.$$

For a polarization type D, let G_D be the following subgroup of $Sp(2g, \mathbb{Q})$:

$$G_D = \left\{ M \in Sp(2g, \mathbb{Q}) \colon \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}^{-1} M \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \in M_{2g}(\mathbb{Z}) \right\}.$$

Notice that $G_{I_g} = Sp(2g, \mathbb{Z})$.

Without going into details of polarizations we just state the following two facts that follow from [3, Proposition 8.1.3 and Remark 8.1.4].

Fact 3.5. *Let D be a polarization type, $A \in GL(g, \mathbb{C})$, and $\tau, \tau' \in \mathbb{H}_g$. The map $z \mapsto Az$ induces an isomorphism (as polarized abelian varieties) between the polarized abelian varieties $\mathcal{E}^D_{\tau'}$ and $\mathcal{E}^D_{\tau}$ if and only if there is $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_D$ such that $\tau' = M \cdot \tau$ and $A = {}^t(\gamma\tau + \delta)$.*

Remark 3.6. For $A \in GL(g, \mathbb{C})$ the matrix M in the previous fact is the unique matrix $M \in Gl(2g, \mathbb{R})$ such that $A(\tau, I_g) = (\tau, I_g)^t M$, namely the following diagram is commutative.

$$\begin{array}{ccc} \mathbb{R}^{2g} & \xrightarrow{x \mapsto (\tau', I_g)x} & \mathbb{C}^g \\ {\scriptstyle x \mapsto {}^tMx} \downarrow & & \downarrow {\scriptstyle z \mapsto Az} \\ \mathbb{R}^{2g} & \xrightarrow[x \mapsto (\tau, I_g)x]{} & \mathbb{C}^g \end{array}$$

3.3. **The Siegel fundamental set $\mathfrak{F}_g$ for the action of $Sp(2g, \mathbb{Z})$.** We fix a positive $g \in \mathbb{N}$ and let $n = g(g+1)/2$. We will identify $Sym(g, \mathbb{R})$ with $\mathbb{R}^n$, and $Sym(g, \mathbb{C})$ with $\mathbb{C}^n$.

3.3.1. *Minkowski reduced matrices.*

Definition 3.7. A real symmetric matrix $\beta = (\beta_{i,j}) \in Sym(g, \mathbb{R})$ is called *Minkowski reduced* if it satisfies the following conditions.

M(I) $\beta_{1,1} > 0$;

M(II) $\beta[a] \geq \beta_{k,k}$, for all $k = 1, \dots g$, and for all ${}^ta = (a_1, \dots, a_g) \in \mathbb{Z}^g$, with $\gcd(a_k, \dots, a_g) = 1$.

M(III) $\beta_{k,k+1} \geq 0$ for all $k = 1, \dots, g-1$.

We denote by $\mathfrak{M}_g$ the set of all Minkowski-reduced $g \times g$ real matrices.

Remark 3.8. Note that the definition above differs with the definition in [11, p.191] or in [26, p.128] in that condition M(I) here is replaced there by the assumption that the matrix β is positive definite. However, if a matrix β is positive definite then $\beta_{1,1} > 0$ and conversely, Siegel shows in [24, Theorem 3] that every matrix satisfying M(I), M(II) is positive definite, so the definitions are equivalent. It follows that the set $\mathfrak{M}_g$ is closed in the set of all real symmetric positive-definite $g \times g$-matrices.

Here are some basic facts about Minkowski reduced matrices (see references below).

Fact 3.9. *(a) There are positive real constants c, c' such that for every real matrix $\beta \in \mathfrak{M}_g$, and for every $x \in \mathbb{R}^g$,*

$$c \sum_{i=1}^{g} \beta_{i,\,i} x_i^2 \leq \beta[x] \leq c' \sum_{i=1}^{g} \beta_{i,\,i} x_i^2$$

(b) For every matrix $\beta \in \mathfrak{M}_g$ and $1 \leq i, j \leq g$, we have $|\beta_{i,\,j}| \leq \frac{1}{2}\beta_{i,\,i}$.
(c) For every matrix $\beta \in \mathfrak{M}_g$, we have $0 < \beta_{1,\,1} \leq \beta_{2,\,2} \leq \cdots \leq \beta_{g,\,g}$.
(d) There are finitely many inequalities in M(II), which together with M(I) and M(III) define $\mathfrak{M}_g$. In particular, the set $\mathfrak{M}_g$ is semialgebraic.
(e) The dimension of $\mathfrak{M}_g$ is n.

For (a), (b), and (c) above, see [26, p. 132, Proposition 1(c,a)]. Clause (d) is exactly [26, p. 130, Theorem 1(2)]. For (e), it follows from [24, Theorem 6] that $\mathfrak{M}_g$ contains a non-empty open subset of positive definite $g \times g$ matrices. It is easy to see that the set of positive definite matrices is open in $Sym(g, \mathbb{R})$, hence the dimension of $\mathfrak{M}_g$ is n.

3.3.2. *The Siegel fundamental set.* For the following definition see [11, p. 194].

Definition 3.10. The Siegel fundamental set $\mathfrak{F}_g$ is the collection of all symmetric $\tau \in Sym(g, \mathbb{C})$ such that

(i) $Im(\tau)$ is in $\mathfrak{M}_g$.
(ii) $|Re(\tau_{i,j})| \leq 1/2$ for all $1 \leq i, j \leq g$.
(iii) $det(Im(\sigma\tau)) \leq det(Im(\tau))$ for all $\sigma \in Sp(2g, \mathbb{Z})$.

As is pointed out in [11, p.194], clause (iii) is equivalent to:

$$\text{For all } \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp(2g, \mathbb{Z}), \text{ we have } |det(\gamma\tau + \delta)| \geq 1. \tag{1}$$

Notice that, since every Minkowski-reduced matrix is positive-definite, we have $\mathfrak{F}_g \subseteq \mathbb{H}_g$.

Here are the main properties that we are going to use.

Fact 3.11.

(a) $\mathfrak{F}_g$ is a closed subset of $\mathbb{C}^n$ (and not only of $\mathbb{H}_g$).
(b) If $\tau = \alpha + i\beta \in \mathfrak{F}_g$ then $\beta_{1,\,1} \geq \frac{\sqrt{3}}{2}$.
(c) There are only finitely many inequalities in (iii) which together with (i) and (ii) define $\mathfrak{F}_g$. In particular, $\mathfrak{F}_g$ is a semialgebraic subset of $\mathbb{H}_g$.

For item (a) see [11, Lemma 16, p. 196]. For (b) see [11, Lemma 15, p. 195], and for (c) we refer to [13, Theorem 1, p.35].

The following fact explains the significance of the set $\mathfrak{F}_g$.

Fact 3.12. *The set $\mathfrak{F}_g \subseteq \mathbb{C}^n$ is the Siegel fundamental set for the action of $Sp(2g, \mathbb{Z})$ on $\mathbb{H}_g$, namely*

(a) $Sp(2g, \mathbb{Z}) \cdot \mathfrak{F}_g = \mathbb{H}_g$ and the there are only finitely many elements $\sigma \in Sp(2g, \mathbb{Z})$ such that $\mathfrak{F}_g \cap \sigma \cdot \mathfrak{F}_g$ is nonempty.
(b) For every compact $X \subseteq \mathbb{H}_g$ the set $\{\sigma \in Sp(2g, \mathbb{Z}) \colon \sigma \cdot \mathfrak{F}_g \cap X \neq \varnothing\}$ is finite.

See [13, Theorem 2, p.34] for the proof of the above fact.

We now describe the fundamental set $\mathfrak{F}_g(G)$ for $G < Sp(2g, \mathbb{Z})$, a subgroup of finite index. Let $\gamma_1, \ldots, \gamma_k \in Sp(2g, \mathbb{Z})$ be representatives of all left co-sets of G in

$Sp(2g,\mathbb{Z})$ (so $Sp(2g,\mathbb{Z}) = \cup_{i=1}^{k} G\gamma_i$). We call a set of the form $\cup_{i=1}^{k}\gamma_i \cdot \mathfrak{F}_g$ *a Siegel fundamental set for* G, and suppressing the dependence on representatives, we will denote this set by $\mathfrak{F}_g(G)$. So

$$\mathfrak{F}_g(G) = \cup_{i=1}^{k}\gamma_i \cdot \mathfrak{F}_g. \tag{2}$$

Obviously, $\mathfrak{F}_g(G)$ is a semialgebraic subset of $\mathbb{H}_g$, and since $\mathfrak{F}_g$ is a fundamental set for the action of $Sp(2g,\mathbb{Z})$, the set $\mathfrak{F}_g(G)$ is a fundamental set for the action of G on $\mathbb{H}_g$, and we have:

Fact 3.13. *(i)* $\mathfrak{F}_g(G)$ *is a closed subset of* $\mathbb{H}_g$*;*
(ii) $G \cdot \mathfrak{F}_g(G) = \mathbb{H}_g$*;*
(iii) there are only finitely many $\sigma \in G$ *such that* $\mathfrak{F}_g(G) \cap \sigma \cdot \mathfrak{F}_g(G)$ *is nonempty.*
(iv) For every compact $X \subseteq \mathbb{H}_g$ *the set* $\{\sigma \in G \colon \sigma \cdot \mathfrak{F}_g(G) \cap X \neq \varnothing\}$ *is finite.*

4. The classical theta functions and embeddings of abelian varieties

4.1. **The theta functions.** As pointed out earlier, we identify the set $Sym(g,\mathbb{C})$ with $\mathbb{C}^n$ (for $n = g(g+1)/2$), and view $\mathbb{H}_g$ as a subset of $\mathbb{C}^n$.

For $(z,\tau) \in \mathbb{C}^g \times \mathbb{H}_g$ let

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}^g} \exp\big(\pi i({}^t n\tau n + 2\,{}^t n z)\big).$$

It is known (see for example [15, p. 118]) that the above series is convergent and ϑ is holomorphic on $\mathbb{C}^g \times \mathbb{H}_g$. It is also immediate from the definition that $\vartheta(z,\tau)$ is $\mathbb{Z}^g$-periodic in z and $(2\mathbb{Z})^n$-periodic in τ.

Definition 4.1. For $a, b \in \mathbb{R}^g$, the associated *Riemann Theta function* is the function $\vartheta\begin{bmatrix} a \\ b \end{bmatrix} \colon \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{C}$ defined by:

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \exp\big(\pi i({}^t a\tau a + 2\,{}^t a(z+b))\big)\vartheta(z+\tau a+b,\tau).$$

4.2. **Embeddings of abelian varieties.** We fix a polarization type $D = Diag(d_1,\dots,d_g)$. We also fix a set of representatives $\{c_0,\dots,c_N\}$ of the cosets of $\mathbb{Z}^g$ in the group $D^{-1}\mathbb{Z}^g$.

Remark 4.2. We could take $\{c_0,\dots,c_N\}$ to be the set of all vectors $c \in D^{-1}\mathbb{Z}^g$ whose components lie in the interval $[0,1)$. In particular we have $N = d_1 \cdot d_2 \cdot \cdots d_g - 1$.

The following fact is a consequence of the classical Lefschetz Theorem (see [15, Theorem 1.3 p.128]).

