RETURNING TO SEMI-BOUNDED SETS

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Abstract. An o-minimal expansion of an ordered group is called semi-bounded if there is no definable bijection between a bounded and an unbounded interval in it (equivalently, it is an expansion of the group by bounded predicates and group automorphisms). It is shown that every such structure has an elementary extension $\mathcal{N}$ such that either $\mathcal{N}$ is a reduct of an ordered vector space, or there is an o-minimal structure $\widehat{\mathcal{N}}$, with the same universe but of different language from $\mathcal{N}$, with (i) Every definable set in $\mathcal{N}$ is definable in $\widehat{\mathcal{N}}$, and (ii) $\widehat{\mathcal{N}}$ has an elementary substructure in which every bounded interval admits a definable real closed field.

As a result certain questions about definably compact groups can be reduced to either ordered vector spaces or expansions of real closed fields. Using the known results in these two settings, the number of torsion points in definably compact abelian groups in expansions of ordered groups is given. Pillay's Conjecture for such groups follows.

1. Introduction

An expansion of an ordered abelian group or an ordered vector space by bounded predicates is sometimes called *a semi-bounded structure* (a combination of semi-linear and bounded). The definable sets in such a structure are called semi-bounded sets. Structural results about semi-bounded sets can be found in [21], [17], [22],[13], [5] (in the o-minimal setting) and [1] (in arbitrary ordered abelian groups). Some results in [15] apply as well.

In this paper I return to the semi-bounded setting, in order to reduce a question about the torsion points of a definably compact groups in o-minimal expansions of ordered groups to similar results in expansions of real closed fields, [9], and in ordered vector spaces, [11].

The idea is as follows: Let $\mathcal{M} = \langle M, <, +, \cdots \rangle$ be a semi-bounded structure whose theory is assumed to be not linear (see [13]). By the Trichotomy Theorem, [19], a real closed field is defined on some open fixed interval $I \subseteq M$. An interval $J \subseteq M$ will be called *short* if it is

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in definable bijection with I; otherwise it is called *long*. The structure $\mathcal{M}$ will be called *short* is every bounded interval in M is short.

As will be observed, every definably compact group in a short model is contained in a finite cartesian product of some bounded interval and therefore definable in an o-minimal expansion of a real closed field. Hence, all results about definably compact groups in expansions of real closed fields hold when the model is short.

Given an arbitrary semi-bounded structure $\mathcal{M}$ whose theory is not linear, one can find $\mathcal{N} \succ \mathcal{M}$, and a new o-minimal structure $\widehat{\mathcal{N}}$, with the same universe as $\mathcal{N}$, basically by extending all partial 0-definable linear maps defined on long intervals to global linear maps, and at the same time restricting $dcl(\emptyset)$. Furthermore, every 0-definable set in $\mathcal{N}$ is still definable in $\widehat{\mathcal{N}}$, possibly over parameters. Having done that, the set of short elements $D \subseteq N$ becomes an elementary substructure of $\widehat{\mathcal{N}}$.

Now, every $\mathcal{N}$-definable group is definable in $\widehat{\mathcal{N}}$ and because $\widehat{\mathcal{N}}$ has a short elementary substructure D, one can transfer the Edmundo-Otero result, [9], about the torsion points of definable groups in expansions of real closed fields to groups definable in $\widehat{\mathcal{N}}$ and hence in $\mathcal{N}$.

Together with the result of Eleftheriou and Starchenko, [11], on definable groups in ordered vector spaces, one obtains (see Theorem 7.6 below):

Theorem 1.1. *If G is a definably connected, definably compact abelian group in an o-minimal expansion of an ordered group then for every k,*

$$Tor_k(G) = (\mathbb{Z}/k\mathbb{Z})^n.$$

Since this is the only missing ingredient for proving Pillay's Conjecture for definable groups in o-minimal expansions of groups, one may conclude the conjecture in this setting as well (see Section 8).

Remark 1.2. *The treatment of semi-bounded sets suggested here does not make use of the known structure theorems for definable sets in semi-bounded structures (see* [17] *and* [5]*), where the analysis is given in terms of bounded sets and unbounded intervals. Instead, bounded sets are replaced by those bounded sets that are contained in D^n and unbounded intervals are replaced by long intervals. At the end of the paper several conjectures are made about possible structure theorems for definable sets and groups in semi-bounded structures.*

Notation The letters $\mathcal{M}, \mathcal{N}, \mathcal{D}$ are used for structures whose universe, respectively, is M, N, D.

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2. The basic definition and properties

As is shown by Edmundo in [5], semi-boundedness has several equivalent definitions. Here I use the following:

Definition 2.1. *A* semi-bounded structure $\mathcal{M} = \langle M, <, +, \cdots \rangle$ *is an o-minimal expansion of an ordered group without poles. Namely, there is no definable bijection between a bounded interval and an unbounded interval. Note that this is a property which is preserved in elementarily equivalent structures.*

Example 2.2. *(1) Every ordered vector space is semi-bounded. This follows from quantifier elimination.*

(2) The expansion $\mathbb{R}_{bdd}$ *of the ordered group of real numbers by restricted multiplication is a semi-bounded structure in which every interval admits a definable real closed field. In fact, every bounded semi-algebraic set is definable in* $\mathbb{R}_{bdd}$*. See* [17] *for details.*

(3) Any elementary extension of $\mathbb{R}_{bdd}$ *is still semi-bounded, but only intervals of finite size are in definable bijection with* $(0,1)$ *(indeed, this follows from 2.5 below). Hence (see 3.3) only those intervals admit a real closed field structure.*

2.1. **Expansions of ordered groups.**

Given $\mathcal{M}$ an o-minimal expansion of an ordered group, there are three possibilities for the theory of $\mathcal{M}$:

(a) $Th(\mathcal{M})$ **is linear** (see [13]). In this case, by the same paper there exists $\mathcal{N} \equiv \mathcal{M}$, with $\mathcal{N}$ a reduct of an ordered vector space over an ordered division ring (with the same addition and linear ordering as the underlying group of $\mathcal{N}$).

(b) $Th(\mathcal{M})$ **is not linear**. In this case, as is not hard to see, the theory of every interval in $\mathcal{M}$ (with the induced structure) is not linear either. It follows that no interval in $\mathcal{M}$ is elementarily equivalent to a reduct of an interval in an ordered vector space and therefore, by Trichotomy Theorem ([19] Theorem 1.2), a real closed field whose ordering agrees with that of $\mathcal{M}$, is definable on some interval $(-a, a)$. There are two sub-cases to consider:

(b1) $\mathcal{M}$ **is semi-bounded**.

(b2) $\mathcal{M}$ **is not semi-bounded**. In this case, one can endow the whole structure $\mathcal{M}$ with a definable real closed field R (but the underlying group addition might not coincide with that of the field). Indeed, this is claimed in [19], but the reference there is not precise, so I spell out the argument: Assume that $\sigma : (b_1, b_2) \to (c, +\infty)$ is a definable map with $\lim_{t\to b_2} \sigma(t) = +\infty$. Without loss of generality, $b_2 - b_1 < a$.

Using translation, it can be assumed that $b_1 = 0$ and $b_2 < a$. However, being inside a real closed field, the intervals $(0, a)$ and $(0, b_2)$ are in definable bijection, so (c, ∞) (and therefore also $(0, \infty)$) is definably bijective to the positive elements of R. This is clearly enough to get a real closed field on the whole of $\mathcal{M}$.

2.2. **Model theoretic preliminaries.** *Assume now that $\mathcal{M}$ is an o-minimal expansion of an ordered group, which is semi-bounded.*

An immediate corollary of this assumption is: If $f : (a, b) \to M$ is a definable function on a bounded interval then f is bounded on (a, b) and therefore the limit of $f(t)$ as t tends to a (or b) exists in M.

Proposition 2.3. *If $\mathcal{M} \prec \mathcal{N}$ and $\mathcal{M}_1$ is the convex hull of $\mathcal{M}$ in $\mathcal{N}$ then $\mathcal{M}_1 \prec \mathcal{N}$.*

Proof. Without loss of generality, the language contains a constant for every element of M. It is sufficient to see that $dcl_{\mathcal{N}}(M_1) = M_1$. Equivalently, for every M-definable function $F(\bar{x})$ in $\mathcal{N}$, and every $\bar{a}$ from M_1, $F(\bar{a}) \in M_1$. Use induction of the number of variables in F.

Assume that $F(\bar{w}, y)$ is of $n+1$ variables, $n \geq 0$, and $\bar{a}$ and b are from M_1 with $(\bar{a}, b) \in domF$. Let $f_{\bar{w}}(y) = F(\bar{w}, y)$. By partitioning the graph of F, we may also assume that for every $\bar{w}$, the domain of $f_{\bar{w}}$ is either empty, or it is an open (bounded or unbounded) interval. Also, without loss of generality, every $f_{\bar{w}}$ is monotonely increasing (the decreasing case is handled similarly).

Assume first that $domf_{\bar{a}} = N$. In this case, Since b is in the convex hull of M, there are $b_1 < b < b_2$, $b_1, b_2 \in M$, and hence $f_{\bar{a}}(b_1) \leq f_{\bar{a}}(b) \leq f_{\bar{a}}(b_2)$. Since $f_{\bar{a}}(b_1), f_{\bar{a}}(b_2) \in dcl_{\mathcal{N}}(\bar{a})$ one may use induction to conclude that they are in M_1, so by convexity so is $f_{\bar{a}}(b)$.

If $domf_{\bar{a}} = (c, +\infty)$, for $c \in M$ then c is in $dcl_{\mathcal{N}}(\bar{a})$ hence, by induction it is in M_1. One can now find $b_2 \in M$ such that $c < b < b_2$. By the comment preceding the proposition, we may assume that f is defined on the closed interval $[c, b_2]$ (this is precisely where semi-boundedness is used!). We now proceed as above. The remaining case is handled similarly. □

Recall that for ordered structures $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is said to be *Dedekind complete in* $\mathcal{N}$ if for every element $n \in N$, if $m_1 < n < m_2$ for some $m_1, m_2 \in M$ then n has a standard part in M. Namely, there exists $m \in M$ with no element of M strictly between n and m. Note that if $\mathcal{M}_1$ is convex in $\mathcal{N}$ then it is clearly Dedekind complete in it. The following powerful theorem of Marker and Steinhorn [14] will be used below:

Theorem 2.4. *If $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ which is moreover Dedekind complete in $\mathcal{N}$ then for every N-definable set $X \subseteq N^k$, the set $X \cap M^k$ is definable in $\mathcal{M}$.*

Corollary 2.5.

(1) *Assume that $F : S \times (a, b) \to M$ is a definable map such that for every $s \in S$, the map $f_s(x) = F(s, x)$ is a bijection between the bounded interval (a, b) and $(0, d_s)$ for $d_s > 0$. Then there exists an $m \in M$ such that for every $s \in S$, $d_s < m$.*

(2) *If $\mathcal{M} \prec \mathcal{N}$, $a < b$ in M, $c < d$ in N and (a, b) is in definable bijection with (c, d) then there exists $m \in M$ such that $d - c < m$.*

Proof. (1) If not, then in an elementary extension $\mathcal{N}$ of $\mathcal{M}$, there exist $s \in S$ and $d_s \in \mathcal{N}$ greater than all elements of $\mathcal{M}$ such that $f_s : (a, b) \to (0, d_s)$ is a definable bijection. Let $\mathcal{M}_1$ be the convex hull of $\mathcal{M}$. Then by Proposition 2.3, $\mathcal{M}_1$ is an elementary substructure of $\mathcal{N}$, which is obviously Dedekind complete in $\mathcal{N}$.