Fact 4.3 (Lefschetz Theorem)**.**
(a) Assume $d_1 \geq 2$.
The functions $\left\{\vartheta\begin{bmatrix} c_0 \\ 0 \end{bmatrix}(z,\tau),\dots,\vartheta\begin{bmatrix} c_N \\ 0 \end{bmatrix}(z,\tau)\right\}$ *have no zero in common, and*

$$\varphi^D(z,\tau) = \left(\vartheta\begin{bmatrix} c_0 \\ 0 \end{bmatrix}(z,\tau) \,:\, \vartheta\begin{bmatrix} c_1 \\ 0 \end{bmatrix}(z,\tau) : \,\dots\, : \vartheta\begin{bmatrix} c_N \\ 0 \end{bmatrix}(z,\tau)\right)$$

is a well-defined holomorphic map from $\mathbb{C}^g \times \mathbb{H}_g$ *into* $\mathbb{P}^N(\mathbb{C})$.
For each $\tau \in \mathbb{H}_g$ *the map* $\varphi^D_\tau \colon z \mapsto \varphi^D(z,\tau)$ *is* Λ^D_τ*-periodic, hence induces a holomorphic map from the abelian variety* $\mathcal{E}^D_\tau$ *into* $\mathbb{P}^N(\mathbb{C})$.

(b) Assume $d_1 \geq 3$.
For each $\tau \in \mathbb{H}_g$ the map φ^D_τ is an immersion on $\mathbb{C}^g$ that induces an analytic embedding of the abelian variety $\mathcal{E}^D_\tau$ into the projective space $\mathbb{P}^N(\mathbb{C})$, whose image is an algebraic variety.

¿From now on, in the case $d_1 \geq 2$, we will denote by $\varphi^D(z,\tau)$ the map

$$\varphi^D(z,\tau) = \left(\vartheta\begin{bmatrix}c_0\\0\end{bmatrix}(z,\tau) : \vartheta\begin{bmatrix}c_1\\0\end{bmatrix}(z,\tau) : \dots : \vartheta\begin{bmatrix}c_N\\0\end{bmatrix}(z,\tau)\right),$$

and for $\tau \in \mathbb{H}_g$, we will denote by $\varphi^D_\tau(z)$ the map $\varphi^D_\tau \colon z \mapsto \varphi^D(z,\tau)$ from $\mathbb{C}^g$ into $\mathbb{P}^N(\mathbb{C})$.

5. Definability of holomorphic $\mathbb{Z}^n$-invariant maps

The main result is in this section is a general theorem (see Theorem 5.8 below) about definability in the structure $\mathbb{R}_{an,exp}$ of certain periodic holomorphic functions on "truncated" tube domains. The theorem will then be applied to prove the definability of the theta functions on a restricted domain. We first review the basics of polyhedral cones and tube domains.

5.1. **Polyhedral cones and Tube domains.**

Notation 5.1. For vectors $v, u \in \mathbb{R}^m$ we write $v > u$ ($v \geq u$) if $v_i > u_i$ ($v_i \geq u_i$) for all $i = 1, \dots, m$.

Let $\mathbb{F}$ be the field of complex or real numbers, and let $L \colon \mathbb{F}^m \to \mathbb{F}^n$ be a linear map. We say that L is *integral* if the standard matrix of L (i.e. the matrix representing L with respect to the standard bases) has integer coefficients. Clearly, L is integral if and only if it maps $\mathbb{Z}^m$ into $\mathbb{Z}^n$.

An nonempty subset $C \subseteq \mathbb{R}^n$ is called *a polyhedral cone* if $C = \{x \in \mathbb{R}^n \colon Ax \geq 0\}$ for some $m \times n$-matrix A. The cone C is called *integral* if one can choose A in $M_{m\times n}(\mathbb{Z})$. The dimension $\dim(C)$ of a polyhedral cone C is the dimension of its linear span. We call $(\mathbb{R}^{\geq 0})^n$ *the standard polyhedral cone* in $\mathbb{R}^n$ and denote it by $\mathbb{R}_n$.

By the Minkowski-Weyl theorem (see for example [27, Theorem 1.3]), for every polyhedral cone $C \subseteq \mathbb{R}^n$, there are $v_1, \dots, v_k \in C$, such that

$$C = \left\{\sum \lambda_j v_j : \lambda_j \geq 0\right\}.$$

Moreover, since Fourier-Motzkin elimination used in the proof of [27, Theorem 1.3] works over $\mathbb{Q}$, if C is integral then $v_1, \dots, v_k$ can be chosen in $\mathbb{Z}^n$. Notice, that in this case $\dim(C)$ is the linear dimension of $\{v_1, \dots, v_k\}$.

Stated differently,

Fact 5.2. *If $C \subseteq \mathbb{R}^n$ is a polyhedral cone of dimension k then there is a linear map $L \colon \mathbb{R}^m \to \mathbb{R}^n$ of rank k such that C equals the image of $\mathbb{R}_m$ under L. Moreover, if C is integral then L can be chosen to be integral as well.*

Remark 5.3. In the above setting the relative interior of C (i.e. the interior of C in its linear span) is $L((\mathbb{R}^{>0})^m)$ (see for example [28, Proposition 2.1.12]).

Definition 5.4. For $S \subseteq \mathbb{R}^n$ the following complex subset is called *the tube domain associated to S*:

$$\mathcal{T}(S) = \{x + iy \in \mathbb{C}^n : y \in S\}.$$

Also, for a positive real $R > 0$ we will denote by $\mathcal{T}_{\leq R}(S)$ *the truncated tube domain*

$$\mathcal{T}_{\leq R}(S) = \{x + iy \in \mathbb{C}^n : \bigwedge_{i=1}^{n} |x_i| \leq R,\, y \in S\} =$$

$$\{z \in \mathcal{T}(S)\colon \bigwedge_{i=1}^{n} |Re(z_i)| \leq R\}.$$

Notation 5.5. For a linear map $L\colon \mathbb{R}^k \to \mathbb{R}^n$ we denote by $L_{\mathbb{C}}$ the complexification of L, i.e. the $\mathbb{C}$-linear map $L_{\mathbb{C}}\colon \mathbb{C}^k \to \mathbb{C}^n$ defined by $L_{\mathbb{C}}(x+iy) = L(x) + iL(y)$ for $x, y \in \mathbb{R}^k$.

Claim 5.6. *Let $S_1 \subseteq \mathbb{R}^m, S_2 \subseteq \mathbb{R}^n$ be sets and $L : \mathbb{R}^m \to \mathbb{R}^n$ a linear map with $L(S_1) = S_2$. If S_2 has non-empty interior then $L_{\mathbb{C}}$ maps $\mathcal{T}(S_1)$ onto $\mathcal{T}(S_2)$.*

Proof. Since the image of L contains an open subset of $\mathbb{R}^n$ its rank equals n, and therefore $L(\mathbb{R}^m) = \mathbb{R}^n$. The result now follows from the definition of a tube domain. □

5.2. **Definability of holomorphic $\mathbb{Z}^n$-invariant maps.** In this section we prove the following general theorem about definability in the structure $\mathbb{R}_{an,exp}$ of certain periodic holomorphic functions on truncated tube domains.

Notation 5.7. Let $C \subseteq \mathbb{R}^n$ be a subset.

(1) For a vector $v \in \mathbb{R}^n$, we denote by $C\langle v\rangle$ the translate of C by the vector $-v$, namely
$$C\langle v\rangle = C - v = \{x \in \mathbb{R}^n \colon x + v \in C\},$$
(2) For a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a real number $d \in \mathbb{R}$, we use the following notations:
$$C^{f>d} = \{x \in C\colon f(x) > d\}, \quad C^{f\geq d} = \{x \in C\colon f(x) \geq d\}.$$

In particular, $C\langle v\rangle^{f>d}$ denotes the set
$$C\langle v\rangle^{f>d} = \{x \in \mathbb{R}^n \colon x + v \in C, f(x) > d\}.$$

Theorem 5.8. *Let $C \subseteq \mathbb{R}^n$ be an integral polyhedral cone with $\dim(C) = n$, let $\ell : \mathbb{R}^n \to \mathbb{R}$ be an integral linear function which is positive on $Int(C)$, let $d_0 > 0$ be a positive real number, and $v \in Int(C)$.*

Let $U \supseteq \mathcal{T}\big(C\langle v\rangle^{\ell>d_0}\big)$ be an open subset of $\mathbb{C}^n$, and $\theta\colon U \to \mathbb{C}$ a holomorphic, $\mathbb{Z}^n$-periodic function bounded on $\mathcal{T}\big(C\langle v\rangle^{\ell>d_0}\big)$.

Then, for any real $d > d_0$, $u \in C$ with $u < v$, and for any real $R > 0$, the restriction of θ to the closed set $\mathcal{T}_{\leq R}\big(C\langle u\rangle^{\ell\geq d}\big)$ is definable in the structure $\mathbb{R}_{an,exp}$.

Moreover, there is a definable open $V \subseteq \mathbb{C}^n$ with $V \supseteq \mathcal{T}_{\leq R}\big(C\langle u\rangle^{\ell\geq d}\big)$, such that the restriction $\theta \restriction V$ is definable in $\mathbb{R}_{an,exp}$.

Proof. We fix $d > d_0 > 0$ and $u < v$ in C as in the theorem. We first consider a special case.

The special case: $C = \mathbb{R}_n$

Notice that because of the $\mathbb{Z}^n$-periodicity of θ, it is sufficient to prove the result for $R = 1/2$.

We use $\mathbf{e}(z)$ to denote the complex exponential function $\mathbf{e}(z) = e^{2\pi iz}$ and we use $\mathbf{e}_n : \mathbb{C}^n \to \mathbb{C}^n$ to denote the map which is $\mathbf{e}(z_i)$ in each coordinate. The map $\mathbf{e}_n$

is a surjective group homomorphism from $\langle \mathbb{C}^n, +\rangle$ to $\langle (\mathbb{C}^*)^n, \cdot\rangle$ which is locally a bi-holomorphism, with kernel $\mathbb{Z}^n$.

Since θ is $\mathbb{Z}$-periodic, it factors through the map $\mathbf{e}_n$, i.e. there is a holomorphic map $\theta^* \colon \mathbf{e}_n(U) \to \mathbb{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
 & U \arrow[dl, "\theta"'] \arrow[d, "\mathbf{e}_n"] \\
\mathbb{C} & \mathbf{e}_n(U) \arrow[l, "\theta^*"']
\end{tikzcd}$$

Since $d > d_0$ and $u < v$ we have $C\langle u\rangle^{\ell \geq d} \subseteq C\langle v\rangle^{\ell > d_0}$, hence $\mathcal{T}\big(C\langle u\rangle^{\ell\geq d}\big) \subseteq U$.

To simplify notation, let $T_v^{>d_0} = \mathcal{T}\big(C\langle v\rangle^{\ell>d_0}\big)$, $T_u^{\geq d} = \mathcal{T}_{\leq 1/2}\big(C\langle u\rangle^{\ell\geq d}\big)$, $\widetilde{T}_v^{>d_0} = \mathbf{e}_n(T_v^{>d_0})$, and $\widetilde{T}_u^{\geq d} = \mathbf{e}_n(T_u^{\geq d})$.