Let Γ be the intersection of the graph of f_s with $M_1 \times M_1$. By Theorem 2.4, Γ is definable in $\mathcal{M}_1$ and it is still the graph of a definable function. Moreover, because f_s was a bijection, for every $y > 0$ in $\mathcal{M}_1$ there exists $x \in (a, b) \subseteq M_1$ such that $f_s(x) = y$. Therefore there exists in $\mathcal{M}_1$ a surjective map between a sub-interval of (a, b) and the interval $(0, +\infty)$. This is impossible because $\mathcal{M}_1$ and $\mathcal{M}$ are elementarily equivalent so $\mathcal{M}_1$ must be semi-bounded as well. □

(2) This easily follows from (1).

3. Short and long intervals

$\mathcal{M}$ is assumed to be semi-bounded and in addition $Th(\mathcal{M})$ not linear.

Fix an element, call it $1 > 0$, such that a real closed field, whose universe is $(0, 1)$ and whose ordering agrees with the $\mathcal{M}$-ordering, is definable in $\mathcal{M}$. Assume from now on that $1 \in dcl(\emptyset)$.

Definition 3.1. Two open intervals (a, b) and (c, d) are called *equivalent* if there exists a definable bijection between them.

An element $a \in M$ is called *short* if either $a = 0$ or $(0, |a|)$ and $(0, 1)$ are equivalent; otherwise it is called *tall*. An interval (a, b) is called *short* if $b - a$ is short, otherwise it is called *long*.

The following lemma can be proved using standard o-minimal arguments, together with the fact that every definable function on a bounded interval has a limit at the endpoints of the interval.

Lemma 3.2. *If (a, b) and (c, d) are equivalent intervals then there exists a definable and continuous, strictly monotone bijection between them (if the intervals are bounded one can always choose the bijection to be increasing).*

Corollary 3.3. *For any interval $I \subseteq M$, I is short if and only if I admits a definable real closed field whose ordering agrees with that of $\mathcal{M}$.*

Proof. If I is short then, by the last lemma it has a definable order-preserving bijection with $(0, 1)$ so admits a definable real closed field. For the converse, if I admits a real closed field structure, then after translation one may assume that either $(0, 1) \subseteq I$ or $I \subseteq (0, 1)$. In both cases one gets an interval inside another real closed field so the two are in definable bijections. (Actually, by [16], the fields on $(0, 1)$ and I and are also definably isomorphic but this will not be required here). $\square$

Lemma 3.4. (1) *If I is a short interval then it is definably bijective with any subinterval of I. In particular, if a is short and $0 < |b| < |a|$ then b is short.*
(2) *If (a, b) and (b, c) are short interval then so is (a, c).*
(3) *If a and b are short elements then so are $a + b$ and $-a$.*

Proof. (1) By the last lemma, I admits a reals closed field structure whose ordering agrees with the $\mathcal{M}$-ordering. In a real closed field any two 1-dimensional open intervals are definably bijective.

(2) Since (a, b) is in bijection with $(0, 1)$ is it also in bijection with $(0, 1/2)$, and similarly, (b, c) is in bijection with $(1/2, 1)$.

(3) This is immediate from (2). $\square$

Lemma 3.5. *Assume that $f : X \to M$ is a definable continuous function whose domain X is a definably connected set, contained in a cartesian product of short intervals. Then $f(X)$ is contained in a short interval.*

Proof. If not, then by definable choice there is a definable curve in X which is in bijection with a long interval in M. Using projections one gets a bijection between short and long intervals. Contradiction. $\square$

Proposition 3.6. *The set D of all short elements in $\mathcal{M}$ is a convex subgroup of M. If $\mathcal{M}$ is $|T|^+$-saturated then $D \neq M$, and in particular, D is not definable.*

Proof. By 3.4, D is a convex subgroup. It is left to see that when $\mathcal{M}$ is $|T|^+$-saturated then $D \neq M$ (saturation is important since, for example, in the expansion of the additive reals by restricted multiplication we have $D = M$). Assume towards contradiction that $D = M$.

Consider the type $p(x)$ which says, for every $\emptyset$-definable family of injections from $(0,1)$ into M, that none of these maps is a bijection between $(0,1)$ and $(0,x)$. By our assumptions, this type is inconsistent, hence there are finitely many definable families of injections from $(0,1)$ into M such that for every $x \in M$, one such injection gives a bijection between $(0,1)$ and $(0,x)$. It follows that there exists a definable family of maps $\{f_s : s \in S\}$ such that each f_s is a bijection between the intervals $(0,1)$ and $(0,d_s)$, and $\{d_s : s \in S\} = M$. This contradicts Corollary 2.5. $\square$

Here are several corollaries:

Corollary 3.7. (1) *Let $\{f_s : s \in S\}$ be a 0-definable family of bijections, $f_s : (0,1) \to I_s$, $I_s \subseteq M$ an open interval. Then the set $\{|I_s| : s \in S\}$ is bounded above and its supremum is a short element in $dcl(\emptyset)$.*

(2) *Let $\{I_s : s \in S\}$ be a 0-definable family of intervals in $\mathcal{M}$.*
(i) If all intervals are short and $\mathcal{M}$ is $|T|^+$-saturated then there exists a short m in $dcl(\emptyset)$, such that the length of every I_s is at most m.
(ii) If all intervals are long then there exists a tall $b \in dcl(\emptyset)$ such that the length of every I_s is not less than b (no saturation is required).

(3) *For every short element $0 < a \in M$ there exists a short $m \in dcl(\emptyset)$ with $a < m$.*

Proof. (1) Let $J = \{|I_s| : s \in S\}$. By 2.5, the set J is bounded above, hence its supremum b exists in $dcl(\emptyset)$. b must be short because $b/2$ is a shorter than some element of J and all elements of J are short.

(2) (i) Because $\mathcal{M}$ is $|T|^+$-saturated there exists a definable family of bijections $f_s : (0,1) \to I_s$, $s \in S$. Now apply (1).

(ii) Let $J = \{|I_s| : s \in S\}$. Without loss of generality, J is closed upward, hence of the form (b,∞) or $[b,\infty)$, for $b \in M_0$. Clearly, if $b \in J$ then it is tall and we are done, so assume $b \notin J$. Now b must be tall because $2b$ is greater than some element of J and all elements of J are tall.