We have a commutative diagram where all vertical arrows are surjective maps.

$$\begin{tikzcd}
 & T_v^{>d_0} \arrow[dl, "\theta"'] \arrow[d, "\mathbf{e}_n"] & T_u^{\geq d} \arrow[l, hook', "\mathrm{id}"'] \arrow[d, "\mathbf{e}_n"] \\
\mathbb{C} & \widetilde{T}_v^{>d_0} \arrow[l, "\theta^*"'] & \widetilde{T}_u^{\geq d} \arrow[l, hook', "\mathrm{id}"']
\end{tikzcd}$$

Note that the restriction of $\mathbf{e}(z)$ to the set $\{z \in \mathbb{C} \colon |Re(z)| \leq 1/2\}$ is definable in $\mathbb{R}_{an,exp}$, since it only requires the functions $\exp(x) \colon \mathbb{R} \to \mathbb{R}$, $\sin(x) \restriction [-\pi, \pi]$, and $\cos(x) \restriction [-\pi, \pi]$. Hence, the restriction of $\mathbf{e}_n$ to $T_u^{\geq d}$ is definable in $\mathbb{R}_{an,exp}$ and it is sufficient to show that the restriction of θ^* to $\widetilde{T}_u^{\geq d}$ is definable in $\mathbb{R}_{an}$.

Let $v_1, \ldots, v_n \in \mathbb{R}$ be the components of v, i.e. $v = (v_1, \ldots, v_n)$, and $l_1, \ldots, l_n \in \mathbb{Z}$ be the components of ℓ, i.e. $\ell(x) = l_1x_1 + \cdots + l_nx_n$. Notice that since by assumptions ℓ is positive on $Int(\mathbb{R}_n)$, all l_i are positive integers. Let $u_1, \ldots u_n \in \mathbb{R}$ be the components of u. We have $0 \leq u_i < v_i$ for $i = 1, \ldots n$.

Recall that

$$T_v^{>d_0} = \{x + iy \in \mathbb{C}^n \colon \bigwedge_{i=1}^{n} y_i \geq -v_i, \quad l_1y_1 + \cdots + l_ny_n > d_0\} \text{ and}$$

$$T_u^{\geq d} = \{x + iy \in \mathbb{C}^n \colon \bigwedge_{i=1}^{n} |x_i| \leq 1/2, \bigwedge_{i=1}^{n} y_i \geq -u_i\,, \, l_1y_1 + \cdots + l_ny_n \geq d\}.$$

It is not hard to compute the images of $T_v^{>d_0}$ and $T_v^{>d_0}$ under $\mathbf{e}_n$ and obtain

$$\widetilde{T}_v^{>d_0} = \{q \in \mathbb{C}^n \colon \bigwedge_{i=1}^{k} 0 < |q_i| \leq e^{2\pi v_i},\, |q_1|^{l_1} \cdots |q_n|^{l_n} < e^{-2\pi d_0}\},$$

$$\widetilde{T}_u^{\geq d} = \{q \in \mathbb{C}^n \colon \bigwedge_{i=1}^{k} 0 < |q_i| \leq e^{2\pi u_i},\, |q_1|^{l_1} \cdots |q_n|^{l_n} \leq e^{-2\pi d}\}.$$

Let

$$O = \{z \in \mathbb{C}^n \colon \bigwedge_{i=1}^{n} |q_i| < e^{2\pi v_i},\, |q_1|^{l_1} \cdots |q_n|^{l_n} < e^{-2\pi d_0}\}$$

and

$$Z = \{q \in \mathbb{C}^n \colon \prod_{i=1}^{n} q_i = 0\}.$$

Obviously, O is an open subset of $\mathbb{C}^n$ and $O \setminus Z \subseteq \widetilde{T}_v^{>d_0}$. Since θ is bounded on $T_v^{>d_0}$, the function θ^* is bounded on $O \setminus Z$, and therefore, by Riemann's removable singularities theorem for complex functions of several variables, θ^* has holomorphic extension $\Theta^* \colon O \to \mathbb{C}$.

Consider the set

$$W = \{q \in \mathbb{C}^n \colon \bigwedge_{i=1}^{k} |q_i| \leq e^{2\pi u_i},\ |q_1|^{l_1} \cdots |q_n|^{l_n} \leq e^{-2\pi d}\}.$$

Obviously, W is a closed subset of $\mathbb{C}^n$, and, since all l_i are positive, it is also bounded, hence compact. Since $u_i < v_i$, and $d > d_0$, we have $W \subseteq O$. Clearly, W is a semialgebraic set, therefore the restriction of Θ^* to W is definable in $\mathbb{R}_{an}$.

The set $\widetilde{T}_u^{\geq d} = W \setminus Z$ is semialgebraic. Hence the restriction of Θ^* to $\widetilde{T}_u^{\geq d}$ is definable in $\mathbb{R}_{an}$. This finishes the proof of the special case.

General Case

Let $L \colon \mathbb{R}^k \to \mathbb{R}^n$ be an integral linear map, as in Claim 5.2, sending $\mathbb{R}_k$ onto the cone C. By Remark 5.3 there is $v' \in Int(\mathbb{R}_k)$ with $L(v') = v$. Since L is a linear map it maps $\mathbb{R}_k\langle v'\rangle$ onto $C\langle v\rangle$, and, by Claim 5.6, $L_{\mathbb{C}}$ maps $\mathcal{T}(\mathbb{R}_k)$ onto $\mathcal{T}(C)$ and $\mathcal{T}(\mathbb{R}_k\langle v'\rangle)$ onto $\mathcal{T}(C\langle v\rangle)$.

Let $\ell' = \ell \circ L$, $\theta' = \theta \circ L_{\mathbb{C}}$, and $U' = L_{\mathbb{C}}^{-1}(U)$.

Using Claim 5.6 again we obtain that $L_{\mathbb{C}}$ maps $\mathcal{T}(\mathbb{R}_k\langle v'\rangle^{\ell'>d_0})$ onto $\mathcal{T}(C\langle v\rangle^{\ell>d_0})$, hence U' contains $\mathcal{T}(\mathbb{R}_k\langle v'\rangle^{\ell'>d_0})$ and θ' is bounded on $\mathcal{T}(\mathbb{R}_k\langle v'\rangle^{\ell'>d_0})$.

Because θ' is $\mathbb{Z}^n$-periodic, we can now use the special case and obtain that the restriction of θ' to the set $\mathcal{T}_{\leq R'}(\mathbb{R}_k\langle u'\rangle^{\ell'\geq d})$ is definable in $\mathbb{R}_{an,exp}$. It follows then that the restriction of θ to $\mathcal{T}_{\leq R}(C\langle u\rangle^{\ell\geq d})$ is definable as well.

This finishes the proof of the main part of the theorem.

For the "moreover" part of the statement, we can take V to be the interior of the set $\mathcal{T}_{\leq R'}(C\langle u'\rangle^{\ell\geq d'})$ for any $R' > R$, $d_0 < d' < d$, and $u' \in C$ with $u < u' < v$.

This concludes the proof of of Theorem 5.8. □

6. Definability of theta functions

Our ultimate goal in this section is to show that the restrictions of Riemann Theta functions $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)$ to an appropriate sub-domain of $\mathbb{C}^g \times \mathbb{H}_g$ is definable in $\mathbb{R}_{an,exp}$. Towards this goal we need to establish the assumptions of Theorem 5.8 and mainly establish the boundedness of some variations of $\vartheta(z,\tau)$. We use ideas from similar boundedness proofs in [15].

We fix a positive integer g and let $n = g(g+1)/2$. We identify the set of all real symmetric $g \times g$-matrices $Sym(g,\mathbb{R})$ with $\mathbb{R}^n$. We will also identify $Sym(g,\mathbb{C})$ with $\mathbb{C}^n$ and with $Sym(g,\mathbb{R}) + iSym(g,\mathbb{R})$. In particular for a set $S \subseteq Sym(g,\mathbb{R})$ we view the corresponding cone $\mathcal{T}(S)$ as a subset of $Sym(g,\mathbb{C})$.

Definition 6.1. Let $\overline{\mathfrak{M}}_g \subseteq Sym(g,\mathbb{R})$ be the integral polyhedral cone given by $\beta_{1,1} \geq 0$, $\beta_{k,k+1} \geq 0, k = 1,\dots,g-1$, and the finitely many inequalities needed for the condition M(III) in Definition 3.7 (See Fact 3.9(d)).

It is immediate from the definition that

$$\mathfrak{M}_g = \overline{\mathfrak{M}}_g \cap \{\beta \in Sym(g,\mathbb{R})\colon \beta_{1,1} > 0\},$$

and the interior of $\overline{\mathfrak{M}}_g$ is contained in $\mathfrak{M}_g$. In particular, the dimension of $\overline{\mathfrak{M}}_g$ is n.

Notation 6.2. For a real $m > 0$, we denote by C_m the following integral polyhedral cone in $\mathbb{R}^g \times \mathbb{R}^n$:

$$C_m = \left\{ (y,\beta) \in \mathbb{R}^g \times Sym(g,\mathbb{R})\colon \beta \in \overline{\mathfrak{M}}_g\,,\ \bigwedge_{i=1}^{g} |y_i| \leq m\beta_{i,i} \right\}.$$

Let $\ell_{1,1}\colon \mathbb{R}^g \times \mathbb{R}^n \to \mathbb{R}$ be the integral linear function $\ell_{1,1}\colon (y,\beta) \mapsto \beta_{1,1}$.

Notice that for every $d > 0$ we have $\mathcal{T}(C_m^{\ell_{1,1}\geq d}) \subseteq \mathbb{C}^g \times \mathbb{H}_g$.

Before proving definability of $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}$ we need to establish the boundedness of an auxiliary function.

Proposition 6.3. *Let $m,d > 0$ be real numbers. Then there is a positive $k \in \mathbb{N}$ and $\beta^* \in Int(\overline{\mathfrak{M}}_g)$ such that for $\beta^*_0 = (0,\beta^*)$ the set $\mathcal{T}(C_m\langle\beta^*_0\rangle^{\ell_{1,1}>d})$ is contained in $\mathbb{C}^g \times \mathbb{H}_g$, and the function*

$$\theta(z,\tau) = \exp\bigl(2\pi i k\tau_{g,g}\bigr)\vartheta(z,2\tau)$$

is holomorphic and bounded on $\mathcal{T}(C_m\langle\beta^*_0\rangle^{\ell_{1,1}>d})$.

Proof. Recall that

$$\mathcal{T}(C_m\langle\beta^*_0\rangle^{\ell_{1,1}>d}) = \\ \{(z,\tau) \in \mathbb{C}^g \times \mathbb{C}^n\colon (Im(z), Im(\beta) + \beta^*) \in C_m,\ Im(\tau_{1,1}) > d\}.$$

Let c, c' be real constants as in Fact 3.9(a). Namely, for every real matrix $\beta \in \mathfrak{M}_g$ and for every $x \in \mathbb{R}^g$,

$$c\sum_{i=1}^{g} \beta_{i,\,i}x_i^2 \leq \beta[x] \leq c'\sum_{i=1}^{g} \beta_{i,\,i}x_i^2.$$

We take $k \in \mathbb{Z}$ and $\beta^* \in Int(\overline{\mathfrak{M}}_g)$ satisfying

$$k > \frac{m^2 g}{2c} \quad \text{and} \quad \beta^*_{g,g} < \frac{cd}{2c'}.$$

Notice that since $\overline{\mathfrak{M}}_g$ is a cone of dimension n such β^* exists.

Since $\beta^* \in Int(\overline{\mathfrak{M}}_g)$, we have $\beta^* \in \mathfrak{M}_g$, hence by Fact 3.9(c)

$$0 < \beta^*_{1,1} \leq \ldots \leq \beta^*_{i,i} \leq \ldots \leq \beta^*_{g,g} < \frac{cd}{2c'}.$$

Let $(z,\tau) \in \mathcal{T}(C_m\langle\beta^*_0\rangle^{\ell_{1,1}>d})$ with $z = x + iy$. For $\beta = Im(\tau) + \beta^*$ we have $(y,\beta) \in C_m$.

Since $(x+iy,\tau) \in \mathcal{T}(C_m\langle\beta^*_0\rangle^{\ell_{1,1}>d})$, we have $\beta_{1,1} - \beta^*_{1,1} > d$, hence $\beta_{1,1} > d$, $\beta \in \mathfrak{M}_g$, and, by Fact 3.9(c),

$$d < \beta_{1,1} \leq \ldots \leq \beta_{i,i} \leq \ldots \leq \beta_{g,g}.$$

For any $v \in \mathbb{R}^g$ we have

$$
\begin{aligned}
Im(\tau)[v] &= (\beta - \beta^*)[v] = \beta[v] - \beta^*[v] \\
&\geq c \sum \beta_{i,i} v_i^2 - c' \sum \beta^*_{i,i} v_i^2 \\
&= \sum (c\beta_{i,i} - c'\beta^*_{i,i}) v_i^2 \geq \sum \left(c\beta_{i,i} - c' \frac{cd}{2c'} \right) v_i^2 \\
&= \sum \left(c(\beta_{i,i} - \frac{d}{2}) \right) v_i^2 \geq \frac{c}{2} \sum \beta_{i,i} v_i^2.
\end{aligned}
\tag{3}
$$

Thus $Im(\tau)$ is a positive definite symmetric matrix, hence $\tau \in \mathbb{H}_g$ and $\mathcal{T}(C_m \langle \beta_0^* \rangle^{\ell_{1,1}>d})$ is contained in $\mathbb{C}^g \times \mathbb{H}_g$.

Since ϑ is holomorphic on $\mathbb{C}^g \times \mathbb{H}_g$, we obtain that θ is also holomorphic on $\mathcal{T}(C_m \langle \beta_0^* \rangle^{\ell_{1,1}>d})$.

To show the boundedness of θ on $\mathcal{T}(C_m \langle \beta_0^* \rangle^{\ell_{1,1}>d})$ we use ideas form [15, p. 118 Proposition 1.1.].

$$
\begin{aligned}
|\theta(z,\tau)| &= \big|\exp\big(2\pi i k \tau_{g,g}\big)\big| \cdot \Big| \sum_{n \in \mathbb{Z}^g} \exp\big(\pi i (2\tau[n] + 2\, {}^t n z)\big) \Big| \\
&\leq |\exp\big(2\pi i k \tau_{g,g}\big)| \cdot \sum_{n \in \mathbb{Z}^g} \big|\exp\big(\pi i (2\tau[n] + 2\, {}^t n z)\big)\big| \\
&= \exp\big(-2\pi k Im(\tau_{g,g})\big) \cdot \sum_{n \in \mathbb{Z}^g} \exp\big(-\pi(\, 2Im(\tau)[n] + \, 2^t n y\,)\big) \\
&= \sum_{n \in \mathbb{Z}^g} \exp\big(-\pi\big(\, 2Im(\tau)[n] + \, 2^t n y \, + 2k Im(\tau_{g,g})\,\big)\big).
\end{aligned}
\tag{4}
$$

Because $(y, \beta) \in C_m$ we have $|y_i| \leq m\beta_{i,i}$. Since $Im(\tau_{g,g}) = \beta_{g,g} - \beta^*_{g,g}$ and $\beta^*_{g,g} > 0$ we also have $Im(\tau_{g,g}) > \beta_{g,g}$. It follows from (3) that for $n \in \mathbb{Z}^g$

$$
\begin{aligned}
2Im(\tau)[n] + \, 2^t n y \, + 2k Im(\tau_{g,g}) &\geq \sum_{i=1}^{g} \big(c\beta_{i,i} n_i^2 - 2|n_i|\beta_{i,i} m \big) + 2k\beta_{g,g} \\
&= \sum_{i=1}^{g} \left(c\beta_{i,i} \left(|n_i| - \frac{m}{c} \right)^2 - \beta_{i,i} \frac{m^2}{c} \right) + 2k\beta_{g,g} \\
&\geq \sum_{i=1}^{g} \left(cd \left(|n_i| - \frac{m}{c} \right)^2 - \beta_{g,g} \frac{m^2}{c} \right) + 2k\beta_{g,g} \\
&= \sum_{i=1}^{g} cd \left(|n_i| - \frac{m}{c} \right)^2 + \left(2k\beta_{g,g} - g\beta_{g,g} \frac{m^2}{c} \right),
\end{aligned}
$$

and since $k > \frac{gm^2}{2c}$ we obtain

$$
2Im(\tau)[n] - \, 2^t n y \, + k Im(\tau_{g,g}) > \sum_{i=1}^{g} cd \left(|n_i| - \frac{m}{c} \right)^2 .
$$

The above inequality together with (4) imply

$$
\begin{aligned}
|\theta(z,\tau)| \leq \sum_{n\in\mathbb{Z}^g} \exp\big(-\pi \sum_{i=1}^{g} cd(|n_i| - m/c)^2\big) \\
= \sum_{n\in\mathbb{Z}^g} \prod_{i=1}^{g} \exp\big(-\pi cd(|n_i| - m/c)^2\big) \\
\leq \left(\sum_{n\in\mathbb{Z}} \exp\big(-\pi cd(|n| - m/c)^2\big) \right)^g .
\end{aligned}
$$

Since the series $\sum_{n\in\mathbb{N}} \exp\big(-\pi cd(|n|-m/c)^2\big)$ converges like $\int_{-\infty}^{+\infty} e^{-ax^2}\, dx$ and does not depend on (z,τ), the function $\theta(z,\tau)$ is bounded on $\mathcal{T}(C_m\langle\beta_0^*\rangle^{\ell_{1,1}>d})$. □

Proposition 6.4. *For all real numbers $m, d, R > 0$ there is an open set $W \subseteq \mathbb{C}^g \times \mathbb{C}^n$ containing the set $\mathcal{T}_{\leq R}(C_m^{\ell_{1,1}\geq d})$ such that the restriction of the function $\vartheta(z,\tau)$ to W is definable in $\mathbb{R}_{an,exp}$.*

Proof. Recall that

$$
\mathcal{T}_{\leq R}(C_m^{\ell_{1,1}\geq d}) = \left\{ (z,\tau) \in \mathcal{T}(C_m^{\ell_{1,1}\geq d})\colon \bigwedge_{i=1}^{g} \big|Re(z_i)\big| \leq R, \bigwedge_{i\leq i,j\leq g} \big|Re(\tau_{i,j})\big| \leq R \right\}.
$$

Let $m, d, R > 0$ be real numbers.

Since the function $\vartheta(z,\tau)$ is $\mathbb{Z}^g$-periodic in z and $(2\mathbb{Z})^n$-periodic in τ, for any $k \in \mathbb{N}$ the function $(z,\tau) \mapsto \exp\big(2\pi ik\tau_{g,g}\big)\vartheta(z, 2\tau)$ is $\mathbb{Z}^g \times \mathbb{Z}^n$-periodic.

Let $k \in \mathbb{N}$ and $\beta^* \in Int(\overline{\mathfrak{M}}_g)$ be as in Proposition 6.3. We are going to apply Theorem 5.8 to the function

$$
\theta(z,\tau) = \exp\big(2\pi ik\tau_{g,g}\big)\vartheta(z, 2\tau).
$$

The set $U = \mathbb{C}^g \times \mathbb{H}_g$ is an open subset of $\mathbb{C}^g \times \mathbb{C}^n$, and $\theta(z,\tau)$ is holomorphic and $\mathbb{Z}^g \times \mathbb{Z}^n$-periodic on U. The integral cone we take is C_m and the linear function $\ell_{1,1}(y,\beta) = \beta_{1,1}$. For $\beta_0^* = (0, \beta^*)$, by Proposition 6.3, $\mathcal{T}(C_m\langle\beta_0^*\rangle^{\ell>d}) \subseteq U$ and the function $\theta(z,\tau)$ is bounded on $\mathcal{T}(C_m\langle\beta_0^*\rangle^{\ell_{1,1}>d})$. Also, since $\beta^* \in Int(\overline{\mathfrak{M}}_g)$, it follows from the definition of C_m that $\beta_0^* \in Int(C_m)$.

Thus all assumptions of Theorem 5.8 hold, and there is a definable open $V \subseteq \mathbb{C}^g \times \mathbb{C}^n$ containing $\mathcal{T}_{\leq R}(C_m\langle\beta_0^*\rangle)^{\ell_{1,1}\geq 2d}$ such that the restriction of $\theta(z,\tau)$ to V is definable in $\mathbb{R}_{an,exp}$. Since C_m is a cone and $\beta_0^* \in C_m$, we have $C_m\langle\beta_0^*\rangle \subseteq C_m$, hence $\mathcal{T}_{\leq R}(C_m^{\ell_{1,1}\geq 2d}) \subseteq V$.

Intersecting V with the set $\{(z,\tau) \in \mathbb{C}^g \times \mathbb{C}^n\colon |Re(\tau_{g,g})| < 2R\}$ if needed, we can assume that $|Re(\tau_{g,g})| < 2R$ on V.

It is easy to see that the restriction of the function $v \mapsto \exp\big(2\pi ikv\big)$ to the set $\{v \in \mathbb{C}\colon |Re(v)| < 2R\}$ is definable in the structure $\mathbb{R}_{an,exp}$, hence the restriction of the function $(z,\tau) \mapsto \vartheta(z, 2\tau)$ to V is also definable.

It is immediate that the restriction of the function $\vartheta(z,\tau)$ to the set $W = \{(z,\tau)\colon (z, \frac{1}{2}\tau) \in V\}$ is definable in $\mathbb{R}_{an,exp}$, and it is not hard to see that W contains the set $\mathcal{T}_{\leq 1/2R}(C_{1/2m}^{\ell_{1,1}\geq 4d})$.

Since m, d, R were arbitrary, the proposition follows. □

Theorem 6.5. *For all $a, b \in \mathbb{R}^g$ and all real numbers $m, d, R > 0$ there is an open set $U \subseteq \mathbb{C}^g \times \mathbb{C}^n$ containing $\mathcal{T}_{\leq R}\big(C_m^{\ell_{1,1} \geq d}\big)$ such that the restriction of the function $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)$ to U is definable in $\mathbb{R}_{an,exp}$.*

Proof. We fix $a, b \in \mathbb{R}^g$ and $m, d, R > 0$. Recall that

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \exp\big(\pi i({}^t a\tau a + 2\,{}^t a(z+b))\big)\vartheta(z + \tau a + b, \tau).$$

For $u \in \mathbb{R}^g$ we denote by $\|u\|_s$ the sup-norm of u, namely $\|u\|_s = \sup\{|u_i| \colon i = 1, \ldots, g\}$.