(3) Because a is short, there exists an $\emptyset$-definable family of maps $\mathcal{F} = \{f_s : s \in S\}$, with each $f_s : (0,1) \to (0, a_s)$ a definable bijection, and $a = a_{s_0}$ for some $s_0 \in S$. Now apply (1). $\square$

4. Affine and linear functions

Here $\mathcal{M}$ is a semi-bounded and $Th(\mathcal{M})$ not linear.

Some of the results in this section, such as 4.4 and 4.8, were proved in [15] for unbounded intervals instead of long ones.

Definition 4.1. A function $f : (a,b) \to M$ is called *linear* on (a,b) if for every $x, y \in (a,b)$, if $x + y \in (a,b)$ then $f(x) + f(y) = f(x+y)$. The function is *affine* if for some (all) $c \in (a,b)$, the function $f_c(x) = f(c+x) - f(c)$ is linear on $(a-c, b-c)$.

$f : (a,b) \to M$ is called *locally affine* if for every $x \in (a,b)$ there exists a neighborhood on which f is affine.

Two functions f, g defined on a neighborhood of 0 are said to have *the same germ at* 0 if there is $\epsilon > 0$ such that $f|(-\epsilon, \epsilon) = g|(-\epsilon, \epsilon)$.

Here are some facts about affine and linear functions:

Fact 4.2.
(1) *If f is affine on (a,b) then for every $c, d \in M$, the function $f(x+c) + d$ is also affine on $(a-c, b-c)$.*
(2) *If f is affine on (a,b) then for every $c, d \in (a,b)$, the functions f_c and f_d, defined above, have the same germ at* 0.
(3) *If f is affine on (a,b) then it is continuous.*
(4) *If f is affine on (a,b) then it is either constant or strictly monotone.*

Proof. (1) This follows easily from the definition.
(2) Assume that $c < d$. Then by assumption, f_c is linear on $(a-c, b-c)$ and therefore, for all x, y such that $x + y \in (a-c, b-c)$,

$$f(c+x+y) = f(c+x) + f(c+y) - f(c).$$

Letting $y = d - c$ one gets

$$f(d+x) - f(d) = f(c+x) - f(c)$$

for all x near 0, hence f_c and f_d have the same germ at 0.
(3) Consider $c \in (a,b)$. By o-minimality, there exists $c' \in (a,b)$ such that f is continuous at c', and by (2) the germs of f_c and $f_{c'}$ at 0 are the same. However, $f_{c'}$ is continuous at 0, and therefore f_c is continuous at 0 as well. Clearly, this implies that f is continuous at c.
(4) Assume that f is not constant. Then, by o-minimality, there exists $c \in (a,b)$ such that f is strictly monotone, say increasing, near c. This implies that f_c is strictly increasing at 0, and therefore, by (2), for all

$d \in (a,b)$, f_d is strictly increasing near 0. This in turn implies that f is increasing near d for all d. O-minimality implies that f is strictly increasing on (a,b). $\square$

Lemma 4.3. *For a definable $f:(a,b)\to M$, the following are equivalent:*

(1) *f is affine on (a,b).*
(2) *f is locally affine at every $c \in (a,b)$.*
(3) *For every $c,d \in (a,b)$ the germs of f_c and f_d at 0 are the same.*

Proof. (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3): Assume that f is locally affine at every point of (a,b) and consider the definable equivalence relation $c \sim d$ if f_c and f_d have the same germ at 0. By 4.2(2) and the fact that f is locally affine, every $\sim$-class is open. But then every $\sim$-class is also closed (its complement is open) hence by definable connectedness there is exactly one such class.

(3) $\Rightarrow$ (1): Note that by 4.2 (3), f is continuous on (a,b). One needs to show, for every $d \in (a,b)$, that the function $f_d(x) = f(d+x) - f(d)$ is linear on $(a-d, b-d)$. It is easy to see that for $d \in (a,b)$, the function f_d also satisfies assumption (3). Hence, one may assume that $0 \in (a,b)$ and that $f(0)=0$ and the goal is to show that f is linear.

By assumption, for all $c \in (a,b)$, in there is $J \ni 0$ such that for all $x \in J$,

$$f(c+x) - f(c) = f(0+x) - f(0) = f(x). \tag{1}$$

Given an arbitrary c in (a,b), let

$$C = \{x \in (a,b) : x+c \in (a,b) \& f(x+c) = f(x) + f(c)\}.$$

It is sufficient to show that $C = (a-c, b-c)$.

By continuity, C is closed in $(a-c, b-c)$. It clearly contains 0 hence it is sufficient to show that it is also open. Fix $d \in C$ and let $J \ni 0$ be small enough so that $f(c+d+y) - f(c+d) = f(y)$ and also $f(d+y) - f(d) = f(y)$, for all $y \in J$. Then, since $d \in C$, for all $y \in J$,

$$\begin{aligned} f(c+d+y) - f(c) &= f(c+d+y) - f(c+d) + f(c+d) - f(c) = \\ &= f(y) + f(d) = f(y+d). \end{aligned}$$

It follows that the set $d+J$ is contained in C and therefore C is open. Hence, f is affine on (a,b). $\square$

Two affine functions $f_1 : I \to M$ and $f_2 : J \to M$ are said to be *equivalent* if the associated linear functions $f_1(a+x) - f_1(a)$ and $f_2(b+x) - f(b)$, $a \in I$, $b \in J$, have the same germ at 0

Lemma 4.4. *If (a, b) is an interval in M $(a, b \in M \cup \{\pm\infty\})$ and $f : (a, b) \to M$ is 0-definable then there are $a = a_0 < \cdots < a_n = b$ in $dcl(\emptyset)$ such that whenever $I = (a_i, a_{i+1})$ is long the restriction of f to I is affine.*

Proof. The function f can be assumed to be continuous and strictly increasing (the decreasing case is handled in the same way). The set of all x such that f is affine near x is 0-definable, and therefore there is a 0-definable partition $a = a_0 < \cdots < a_n = b$ such that on each (a_i, a_{i+1}) either f is locally affine (hence, by 4.3, affine on the whole interval) or f is nowhere affine. It is sufficient to see that whenever the latter occurs then the interval must be short. Assume towards a contradiction that f is nowhere affine on (a_i, a_{i+1}) and that the interval is long. Notice that the interval remains long in any elementary extension hence one may assume that $\mathcal{M}$ is sufficiently saturated.