Let $f_1 \colon \mathbb{C}^g \times \mathbb{C}^n \to \mathbb{C}$ be the map

$$f_1(z, \tau) = {}^t a\tau a + 2\,{}^t a(z+b).$$

If $(z, \tau) \in \mathcal{T}_{\leq R}\big(C_m^{\ell_{1,1} \geq d}\big)$ then $\|Re(z)\|_s \leq R$ and $|Re(\tau_{i,j})| \leq R$, hence

$$|Re(f_1(z,\tau))| = \left| \sum_{1 \leq i,j \leq g} Re(\tau_{i,j}) a_i a_j + 2\sum_{i=1}^{g} a_i(Re(z_i) + b_i) \right| \leq$$
$$g^2 R\|a\|_s^2 + 2g\|a\|_s(R + \|b\|_s).$$

Hence there is $R_1 > 0$ such that the set $V_1 = \{w \in \mathbb{C} \colon |Re(w)| < R_1\}$ contains the image of $\mathcal{T}_{\leq R}\big(C_m^{\ell_{1,1} \geq d}\big)$ under f_1.

It follows then that the restriction of the function $w \mapsto \exp\big(\pi i w\big)$ to the set V_1 is definable in $\mathbb{R}_{an,exp}$. Let $U_1 = f_1^{-1}(V_1)$. Clearly U_1 is open in $\mathbb{C}^g \times \mathbb{C}^n$, it contains $\mathcal{T}_{\leq R}\big(C_m^{\ell_{1,1} \geq d}\big)$ and the restriction of the function

$$(z, \tau) \mapsto \exp\big(\pi i({}^t a\tau a + 2\,{}^t a(z+b))\big)$$

to U_1 is definable in $\mathbb{R}_{an,exp}$.

Let $f_2 \colon \mathbb{C}^g \times \mathbb{C}^n \to \mathbb{C}^g$ be the map $f_2(z,\tau) = z + \tau a + b$. If $(z, \tau) \in \mathcal{T}_{\leq R}\big(C_m^{\ell_{1,1} \geq d}\big)$ and $w = f_2(z, \tau)$ then for $i = 1, \ldots g$ we have

$$|Re(w_i)| \leq R + gR\|a\|_s + \|b\|_s,$$

and

$$|Im(w_i)| \leq |Im(z_i)| + \sum_{j=1}^{g} |Im(\tau_{i,j})| \cdot \|a\|_s.$$

Applying Fact 3.9(b) and the definition of C_m (see 6.2), we obtain

$$|Im(w_i)| \leq (m + g\|a\|_s) Im(\tau_{i,i}).$$

Let $R_2 = R + gR\|a\|_s + \|b\|_s$ and $m_2 = m + g\|a\|_s$. It is immediate that the image of $\mathcal{T}_{\leq R}\big(C_m^{\ell_{1,1} \geq d}\big)$ under the map $(z, \tau) \mapsto (f_2(z,\tau), \tau)$ is contained in the set $\mathcal{T}_{\leq R_2}\big(C_{m_2}^{\ell_{1,1} \geq d}\big)$. Applying Proposition 6.4 we can find a definable open set $U_2 \subseteq \mathbb{C}^g \times C^n$ such that the restriction of the function $\vartheta(z + \tau a + b, \tau)$ to U_2 is definable in $\mathbb{R}_{an,exp}$. We can take $U = U_1 \cap U_2$. $\square$

7. Definability of complex tori and their embeddings

Notation 7.1. For a $g \times g$-matrix M and $1 \le j \le g$ we will denote by $M_{(j)}$ the j-th column of M.

Notation 7.2. Let D be a polarization type. For $\tau \in \mathbb{H}_g$ we will denote by $E^D_\tau \subseteq \mathbb{C}^g$ the fundamental parallelogram of the lattice Λ^D_τ:

$$E^D_\tau = \left\{ \sum_{i=1}^{g} t_i \tau_{(i)} + \sum_{j=1}^{g} s_j D_{(j)} \colon 0 \le t_i, s_j < 1 \right\},$$

and by $\bar{E}^D_\tau$ its closure in $\mathbb{C}^g$

$$\bar{E}^D_\tau = \left\{ \sum_{i=1}^{g} t_i \tau_{(i)} + \sum_{j=1}^{g} s_j D_{(j)} \colon 0 \le t_i, s_j \le 1 \right\}.$$

Clearly E^D_τ contains a unique representative for every Λ^D_τ-coset, and we will identify the underlying set of $\mathcal{E}^D_\tau$ with E^D_τ. Using this identification it is not hard to equip each $\mathcal{E}^D_\tau$, uniformly in τ, with a semialgebraic $\mathbb{C}$-manifold structure in the sense of [16, Section 2.2] (see also Definition 9.1 in Appendix), and obtain a semialgebraic family of complex-analytic manifolds $\{\mathcal{E}^D_\tau \colon \tau \in \mathbb{H}_q\}$.

Our main goal in this section is to show that in the case $D_{1,1} \ge 2$, for a sufficiently large set $F \subseteq \mathbb{H}_g$, the restriction of φ^D to the set $\{(z,\tau) \in \mathbb{C}^g \times \mathbb{H}_g \colon \tau \in F,\, z \in \bar{E}^D_\tau\}$ is definable in $\mathbb{R}_{an,exp}$, and in particular the family of maps

$$\varphi^D_\tau \colon \mathcal{E}^D_\tau \to \mathbb{P}^N(\mathbb{C}),\ \tau \in F$$

is definable in $\mathbb{R}_{an,exp}$.

As in the previous section we fix a positive integer g and let $n = g(g+1)/2$. We also fix a polarization type $D = Diag(d_1, \dots, d_g)$.

Notation 7.3. Let V be a subset of $\mathbb{H}_g$.

(1) We will denote by $\mathcal{X}^D(V)$ the following subset of $\mathbb{C}^g \times \mathbb{H}_g$:

$$\mathcal{X}^D(V) = \{(z,\tau) \in \mathbb{C}^g \times \mathbb{H}_g \colon \tau \in V,\, z \in \bar{E}^D_\tau\}.$$

(2) For a real $K > 0$ we will denote by $\mathcal{X}^D_{<K}(V)$ the following subset of $\mathbb{C}^g \times \mathbb{H}_g$:

$$\mathcal{X}^D_{<K}(V) = \big\{(z,\tau) \in \mathbb{C}^g \times \mathbb{H}_g \colon \tau \in V,\, z = \tau r + D r' \\ \text{for some } r, r' \in \mathbb{R}^g \text{ with } \|r\|_s < K, \|r'\|_s < K\big\}.$$

Remark 7.4. Note that for $K > 1$, we have $\mathcal{X}^D(V) \subseteq \mathcal{X}^D_{<K}(V)$.

It is not hard to see that for a closed subset $V \subseteq \mathbb{H}_g$ the set $\mathcal{X}^D(V)$ is closed in $\mathbb{C}^g \times \mathbb{H}_g$, and if $V \subseteq \mathbb{H}_g$ is open then for any $K > 0$ the set $\mathcal{X}^D_{<K}(V)$ is an open subset of $\mathbb{C}^g \times \mathbb{H}_g$.

7.1. Definability of theta functions over the Siegel fundamental domain.

We first show that restrictions of theta functions to the set $\mathcal{X}^D(\mathfrak{F}_g)$ is definable in $\mathbb{R}_{an,exp}$.

Claim 7.5. *Let $D = Diag(d_1, \dots, d_g)$ be a polarization type. For any real $K > 0$ there are $d, m, R > 0$ such that $\mathcal{X}^D_{<K}(\mathfrak{F}_g) \subseteq \mathcal{T}_{\le R}\big(C^{\ell_{1,1} \ge d}_m\big)$.*

Proof. Let $(z,\tau) \in \mathcal{X}^D_{<K}(\mathfrak{F}_g)$. Then $\tau = \alpha + i\beta \in \mathfrak{F}_g$ and $z = \tau r + Dr'$ for some $r, r' \in \mathbb{R}^g$ with $\|r\|_s < K$ and $\|r'\|_s < K$. If $x = Re(z)$ and $y = Im(z)$ then $x = \alpha r + Dr'$ and $y = \beta r$, with $\beta \in \mathfrak{M}_g$.

For each $i = 1, \ldots, g$, we have $|y_i| \le \sum_{j=1}^g |\beta_{i,j}|\,|r_j|$, hence, by Fact 3.9(b), $|y_i| < \frac{1}{2} gK\beta_{i,i}$. Thus $(y,\beta) \in C_m$ for $m = \frac{1}{2}gK$ and, by Fact 3.11(b), $(z,\tau) \in \mathcal{T}(C_m^{\ell_{1,1}\ge d})$ for $d = \frac{\sqrt{3}}{2}$.

For $i = 1, \ldots, g$ we also have $|x_i| \le \sum_{j=1}^g |\alpha_{i,j}||r_j| + d_i|r_j|'$. Since $d_1 \le d_2 \le \ldots \le d_g$ and $\tau \in \mathfrak{F}_g$, it follows that $|x_i| < \frac{1}{2}gK + d_gK$, and we can take $R = \frac{1}{2}gK + d_gK$. □

Applying Theorem 6.5 and using Remark 7.4 we obtain the following two corollaries.

Corollary 7.6. *For any $a, b \in \mathbb{R}^g$ and $K > 0$ there is an open set $U \subseteq \mathbb{C}^g \times \mathbb{H}_g$ containing $\mathcal{X}^D_{<K}(\mathfrak{F}_g)$ such that the restriction of the function $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)$ to U is definable in $\mathbb{R}_{an,exp}$.*

Corollary 7.7. *For every $a, b \in \mathbb{R}^g$ there is an open set $U \subseteq \mathbb{C}^g \times \mathbb{H}_g$ containing $\mathcal{X}^D(\mathfrak{F}_g)$ such that the restriction of $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)$ to U is definable in $\mathbb{R}_{an,exp}$.*

7.2. **Definability of the map $\varphi^{\mathbf{D}}(\mathbf{z},\tau)$.** Our next goal is to show that if $d_1 \ge 2$ and $G \leqslant Sp(2g,\mathbb{Z})$ has finite index then the restrictions of the map $\varphi^D(z,\tau)$ to sets $\mathcal{X}^D(\mathfrak{F}_g(G))$ is definable in $\mathbb{R}_{an,exp}$.

We need the following auxiliary claim.

Claim 7.8. *Let $K > 0$, and $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp(2g,\mathbb{R})$. For every $a^1, b^1 \in \mathbb{R}^g$ there is an open set $U \subseteq \mathbb{C}^g \times \mathbb{H}_g$ containing the set $\mathcal{X}^D_{<K}(M \cdot \mathfrak{F}_g)$ such that the restriction of the function*

$$f^{a_1}_{b_1}(z,\tau) = \exp\big(-\pi i({}^t z\gamma\,{}^t\big(\gamma(M^{-1}\cdot\tau)+\delta\big)z)\big)\cdot\vartheta\begin{bmatrix} a^1 \\ b^1 \end{bmatrix}(z,\tau)$$

to U is definable in $\mathbb{R}_{an,exp}$.

Proof. We will use the following fact that follows from the theta transformation formula (see [3, p. 221 and 8.6.1 on p. 227]).