Consider the map $g(x) = f(x+1) - f(x)$, defined on the long interval $J = (a_i, a_{i+1} - 1)$. The function g is continuous and, by our assumption on f, it is positive everywhere. The interval J can be partitioned into finitely many sub-intervals such that g is either constant or strictly monotone on each sub-interval. At least one of those intervals is long.

Claim 4.5. *If $f(x+1) - f(x)$ is constant on a long interval $J' = (a', b')$ then f is affine on a subinterval of J'.*

Proof of Claim Consider the set $Y \subseteq M$ whose elements are all those $y_0 < b' - a'$ such that $f(c_1 + y_0) - f(c_1) = f(c_2 + y_0) - f(c_2)$ for all $c_1, c_2 \in (a', b' - y_0)$. Clearly, $1 \in Y$ and because J' is long, so is every $n \cdot 1$. It follows that Y is infinite and hence contains a nonempty interval (c, d). If one fixes $c_0 \in (c, d)$ and write the elements of (c, d) as $c_0 + t$ for $t \in (c - c_0, d - c_0)$ then for every c_1, c_2 in a suitable interval J'' and for every t sufficiently close to 0, one has

$$f(c_1 + c_0 + t) - f(c_1) = f(c_2 + c_0 + t) - f(c_2)$$

and in particular,

$$f(c_1 + c_0) - f(c_1) = f(c_2 + c_0) - f(c_2).$$

Subtracting the second equation from the first one obtains, for every t sufficiently close to 0,

$$f(c_1 + c_0 + t) - f(c_1 + c_0) = f(c_2 + c_0 + t) - f(c_2 + c_0).$$

Since this is true for every $c_1, c_2 \in J''$ it follows that f is affine on the interval $J'' + c_0$, ending the proof of the claim.

Since f is assumed to be nowhere affine, the function $f(x+1) - f(x)$ is strictly monotone on every long interval in the partition of J. We assume then, without loss of generality, that $f(x+1)-f(x)$ is strictly increasing on J.

Claim 4.6. *There is $d \in D$ such that for every $x \in J$, $g(x) < d$.*

Proof of Claim Indeed, consider the family of maps, $h_x : (0,1) \to M$, $x \in J$, given by $h_x(t) = f(x+t) - f(x)$. This is a definable family of strictly increasing continuous bijections between $(0,1)$ and the interval $(0, g(x))$, hence (clearly, all intervals $(0, g(x))$ are short) by Lemma 3.7, there exists a bound $d \in D$ such that $g(x) < d$ for all $x \in J$, thus proving the claim.

It now follows that the map g, which is injective on J, sends J into the interval $(0, d)$. This is impossible because J is long while $(0, d)$ is short. This ends the proof of the claim and the Lemma. $\square$

Remark 4.7. *Note that two linear functions, defined on the same open interval $I \ni 0$ are equivalent if and only if they agree one at least one nonzero element in their common domain (see for example Proposition 4.1 in* [13]*).*

As in the case for unbounded intervals, one can prove that there is no infinite definable family of non-equivalent linear functions on long intervals:

Lemma 4.8. *If $\mathcal{F} = \{f_s : s \in S\}$ is a 0-definable family of linear functions, $f_s : (0, a_s) \to M$ then there are finitely many 0-definable linear functions $\lambda_1, \dots, \lambda_k$, and a short $b \in dcl(\emptyset)$, such that for every $s \in S$,*
(i) Either $|a_s| < b$, or
(ii) For some $i = 1, \dots, k$, the function f_s is the restriction of λ_i to I_s (in particular, I_s is contained in $dom(\lambda_i)$).

Proof. Fix a family $\mathcal{F}$ as above. First note that if one proves the statement in an elementary extension $\mathcal{N} \succ \mathcal{M}$ then it is true for $\mathcal{M}$ as well, so we may assume that $\mathcal{M}$ is $|T|^+$-saturated.

The equivalence relation on linear functions induces a 0-definable equivalence relation $\sim$ on S and by definable choice there exists a 0-definable set of representatives $S_1 \subseteq S$ for the $\sim$-classes. For $s \in S$, let $I_s = (0, a_s)$.

For every $r \in S_1$, let $J_r = \bigcup_{s \sim r} I_s$, and let $\lambda_r = \bigcup_{s \sim r} f_s$ (this makes sense because of the equivalence). Our goal is to show that there is a finite set $F \subseteq S_1$ such that for all $r \in S_1 \setminus F$, the interval J_r is short. Indeed, if that is proved then, by 3.7 there is an upper bound $b \in M_0$ on the length of all J_r, $r \in S_1 \setminus F$, and therefore $|I_s| < b$ for all $s \sim r \in (S_1 \setminus F)$.

Assume towards contradiction that there are infinitely many $r \in S_1$ for which I_r is long. By continuity arguments (applied to the endpoints of J_s) one may find an infinite definable $S_2 \subseteq S_1$ and a tall ℓ such that for every $r \in S_2$, $(0, \ell) \subseteq J_r$. Since the equivalence class of a linear function is determined by its value at a single non-zero element, it is possible to re-parameterize the family $\{\lambda_r : r \in S_2\}$ by $\lambda_r(\ell)$ and so assume that S_2 is an open interval in M.

Fixing a generic $r_0 \in S_2$ then, by continuity, for r sufficiently close to r_0 the element $a = \lambda_r(\ell) - \lambda_{r_0}(\ell)$ is a short element. The function $\lambda_r(t) - \lambda_{r_0}(t)$ is now a linear function (hence continuous and monotone) sending the long interval $(0, \ell)$ onto the short interval $(0, a)$. Contradiction.