Fact 7.9 (The theta transformation formula)**.** *Assume M, a^1, b^1 are as in the above claim. Then there are $a, b \in \mathbb{R}^g$ and $k, k_1 \in \mathbb{C}$ such that for all $z \in \mathbb{C}^g$ and $\tau_1 \in \mathbb{H}_g$ we have*

$$\vartheta\begin{bmatrix} a^1 \\ b^1 \end{bmatrix}(z, M\cdot\tau_1) =$$
$$k\sqrt{det(\gamma\tau_1+\delta)}\exp\big(\pi i(k_1 + {}^t z\gamma\,{}^t(\gamma\tau_1+\delta)z)\big)\vartheta\begin{bmatrix} a \\ b \end{bmatrix}({}^t(\gamma\tau_1+\delta)z, \tau_1).$$

We apply the theta transformation formula with $\tau = M\cdot\tau_1$ and obtain

$$f^{a_1}_{b_1}(z,\tau) = k\sqrt{det(\gamma\tau_1+\delta)}\exp\big(\pi i k_1\big)\vartheta\begin{bmatrix} a \\ b \end{bmatrix}({}^t(\gamma\tau_1+\delta)z, \tau_1).$$

Since k and k_1 are constants, and the function $\tau \mapsto \sqrt{det(\gamma\tau+\delta)}$ is semialgebraic, it suffices to find an open $U \subseteq \mathbb{C}^g \times \mathbb{H}_g$ containing $\mathcal{X}^D_{<K}(M\cdot\mathfrak{F}_g)$ such that the restriction of the function

$$(z,\tau) \mapsto \vartheta\begin{bmatrix} a \\ b \end{bmatrix}({}^t(\gamma(M^{-1}\cdot\tau)+\delta)z, M^{-1}\cdot\tau)$$

to U is definable in $\mathbb{R}_{an,exp}$.

Clearly the map $g(z,\tau) = ({}^t(\gamma(M^{-1}\cdot\tau)+\delta)z, M^{-1}\cdot\tau)$ is semialgebraic and holomorphic. Thus it is sufficient to show that there is an open set $V \subseteq \mathbb{C}^g \times \mathbb{H}_g$ containing the image of $\mathcal{X}^D_{<K}(M\cdot\mathfrak{F}_g)$ under g such that the restriction of the function $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)$ to V is definable in $\mathbb{R}_{an,exp}$.

Let $(z,\tau) \in \mathcal{X}^D_{<K}(M\cdot\mathfrak{F}_g)$ and $(z_1,\tau_1) = g(z,\tau)$. It is immediate that $\tau_1 \in \mathfrak{F}_g$, $\tau = M\cdot\tau_1$ and $z_1 = {}^t(\gamma\cdot\tau_1+\delta)z$.

Since $(z,\tau) \in \mathcal{X}^D_{<K}(M\cdot\mathfrak{F}_g)$, there are $r,r' \in \mathbb{R}^g$, with $\|r\|_s < K$, $\|r'\|_s < K$, such that $z = \tau r + Dr'$.

Let $r_1, r_1' \in \mathbb{R}^g$ be such that $z_1 = \tau_1 r_1 + Dr_1'$. It follows from Fact 3.5 by direct computations (see also Remark 3.6) that

$$\begin{pmatrix} r_1 \\ r_1' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D^{-1} \end{pmatrix} {}^tM \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} r \\ r' \end{pmatrix}$$

or equivalently $\begin{pmatrix} r_1 \\ Dr_1' \end{pmatrix} = {}^tM \begin{pmatrix} r \\ Dr' \end{pmatrix}$.

Thus $\|r_1\|_s < K\cdot\|M\|_s\cdot\|D\|_s$ and $\|r_1'\|_s < K\cdot\|M\|_s\cdot\|D\|_s$. It follows that for $K' = K\cdot\|M\|_s\cdot\|D\|_s$, the image of $\mathcal{X}^D_{<K}(M\cdot\mathfrak{F}_g)$ under g is contained in the set $\mathcal{X}^D_{<K'}(\mathfrak{F}_g)$. By Corollary 7.6 there is an open set V as needed. □

Corollary 7.10. *Fix a polarization type $D = diag(d_1,\dots,d_g)$ and assume $d_1 \geq 2$. Then*

(1) For every $M \in Sp(2g,\mathbb{R})$ there is an open $U \subseteq \mathbb{C}^g \times \mathbb{H}_g$ containing $\mathcal{X}^D(M\cdot\mathfrak{F}_g)$ such that the restriction of $\varphi^D(z,\tau)$ to U is definable in the structure $\mathbb{R}_{an,exp}$.

(2) Let $G < Sp(2g,\mathbb{Z})$ be a subgroup of finite index and fix $\mathfrak{F}_g(G)$ a Siegel fundamental set for the action of G on $\mathbb{H}_g$. Then there is an open set $U \subseteq \mathbb{C}^g \times \mathbb{H}_g$ containing $\mathcal{X}^D(\mathfrak{F}_g(G))$ such that the restriction of $\varphi^D(z,\tau)$ to U is definable in the structure $\mathbb{R}_{an,exp}$.

Proof. (1) Since $d_1 \geq 2$, by Fact 4.3, the map

$$\varphi^D(z,\tau) = \left(\vartheta\begin{bmatrix} c_0 \\ 0 \end{bmatrix}(z,\tau) : \vartheta\begin{bmatrix} c_1 \\ 0 \end{bmatrix}(z,\tau) : \dots : \vartheta\begin{bmatrix} c_N \\ 0 \end{bmatrix}(z,\tau)\right)$$

is well defined. Since each $f_a^b(z,\tau)$ is a product if $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)$ and a function that does not depend on a,b, we have

$$\left(\vartheta\begin{bmatrix} c_0 \\ 0 \end{bmatrix}(z,\tau) : \vartheta\begin{bmatrix} c_1 \\ 0 \end{bmatrix}(z,\tau) : \dots : \vartheta\begin{bmatrix} c_N \\ 0 \end{bmatrix}(z,\tau)\right) = \\ (f_0^{c_0}(z,\tau) : f_0^{c_1}(z,\tau) : \dots : f_0^{c_N}(z,\tau)),$$

hence $\varphi^D(z,\tau) = (f_0^{c_0}(z,\tau) : f_0^{c_1}(z,\tau) : \ldots : f_0^{c_N}(z,\tau))$. We now use Claim 7.8 and Remark 7.4 to get the required set U.

To see (2) just note that $\mathfrak{F}_g(G)$ is given by finitely many $Sp(2g,\mathbb{Z})$-translates of $\mathfrak{F}_g$ so we can apply (1). □

8. Projective embeddings of some classical families and moduli spaces

In this section we use our previous results to establish the definability of embeddings of some classical families of abelian varieties and their moduli spaces into projective space.

For the rest of this section we fix a polarization type

$$D = Diag(d_1, \ldots, d_g).$$

8.1. Modular forms and the moduli space of principally polarized abelian varieties.

Definition 8.1. Let Γ be a finite index subgroup of $Sp(2g,\mathbb{Z})$. A holomorphic function $\phi : \mathbb{H}_g \to \mathbb{C}$ is called *a modular form of weight k with respect to* Γ, if for every $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ and for every $\tau \in \mathbb{H}_g$,

$$\phi(M \cdot \tau) = det(\gamma\tau + \delta)^k \phi(\tau).$$

Clearly, if $\Gamma_1 \leqslant \Gamma_2 \leqslant Sp(2g,\mathbb{Z})$ then every modular form with respect to Γ_2 is also a modular form with respect to Γ_1. For any integer $k \geq 1$ we let

$$\Gamma_g(k) = \{M \in Sp(2g,\mathbb{Z})\colon M \equiv I_{2g} \mod k\}.$$

Using the definability of the theta functions we can now prove:

Theorem 8.2. *Let $\Gamma \leqslant Sp(2g,\mathbb{Z})$ be a congruence subgroup of $Sp(2g,\mathbb{Z})$, namely a subgroup which contains some $\Gamma_g(n)$. If ϕ is a modular form with respect to Γ then there is a definable open $U \subseteq \mathbb{H}_g$ containing $\mathfrak{F}_g$ such that $\phi \restriction U$ is definable in $\mathbb{R}_{an,exp}$.*

Proof. By Igusa's work [12, Corollary, p. 235] (see also [4, Theorem III.6.2]), each such modular form with respect to the congruence subgroup $\Gamma_g(2k)$ is algebraic over a ring generated by monomials in the so-called theta constants, namely a ring generated by products $\vartheta\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}(0,\tau)\cdots\vartheta\begin{bmatrix} a_k \\ b_k \end{bmatrix}(0,\tau)$, for certain $a_i, b_i \in \mathbb{Q}^g$. Since $\Gamma_g(2n) \leqslant \Gamma_g(n) \leqslant \Gamma$, it follows that every modular form with respect to Γ is algebraic over the same ring. By Corollary 7.6, the restriction of each $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(0,\tau)$ to some open set containing $\mathfrak{F}_g$ is definable in $\mathbb{R}_{an,exp}$. It is thus sufficient to note that every function in the algebraic closure of the ring generated by the theta functions is definable on some open set containing $\mathfrak{F}_g$.

Assume that $h(\tau)$ is the zero of a polynomial over the ring of theta constants. So $h(\tau)$ is a zero of a polynomial

$$p_\tau(w) = \vartheta_n(\tau)w^n + \cdots + \vartheta_1(\tau)w + \vartheta_0(\tau),$$

where each ϑ_i is a polynomial in theta constants. We find a definable set $U \subseteq \mathbb{H}_g$ containing $\mathfrak{F}_g$ on which all ϑ_i's are definable and consider the set

$$H = \{(\tau, w) \in U \times \mathbb{C} : p_\tau(w) = 0\},$$

which is definable in $\mathbb{R}_{an,exp}$. Using o-minimal cell decomposition, we can write U as the union of finitely many definable open simply connected sets and a set of smaller dimension such that on each of these open sets the set H is the union of finitely many graphs of holomorphic functions. Now, on each of the open sets the graph of h equals one of the branches of this cover, hence definable. The rest of the graph of h is obtained by taking topological closure. Thus, $h \restriction U$ is definable in $\mathbb{R}_{an,exp}$. □

By the work of Baily and Borel, [2, Theorem 10.11] (see also [10, Theorem 7]), the quotient $Sp(2g,\mathbb{Z})\backslash\mathbb{H}_g$ can be embedded into projective space using Siegel modular forms as coordinate functions. Namely, there is a holomorphic map $F : \mathbb{H}_g \to \mathbb{P}^N(\mathbb{C})$, given in coordinates by modular forms with respect to $Sp(2g,\mathbb{Z})$, which induces an embedding of $Sp(2g,\mathbb{Z})\backslash\mathbb{H}_g$ into $\mathbb{P}^N(\mathbb{C})$. Moreover, its image is Zariski open in some algebraic variety. Taken together with Theorem 8.2 we can conclude the definability in $\mathbb{R}_{an,exp}$ of a restricted F.

Theorem 8.3. *There is a definable open set $U \subseteq \mathbb{H}_g$ containing $\mathfrak{F}_g$ such that $F \restriction U$ is definable in $\mathbb{R}_{an,exp}$.*

More explicitly, the function F satisfies: for every $\tau, \tau' \in U$, $F(\tau) = F(\tau')$ if and only if τ and τ' are in the same $Sp(2g,\mathbb{Z})$-orbit, and $F(U)$ is a quasi-projective subset of $\mathbb{P}^N(\mathbb{C})$.

8.2. **Embeddings of families.** We consider here the uniform definability of the embedding of abelian varieties into projective space.

The following is immediate from Corollary 7.10.

Corollary 8.4. *Let $G < Sp(2g,\mathbb{Z})$ be a subgroup of finite index and fix $\mathfrak{F}_g(G) \subseteq \mathbb{H}_g$, a Siegel fundamental set for G. If $d_1 \geq 2$ then the family of maps*

$$\{\varphi^D_\tau \colon \mathcal{E}^D_\tau \to \mathbb{P}^N(\mathbb{C}) \colon \tau \in \mathfrak{F}_g(G)\}$$

is definable in the structure $\mathbb{R}_{an,exp}$. In particular, when $d_1 \geq 3$ (see Fact 4.3), the family of projective abelian varieties $\{\varphi^D(\mathcal{E}^D_\tau) \colon \tau \in \mathfrak{F}_g(G)\}$ is definable.

As the next theorem shows, we can omit the restriction on D and still obtain a uniformly definable family of emebeddings of polarized abelian varieties of type D.

Theorem 8.5. *For any polarization type D there is in $\mathbb{R}_{an,exp}$ a definable set $S \subseteq \mathbb{H}_g$, containing a representative for every G_D-orbit, and a definable embedding of the family $\{\mathcal{E}^D_\tau \colon \tau \in S\}$ into projective space. More precisely, for some N there is a definable family of embeddings $\{h_\tau : \mathcal{E}^D_\tau \to \mathbb{P}^N(\mathbb{C}) : \tau \in S\}$.*

Proof. If $d_1 \geq 3$ then the theorem follows from Corollary 8.4 with $h_\tau = \varphi^D_\tau$.

Assume $d_1 < 3$ and let $G = G_D \cap Sp(2g,\mathbb{Z})$. It is known (see Fact 8.7 below) that G has finite index in both groups, so we fix a Siegel fundamental set $\mathfrak{F}_g(G)$ for G. For any $\tau \in \mathbb{H}_g$ the map $z \mapsto 3z$ is an isomorphism from the torus $\mathcal{E}^D_\tau$ onto the torus $\mathcal{E}^{3D}_{3\tau}$, hence we have in $\mathbb{R}_{an,exp}$ a definable embedding of the family $\{\mathcal{E}^D_\tau \colon \tau \in \mathfrak{F}_g(G)\}$ into the family $\{\mathcal{E}^{3D}_{3\tau} \colon \tau \in \mathfrak{F}_g(G)\}$.

Let $\gamma_1, \ldots, \gamma_k \in Sp(2g,\mathbb{Z})$ be such that $\mathfrak{F}_g(G) = \cup \gamma_i \cdot \mathfrak{F}_g$. Let $\tau \in \gamma_i \cdot \mathfrak{F}_g$ and let $\tau_1 = \gamma_i^{-1} \cdot \tau$. Obviously $\tau_1 \in \mathfrak{F}_g$ and $3\tau = \gamma_i \cdot (3\tau_1)$. Since $\tau_1 \in \mathfrak{F}_g$, then it is easy to see that $3\tau_1$ satisfies the conditions (i) and (iii) from the definition of $\mathfrak{F}_g$, and $3\tau_1 \in \mathfrak{F}_g + \beta$ for some $\beta \in M_g(\mathbb{Z})$ with $\|\beta\|_s \leq 2$. By the definition of the action of $Sp(2g,\mathbb{Z})$ on $\mathbb{H}_g$, for any $\beta \in M_g(\mathbb{Z})$ we have $\mathfrak{F}_g + \beta = M_\beta \cdot \mathfrak{F}_g$, where $M_\beta = \begin{pmatrix} I_g & \beta \\ 0 & I_g \end{pmatrix} \in Sp(2g,\mathbb{Z})$.

Thus for every $i = 1, \dots, k$ the family $\{\mathcal{E}^{3D}_{3\tau} \colon \tau \in \gamma_i \cdot \mathfrak{F}_g\}$ is contained in the family $\{\mathcal{E}^{3D}_{\tau} \colon \tau \in S_i\}$, where S_i is the finite union

$$S_i = \bigcup \bigl\{\gamma_i \cdot M_\beta \cdot \mathfrak{F}_g \colon \beta \in M_g(\mathbb{Z}) \text{ with } \|\beta\|_s \le 2\bigr\}.$$

We can now use Corollary 7.10(1) (and Fact 4.3(b)) to uniformly embed each $\{\mathcal{E}^{3D}_{\tau} \colon \tau \in S_i\}$ into a projective space, definably in the structure $\mathbb{R}_{an,exp}$. □

8.3. **On embeddings of moduli spaces and universal families.** Note that as a corollary of Theorem 8.5 we obtain, for every g and D, a definable family $\{h_\tau(\mathcal{E}^D_\tau) : \tau \in \mathfrak{F}_g(G_D)\}$ of g-dimensional projective abelian varieties with polarization type D. Moreover, since τ varies over a fundamental set for G_D, the family contains a representative for *every* g-dimensional polarized abelian variety of type D. However, we do not claim that every such variety appears exactly once in the family. In order to obtain such a family we need to work with certain subgroups of G_D.

Definition 8.6 (see [3, p. 233 (4) and p. 235, Lemma 8.9.1])**.** For a polarization type D let $G_D(D)_0$ be the collection of all $2g \times 2g$ matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ in G_D for which the following hold:

(i) There are $a, b, c, d \in M_g(\mathbb{Z})$ such that $\alpha = I_g + Da$, $\beta = DbD$, $\gamma = c$ and $\delta = I_g + dD$.

(ii) The diagonal elements of the matrices $D^{-1}\alpha\, {}^t\beta D^{-1}$ and $\gamma\, {}^t\delta$ are even integers.

By (i), $G_D(D)_0 < Sp(2g, \mathbb{Q}) \cap M_g(\mathbb{Z}) = Sp(2g, \mathbb{Z})$.

We recall the following.

Fact 8.7. *$G_D(D)_0$ has finite index in both G_D and $Sp(2g, \mathbb{Z})$.*

Proof. By [3, Lemma 8.9.1 (b)] $G_D(D)_0$ has finite index in G_D. It is not hard to check by direct computations that $G_D(D)_0$ contains the principal congruence subgroup $\Gamma_g(2(d_g)^2)$. Since $\Gamma_g(2(d_g)^2)$ has finite index in $Sp(2g, \mathbb{Z})$, $G_D(D)_0$ has finite index as well. □

Let $\Psi^D(\tau) \colon \mathbb{H}_g \to \mathbb{P}^N(\mathbb{C})$ be the map

$$\Psi^D(\tau) = \varphi^D(0, \tau),$$

and $\Phi^D \colon \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$ be the map

$$\Phi^D(z, \tau) = (\varphi^D(z, \tau), \Psi^D(\tau)).$$

We have a commuting diagram

$$\begin{tikzcd} \mathbb{C}^g \times \mathbb{H}_g \arrow[r, "\Phi^D"] \arrow[d, "\pi_2"'] & \mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C}) \arrow[d, "\pi_2"] \\ \mathbb{H}_g \arrow[r, "\Psi^D"] & \mathbb{P}^N(\mathbb{C}). \end{tikzcd}$$

The reason for introducing $G_D(D)_0$ is the following result, which can be seen as also saying that for appropriate D, the map Ψ^D induces an embedding of the quotient space $G_D(D)_0\backslash\mathbb{H}_g$ into projective space, whose image is a quasi-projective variety. For a proof we refer to [3, Theorem 8.10.1. and Remark 8.10.4].

Fact 8.8. *Assume $d_1 \geq 4$, and $2|d_1$ or $3|d_1$. The map $\Psi^D\colon \mathbb{H}_g \to \mathbb{P}^N$ is an immersion, the image of $\mathbb{H}_g$ under Ψ^D is a Zariski open subset of an algebraic variety in $\mathbb{P}^N$, and $\Psi^D(\tau) = \Psi^D(\tau')$ if and only if there exists $M \in G_D(D)_0$ with $M \cdot \tau = \tau'$.*

Moreover, if $\Psi^D(\tau) = \Psi^D(\tau')$ then $\varphi^D_\tau(\mathcal{E}^D_\tau) = \varphi^D_{\tau'}(\mathcal{E}^D_{\tau'})$.

Notation 8.9. We will denote by $\mathfrak{F}^D_g$ a fixed Siegel fundamental set for the action $G_D(D)_0$ on $\mathbb{H}_g$.

Since $G_D(D)_0$ is a subgroup of G_D, by Fact 3.5, every polarized abelian variety of type D is isomorphic to one of $\mathcal{E}^D_\tau$ with $\tau \in \mathfrak{F}^D_g$. Note also that by Fact 8.8, for D as above, if $\mathfrak{F} \subseteq \mathbb{H}_g$ contains $\mathfrak{F}^D_g$ then $\Psi^D(\mathfrak{F}) = \Psi^D(\mathbb{H}_g)$ and $\Phi^D(\mathcal{X}^D(\mathfrak{F})) = \Phi^D(\mathbb{C}^g \times \mathbb{H}_g)$. An immediate corollary of Corollary 7.10(2) is:

Theorem 8.10. *If $d_1 \geq 4$, and $2|d_1$ or $3|d_1$ then there is a definable open set $U \subseteq \mathbb{C}^g \times \mathbb{H}_g$ containing $\mathcal{X}^D(\mathfrak{F}^D_g)$ such that $\Phi^D \restriction U$ is definable in $\mathbb{R}_{an,exp}$.*

8.4. **A new proof for a theorem of Baily.** In this section we demonstrate how o-minimality can be used to provide an alternative proof of the following theorem of Baily, [1].

Theorem 8.11 (Baily)**.** *Assume $d_1 \geq 4$, and $2|d_1$ or $3|d_1$. Then the image of $\mathbb{C}^g \times \mathbb{H}_g$ under Φ^D is a Zariski open subset of an algebraic variety in $\mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$.*

Proof. Let $U \subseteq \mathbb{C}^g \times \mathbb{H}_g$ be the open set from Theorem 8.10. The map $\Phi \restriction U$ is definable in $\mathbb{R}_{an,exp}$ and $\Phi(U) = \Phi(\mathbb{C}^g \times \mathbb{H}_g)$.

Let $X = \Phi^D(\mathcal{X}^D(\mathbb{H}^D_g))$. The main step in proving the theorem is to show that the topological closure $\bar{X}$ of X in $\mathbb{P}^N(\mathbb{C})$ is a projective variety. For that we will use the following fact proved in the Appendix. For $Z \subset \mathbb{C}^k$, we let $Fr(Z) = Fr_{\mathbb{C}^k}(Z) = Cl(Z) \setminus Z$, where $Cl(Z)$ stands for the topological closure of Z in $\mathbb{C}^k$. We write $\dim_{\mathbb{R}}(Z)$ for the o-minimal dimension of Z.

Fact 8.12. *Let $\phi : U \to \mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$ be a finite-to-one holomorphic map from an open $U \subseteq \mathbb{C}^m$ which is definable in $\mathbb{R}_{an,exp}$. Assume that there is a definable set $F \subseteq U$ such that $\phi(U) = \phi(F)$ and such that $\dim_{\mathbb{R}}(Fr(F)) \leq 2m-2$. Then the topological closure of $\phi(U)$ in $\mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$ is an algebraic variety.*

We would like to apply the above result to the map $\phi = \Phi^D \restriction U$ with $F = \mathcal{X}^D(\mathfrak{F}^D_g)$ and U viewed as an open subset of $\mathbb{C}^g \times \mathbb{C}^n$. Since the pre-image of every point under the map Φ^D is a discrete subset of $\mathbb{C}^g \times \mathbb{H}_g$, by o-minimality its intersection with U must be finite, so ϕ^D is finite-to-one on U. It is left to see that $\dim_{\mathbb{R}}(Fr_{\mathbb{C}^g\times\mathbb{C}^n}(F)) \leq 2(g+n) - 2$.