It was therefore shown that for all but finitely many $r \in S_1$, the domain of λ_r is a short interval, whose length is bounded by some short $b \in D$. It is left to see that this finite set of r's is 0-definable. This can be done by considering the 0-definable set of intervals $\{J_r : r \in S_1\}$. If all J_r's are short there is nothing to do. Otherwise, what was shown so far implies that there are only finitely many J_r's of maximal length (possibly infinite). This set is clearly 0-definable so can be omitted, consider the remaining J_r's and repeat the process, until there are no remaining long J_r's in the family. $\square$

Remark 4.9. *In the notation of the last proof, it is possible that S_1 will be infinite, namely that there will be an infinite family of nonequivalent linear maps, all defined on short intervals. This will imply the definability of local multiplication over the group $\langle M, + \rangle$ but does not contradict semi-boundedness.*

The following lemma will not be used in the subsequent arguments. It is included here for a possible future use.

Lemma 4.10. *Assume that $C \subseteq M^{n+1}$ is an open cell, C_1 the projection of C on the first n coordinates. If $F : C \to M$ is a 0-definable function then there are finitely many 0-definable linear functions $\lambda_1, \dots, \lambda_k$, each defined on a long interval, and for every $x \in C_1$, there is a partition of the interval C_x as follows: $a_0(x) < a_1(x) < \cdots < a_r(x)$ (r depending on x), and for every i, either*
(i) The interval $(a_i(x), a_{i+1}(x))$ is short, or
(ii) The function $f_x(y) = F(x, y)$ is affine on $(a_i(x), a_{i+1}(x))$ and the map $t \mapsto f_x(a_i(x) + t) - f_x(a_i(x))$ is the restriction of one of the λ_j's.

Proof. By moving to an elementary extension, we may assume that $\mathcal{M}$ is $|T|^+$-saturated. The initial partition of every C_x is given by Lemma

4.4. As is shown in that proof, the partition of C_x is uniform in x (it partitions C_x according to the points at which $F(x,-)$ is locally affine) and hence there exists a uniform bound on the number of intervals in that partition. For every x, consider all intervals in the partition of C_x on which f_x is nowhere affine. This is a 0-definable family of short intervals, hence by 3.7, there is a short upper bound b on the length of all of these intervals.

The remaining intervals in C_x are those on which f_x is affine and now consider the family of all f_x, restricted to these intervals, as x varies in C_1 (namely, for every $x \in C_1$ there might be finitely many such functions). By translation, one may assume that each such function is linear. Applying 4.8 one obtains finitely many definable linear functions $\lambda_1, \dots, \lambda_k$ and a short element b, such that every interval in this family is either of length less than b or is a restriction of some λ_i, $i = 1, \dots, k$. This implies the lemma. $\square$

5. Changing the language

Assume that $\mathcal{M}$ is semi-bounded and $Th(\mathcal{M})$ is not linear.

Let Λ be the collection of all 0-definable linear functions whose domain is a long interval of the form $(-a_\lambda, a_\lambda)$ (with possibly $a_\lambda = \infty$). For every 0-definable $X \subseteq D^n$ in $\mathcal{M}$, let R_X be an n-place predicate symbol and let $\mathcal{L}_D$ be the collection of all those predicates.

Let
$$\widetilde{\mathcal{L}} = \{<, +, 1\} \cup \mathcal{L}_D \cup \{\lambda : \lambda \in \Lambda\},$$
where each λ is a unary function symbol. Let $\widetilde{\mathcal{M}}$ be the corresponding $\widetilde{\mathcal{L}}$-structure whose universe is M and all other symbols in the language interpreted naturally (with λ taken to be 0 outside $(-a_\lambda, a_\lambda)$).

Obviously, every 0-definable set in $\widetilde{\mathcal{M}}$ is 0-definable in $\mathcal{M}$. The converse is almost true, in the following sense:

Theorem 5.1. *Let $\widetilde{\mathcal{M}}_C$ be the expansion of $\widetilde{\mathcal{M}}$ by a new constant symbol for every element in $dcl_{\mathcal{M}}(\emptyset)$. Then, every 0-definable set in the structure $\mathcal{M}$ is 0-definable in $\widetilde{\mathcal{M}}_C$.*

Proof. This will be done by induction in a usual o-minimal method. It is sufficient to show that every 0-definable $f : U \to M$, where U is an open cell M^n, is 0-definable in $\widetilde{\mathcal{M}}_C$.

Definition 5.2. Let $U \subseteq M^n$ be an open set, $f : U \to M$ a definable function. For $S \subseteq \{1, \dots .n\}$, the function f is *S-bounded* if if for all $i \in S$ there exists $d \in D$ such that $\pi_i(U) \subseteq [-d, d]$ (where π_i is the projection onto the i-th coordinate). In particular, every f is $\emptyset$-bounded.

Note that if $S = \{1, \ldots, n\}$ and f is 0-definable in $\mathcal{M}$ and S-bounded then its domain is contained in D^n and by 3.5, its image is contained in a short interval, so after translation by an element of $dcl_{\mathcal{M}}(\emptyset)$, one sees that the function is 0-definable in $\widetilde{\mathcal{M}}_C$. It is sufficient to prove the following claim:

Claim 5.3. *Let $f : U \to M$ be a 0-definable function in $\mathcal{M}$. If f is S-bounded, for some $S \subsetneq \{1, \ldots, n\}$ and $i \notin S$ then f can be defined using finitely many 0-definable sets in $\widetilde{\mathcal{M}}_C$, together with finitely many 0-definable functions in $\mathcal{M}$ which are $S \cup \{i\}$-bounded.*

Once the claim is proved, then by proceeding to handle the $S \cup \{i\}$-bounded functions one can eventually reach $\{1, \ldots, n\}$-bounded functions, thus proving the theorem.
Proof of Claim 5.3.: Use induction on n:

For $n = 1$: By partitioning $dom f$ in $\mathcal{M}$, we may assume that f is either affine or nowhere affine on its domain. The domain of f is an open interval which is 0-definable hence contains a point $a_0 \in dcl_{\mathcal{M}}(\emptyset)$. If we replace f with $\widetilde{f}(x) = f(a_0 + x) - f(a_0)$ then $\widetilde{f}$ is 0-definable in $\mathcal{M}$, $0 \in dom(\widetilde{f})$ and $\widetilde{f}(0) = 0$.

If $dom\widetilde{f}$ is short then $\widetilde{f}$ is 1-bounded, which implies that it is 0-definable in $\widetilde{\mathcal{M}}$. If $dom\widetilde{f}$ is long then, by 4.4, $\widetilde{f}(x)$ must be affine on some long interval, which implies it is everywhere affine. Since $\widetilde{f}(0) = 0$, it is actually linear and 0-definable (see 4.8) in $\mathcal{M}$, therefore equals $\lambda(x)$ for some $\lambda \in \Lambda$.

In both cases, f is clearly defined using $\widetilde{f}$, $+$, and $a_0 \in dcl(\emptyset)$, hence it is 0-definable in $\widetilde{\mathcal{M}}_C$.

The $n+1$ case: Without loss of generality, $i = n+1 \notin S$. By standard o-minimal methods one may assume the following:
(1) The domain of f is an open cell C in M^{n+1} whose projection in M^n is denoted by C_1:

$$C = \{(x, y) \in C_1 \times M : h_1(x) < y < h_2(x)\},$$

for 0-definable $h_1, h_2 : C_1 \to M \cup \{\pm\infty\}$ such that $h_1 < h_2$ on C_1.
(2) For every $x \in C_1$, the following hold:

(a) The fiber C_x is either M, or of the form $(h_1(x), h_2(x))$ for $h_1(x) \in M$, and $h_2(x) \in M \cup \{+\infty\}$, uniformly in x. (Indeed, if C_x is of the form $(-\infty, b)$ then $f(x, y)$ can be replaced by $f(-x, y)$)).
(b) The function $f_x(t) = f(x, t)$ is continuous and is either constant, strictly increasing in t, or strictly decreasing in t, uniformly in x.
(c) Either, for every $x \in C_1$ the function f_x is nowhere affine, or for every $x \in C_1$ the function f_x is affine on its domain.

By definable choice, there exists a 0-definable $h_0 : C_1 \to M$ such that $h_0(x) \in C_x$ for every $x \in C_1$. By partitioning C_1 (and therefore C) further, we may assume that h_0 is continuous. Now, replace f by

$$\widetilde{f}(x, t) = f(x, h_0(x) + t) - f(x, h_0(x)).$$

The domain of $\widetilde{f}$ is

$$\widehat{C} = \{(x, t) : x \in C_1 \& h_1(x) - h_0(x) < t < h_2(x) - h_0(x)\},$$

hence $(x, 0) \in \widehat{C}$, for every $x \in C_1$. The function $\widetilde{f}$ is still S-bounded and $\widetilde{f}(x, 0) = 0$ for all $x \in C_1$.

By induction, h_0, h_1, h_2, $f(x, 0)$ and $f(x, h_0(x))$ are 0-definable in $\widetilde{\mathcal{M}}_C$. Also, f can clearly be recovered, without parameters, from $\widetilde{f}$ using $h_0(x), h_1(x), h_2(x)$, $f(x, h_0(x))$ and $+$, so it is sufficient to show that $\widetilde{f}$ can be defined using finitely many 0-definable sets in $\widetilde{\mathcal{M}}_C$, together with finitely many 0-definable functions in $\mathcal{M}$ which are $S \cup \{n+1\}$-bounded.

Case 1 For every $x \in C_1$, the function $f_x(t) = f(x, t)$ is nowhere affine.

In this case, by 4.4, every interval $(h_1(x) - h_0(x), h_2(x) - h_0(x))$ is short and hence there exists an upper bound $b \in D$ to the length of all $\widehat{C}_x$. Namely the domain of $\widetilde{f}$ is contained in $C_1 \times (0, b)$, so f is $S \cup \{n+1\}$-bounded.

Case 2 For every $x \in C_1$ the function $f_x(t)$ is affine on its domain.

It follows that every $\widetilde{f}_x$ is linear. By Lemma 4.8, there exists a short element b and there are finitely many functions $\lambda_1, \ldots, \lambda_k \in \Lambda$ such that for every $x \in C_1$, either $|C_x| < b$, or $\widetilde{f}_x$ is a restriction of one of the λ_i's to $\widehat{C}_x$.

By further partitions (using $\lambda_1, \ldots, \lambda_k$), it can be assumed that either for every $x \in C_1$, $\widetilde{f}_x$ is the restriction of some λ_i (same λ_i uniformly in x), or for every $x \in C_1$, $C_x \subseteq (0, b)$.

In the first case, $\widetilde{f}$ is 0-definable in $\mathcal{M}$ using C and functions in Λ, so by induction it is 0-definable $\widetilde{\mathcal{M}_C}$. In the second case, the domain of $\widetilde{f}$ is contained in $C_1 \times (0,b)$ so it is $S \cup \{n+1\}$-bounded. $\square$

Lemma 5.1 shows in particular that if a structure $\mathcal{M}$ has no poles then every definable set is defined using the ordered group structure, global 0-definable linear functions, and finitely many bounded sets (which may include the graphs of linear function on long intervals). This shows that the "no poles" definition of semi-boundedness implies the one from the introduction. The opposite implication is proved using automorphisms (see the proof of Theorem 1.2 in [17]). The equivalence of the two definitions was originally established by Edmundo in [5].

6. Extending partial linear maps to global ones

$\mathcal{M}$ is semi-bounded with a nonlinear theory

By Lemma 5.1, one can assume that $\mathcal{M}$ is an $\mathcal{L}_C$-structure, where

$$\mathcal{L} = \{<, +, 1\} \cup \{R_X \in \mathcal{L}_D\} \cup \{\lambda \in \Lambda\}$$

and C names all elements of $dcl(\emptyset)$.