For every $\gamma \in Sp(2g, \mathbb{R})$, we consider the definable map $\tau \mapsto \gamma\tau$, from the open set $\mathbb{H}_g \subseteq \mathbb{C}^n$ into $\mathbb{C}^n$. By Fact 3.11, the set $\mathfrak{F}_g$ a closed subset of $\mathbb{C}^n$ and therefore, by Theorem 9.2 below, $\dim_{\mathbb{R}}(Fr_{\mathbb{C}^n}(\gamma \cdot \mathfrak{F}_g)) \leq 2n-2$. Since $\mathfrak{F}^D_g$ is a finite union of such translates (see (2)), it follows that $\dim_{\mathbb{R}}(Fr_{\mathbb{C}^n}(\mathfrak{F}^D_g)) \leq 2n-2$. Recall that $F = \mathcal{X}^D(\mathfrak{F}^D_g)$ is defined as

$$\left\{ (\sum_{i=1}^{g} t_i\tau_{(i)} + \sum_{j=1}^{g} s_j D_{(j)}, \tau) \in \mathbb{C}^g \times \mathbb{H}_g : \tau \in \mathfrak{F}^D_g \ , \ 0 \leq t_i, s_j \leq 1 \right\}.$$

Consider the frontier of F inside $\mathbb{C}^g \times \mathbb{C}^n$. It is easy to see that if (z, τ) belongs to this frontier then $\tau \in Fr_{\mathbb{C}^n}(\mathfrak{F}_g^D)$, so by the above, the real dimension of $Fr(F)$ is at most $2g + 2n - 2$, as we wanted.

We can therefore apply Fact 8.12 and conclude that $\bar{X}$ is an algebraic subvariety of $\mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$.

Let $Y = \pi_2(X)$ and $\bar{Y} = Y$ be the topological closure of Y. By Fact 8.8, $\bar{Y}$ is a Zariski closed set and Y is Zariski open in $\bar{Y}$.

It is not hard to see that $Fr(X)$ is disjoint from $\pi_2^{-1}(Y)$, hence $X = \bar{X} \cap \pi_2^{-1}(Y)$. Since Y is Zariski open in $\bar{Y}$, the set X is Zariski open in $\bar{X}$. □

9. Appendix: O-minimality and complex analysis

We review here some basic notions from [16] and [19], and prove the results we used earlier.

Definition 9.1. Let $\mathcal{R} = \langle \mathbb{R}; , +, \cdot, \cdots \rangle$ be an o-minimal expansion of the real field. *A definable n-dimensional $\mathbb{C}$-manifold* is a definable set M equipped with a finite cover of definable sets $M = \bigcup_i U_i$, with each U_i in definable bijection with an open subset of $\mathbb{C}^n$, and such that the transition maps are holomorphic.

For M a manifold and $X \subseteq M$ we write $Cl_M(X)$ for the topological closure of X in M, and $Fr_M(X)$ for the frontier $Cl_M(X) \setminus X$. We fix $\mathcal{R}$ an o-minimal expansion of the field of reals. All definability is in $\mathcal{R}$.

Theorem 9.2. *Assume that $\phi : U \to N$ is a definable holomorphic map from a definable open subset of $\mathbb{C}^n$ into a definable $\mathbb{C}$-manifold.*

(1) Let $F \subseteq U$ be a closed (not necessarily definable) set. Then there is a definable $Y \subseteq N$, with $\dim_{\mathbb{R}}(Y) \leq 2n - 2$ such that $Fr_N(\phi(F)) \subseteq Y$. Moreover, the same result holds if we assume, instead of F being closed, that $Fr_{\mathbb{C}^n}(F)$ is contained in some definable set whose real dimension is $2n - 2$.

(2) Assume that ϕ is a finite-to-one map, and that there is a set $F \subseteq U$ with $\phi(F) = \phi(U)$, such that $Fr_{\mathbb{C}^n}(F)$ is contained in a definable set of dimension $2n - 2$. Then $Cl_N(\phi(U))$ is a complex analytic subset of N. In particular, if $N = \mathbb{P}^k(\mathbb{C})$ then, by Chow's Theorem, $Cl_N \phi(U))$ is a projective variety.

Proof. (1) Assume first that F is closed. Note that if F is compact then $\phi(F)$ is closed so $Fr_N(\phi(F))$ is empty. Hence, the points of $Fr_N(\phi(F))$ arise from the behavior of ϕ "at ∞".

We repeat the argument in [19, Section 7.2]. We write $\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \cup H$, for H a projective hyperplane, and let $\Gamma \subseteq \mathbb{P}^n(\mathbb{C}) \times N$ be the closure of the graph of ϕ. We let $\pi : \mathbb{P}^n(\mathbb{C}) \times N \to N$ be the projection onto the second coordinate. Because F is closed, the frontier of $\phi(F)$ in N is contained $Y = \pi(\Gamma \cap (H \times N))$, namely for every $y \in Fr_N(\phi(F))$ there exists $z \in H$ such that $(z, y) \in \Gamma$. It is therefore sufficient to see that $\dim_{\mathbb{R}}(\Gamma \cap (H \times N)) \leq 2n - 2$.

We let B_{inf} be the set of all $(z, y) \in \Gamma \cap (H \times N)$ such that there are infinitely $y' \in N$ with $(z, y') \in \Gamma$ and let $B_{fin} = \Gamma \cap (H \times N) \setminus B_{inf}$. By [17, Lemma 6.7(ii)], $\dim_{\mathbb{R}}(B_{inf}) \leq 2n-2$. But now, since $\dim_{\mathbb{R}}(H) = 2n-2$ it follows from the definition of B_{fin} that also $\dim_{\mathbb{R}}(B_{fin}) \leq 2n-2$. Since $\Gamma \cap (H \times N) \subseteq B_{inf} \cup B_{fin}$, we have $\dim_{\mathbb{R}}(\Gamma \cap (H \times N)) \leq 2n - 2$, as required.

As for the "moreover" statement, we can repeat the above argument, with H replaced by $H' = Fr_{\mathbb{P}^n(\mathbb{C})}(F)$. The assumption implies H' is contained in a definable subset of $\mathbb{P}^n(\mathbb{C})$ of dimension $2n-2$, so we can repeat the argument.

(2) By [17, Corollary 6.3], there is a definable closed $E \subseteq N$ with $\dim_{\mathbb{R}}(E) \leq \dim_{\mathbb{R}} \phi(U) - 2$, such that $A = \phi(U) \setminus E$ is locally $\mathbb{C}$-analytic in N. Since ϕ is finite-to-one, the intersection of $\phi(U)$ with every open subset of N is either empty or of dimension $2n-2$. Because $\phi(U) = \phi(F)$, we can conclude from (1) then $\dim_{\mathbb{R}} Fr_N(\phi(U)) \leq 2n-2$. In particular, $\dim_{\mathcal{R}} Fr_N(A) \leq 2n-2$.

We can now apply [17, Theorem 4.1] to A and conclude that $Cl(A)$ is an analytic subset of N. It is easy to see that $Cl_N(A) = Cl_N(\phi(U))$. $\square$

References

[1] Walter L. Baily Jr., *On the theory of θ-functions, the moduli of abelian varieties, and the moduli of curves*, Ann. of Math. (2) **75** (1962), 342–381.

[2] Walter L. Baily Jr. and Armand Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528.

[3] Christina Birkenhake and Herbert Lange, *Complex abelian varieties*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 2004.

[4] Jean-Benoît Bost, *Introduction to compact Riemann surfaces, Jacobians, and abelian varieties*, From number theory to physics (Les Houches, 1989), Springer, Berlin, 1992, pp. 64–211.

[5] Lou van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998. MR1633348 (99j:03001)

[6] Lou van den Dries and Christopher Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), no. 2, 497–540.

[7] ———, *On the real exponential field with restricted analytic functions*, Israel J. Math. **85** (1994), 19-56.

[8] Lou van den Dries, Angus Macintyre, and David Marker, *The elementary theory of restricted analytic fields with exponentiation*, Ann. Math. **140** (1994), 183-205.

[9] Andrei Gabrielov, *Projections of semi-analytic sets*, Functional Anal. Appl. **2** (1968), 282-291.

[10] Gerard van der Geer, *Siegel modular forms and their applications*, The 1-2-3 of modular forms (Nordfjordeid, Norway, 2004), Springer, Berlin, 2008, pp. 181–245.

[11] Jun-ichi Igusa, *Theta functions*, Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 194.

[12] ———, *On the graded ring of theta-constants (II)*, American J. of Math. **88** (1966), 221–236.

[13] Helmut Klingen, *Introductory lectures on Siegel modular forms*, Cambridge Studies in Advanced Mathematics, vol. 20, Cambridge University Press, Cambridge, 1990.

[14] Bruno Klingler and Andrei Yafaev, *The André Oort conjecture*, preprint.

[15] David Mumford, *Tata lectures on theta. I*, Progress in Mathematics, vol. 28, Birkhäuser Boston Inc., Boston, MA, 1983. With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman.

[16] Ya'acov Peterzil and Sergei Starchenko, *Mild manifolds and a non-standard Riemann existence theorem*, Selecta Math. (N.S.) **14** (2009), no. 2, 275–298.

[17] ———, *Complex analytic geometry and analytic-geometric categories*, J. Reine Angew. Math. **626** (2009), 39–74.

[18] ———, *Uniform definability of the Weierstrass $\wp$ functions and generalized tori of dimension one*, Selecta Math. (N.S.) **10** (2004), no. 4, 525–550.

[19] ———, *Tame complex analysis and o-minimal structure*, proceedings of the ICM 2010.

[20] Jonathan Pila, *O-minimality and the André-Oort conjecture for $\mathbb{C}^n$*, Ann. of Math. (2) **173** (2011), no. 3, 1779–1840.

Rational points of definable sets and results of Andre-Oort–Manin-Mumford type, IMRN 2009 (2009), 2476-2507,

[21] ———, *ational points of definable sets and results of Andre-Oort–Manin-Mumford type*, IMRN **13** (2009), 2476-2507.

[22] Jonathan Pila and Alex W. Wilkie, *The rational points of a definable set*, Duke Math. J. **133** (2006), no. 3, 591–616.

[23] Jonathan Pila and Umberto Zannier, *Rational points in periodic analytic sets and the Manin-Mumford conjecture*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **19** (2008), no. 2, 149–162.

[24] C. L. Siegel, *Lectures on quadratic forms*, Notes by K. G. Ramanathan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 7, Tata Institute of Fundamental Research, Bombay, 1967.

[25] Carl Ludwig Siegel, *Symplectic geometry*, Academic Press, New York, 1964.

[26] Audrey Terras, *Harmonic analysis and symmetric spaces and applications. II*, Springer-Verlag, Berlin, 1988.

[27] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.

[28] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal, *Fundamentals of convex analysis*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2001. Abridged version of *Convex analysis and minimization algorithms. I* [Springer, Berlin, 1993] and *II*.

University of Haifa
E-mail address: kobi@math.haifa.ac.il

University of Notre Dame
E-mail address: sstarche@nd.edu