For $\lambda \in \Lambda$, denote by $\widehat{\lambda}$ the corresponding equivalence class of the linear function, and let $\widehat{\Lambda}$ be the collection of all those equivalence classes. Notice that $\widehat{\Lambda}$ is a ring under point-wise addition. Moreover, because the image of a long interval under a linear function is also long, $\widehat{\Lambda}$ is closed under composition and compositional inverse, therefore it is an ordered division ring. Actually, as in Corollary 9.3 in [19], since $Th(\mathcal{M})$ is not linear, a real closed field R is definable in a neighborhood of 0, and therefore the compositional group $\widehat{\Lambda} \setminus \{0\}$ can be embedded in $GL_1(R)$ which is commutative. It follows that $\langle \widehat{\Lambda}, +, \circ \rangle$ is an ordered field. Let

$$\widehat{\mathcal{L}} = \{<, +, 1\} \cup \{R_X \in \mathcal{L}_D\} \cup \{\widehat{\lambda} \in \widehat{\Lambda}\}.$$

The following result is a variation of Theorem 6.1 from [13].

Theorem 6.1. *There exists an elementary extension $\mathcal{N} \succ \mathcal{M}$ and an $\widehat{\mathcal{L}}$-structure*

$$\widehat{\mathcal{N}} = \langle \widehat{N}, <, +, 1, \{R_X \in \mathcal{L}_D\}, \{\widehat{\lambda} \in \widehat{\Lambda}\} \rangle,$$

with the same universe and same interpretation for $\{<, +, 1\} \cup \mathcal{L}_D$ as $\mathcal{N}$, in which every $\widehat{\lambda}$ is interpreted as a linear map from N to N which extends all corresponding $\lambda \in \Lambda$, and furthermore:
(i) $\widehat{\mathcal{N}}$ eliminated quantifiers, and is o-minimal.
(ii) The set D of short elements of $\mathcal{N}$ forms an elementary substructure of $\widehat{\mathcal{N}}$.

Note that if every element of $\mathcal{M}$ is short then $\Lambda = \widehat{\Lambda}$ and $\widehat{\mathcal{N}} = \mathcal{N}$.

Proof. Consider the following reduct of $\mathcal{M}$:

$$\mathcal{M}_\Lambda = \langle M, <, +, 1, \{\lambda \in \Lambda\}\rangle,$$

. By Theorem 6.1 in [13], there exists an elementary extension $\mathcal{N}_\Lambda \succ \mathcal{M}_\Lambda$, and an ordered vector space

$$V = \langle N_\Lambda, <, +, 1, \{\widehat{\lambda} \in \widehat{\Lambda}\}\rangle$$

over the field $\widehat{\Lambda}$, with the same universe as $\mathcal{N}_\Lambda$, where every partial linear map $\lambda \in \Lambda$ is extended to a global linear map $\widehat{\lambda} : N_\Lambda \to N_\Lambda$. Indeed, although there is a linearity assumption in Theorem 6.1 from [13], the proof itself is done in the setting of an o-minimal expansion of an ordered group by partial linear functions, as given here.

Let

$$\mathcal{L}_0 = \{<, +, 1\} \cup \{\lambda : \lambda \in \Lambda\}$$

and

$$\mathcal{L}_1 = \{<, +, 1\} \cup \{\lambda \in \Lambda\} \cup \{\widehat{\lambda} \in \widehat{\Lambda}\}\,;\ \mathcal{L}_2 = \mathcal{L} = \{<, +, 1\} \cup \{\lambda \in \Lambda\} \cup \{R_X \in \mathcal{L}_D\},$$

and let $\widehat{\mathcal{L}} = \mathcal{L}_1 \cup \mathcal{L}_2$.

Let T_1 be the $\mathcal{L}_1$-theory of V (i.e. with additional names for all partial linear maps from Λ) and let T_2 be the $\mathcal{L}_2(M)$ theory of $\mathcal{M}$ (i.e. with a name for each element of M).

By Robinson's Consistency Theorem, $T_1 \cup T_2$ is consistent, hence there exists an $\widehat{\mathcal{L}}$-structure

$$\widehat{\mathcal{N}} = \langle \widehat{N}, <, +, 1, \{R_X \in \mathcal{L}_D\}, \{\lambda \in \Lambda\}, \{\widehat{\lambda} \in \widehat{\Lambda}\}\rangle,$$

which is an elementary extension of $\mathcal{M}$ as an $\mathcal{L}_2$-structure and is elementarily equivalent to V_Λ as an $\mathcal{L}_1$-structure. In particular, every global linear map $\widehat{\lambda} : \widehat{N} \to \widehat{N}$ extends all the corresponding equivalent partial linear maps $\lambda \in \Lambda$.

Consider the following (see Proposition 5.1 in [13]):

Proposition 6.2. *Let V be an ordered vector space over a field $\widehat{\Lambda}$, $I = [-a, a]$ a closed interval in V, and let*

$$\mathcal{V} = \langle V, <, +, \{\widehat{\lambda} : \lambda \in \widehat{\Lambda}\}, \{P : P \in \mathcal{P}\}\rangle$$

be an expansion of V by some collection $\mathcal{P}$ of subsets of I^n, for various n. Assume also:
(i) $\mathcal{P}$ contains all those a-definable sets in the ordered vector space V.
(ii) $\mathcal{P}$ is closed under definability in I, namely, every 0-definable set in the structure $\mathcal{I} = \langle I, \{P : P \in \mathcal{P}\}\rangle$ is already in $\mathcal{P}$.

Then $\mathcal{V}$ eliminates quantifiers.

Let us see first why this proposition implies that $\widehat{\mathcal{N}}$ is o-minimal. We let D denote the set of all short elements in the $\mathcal{L}_2$-structure

$$\mathcal{N} = \langle \widehat{N}, <, +, 1, \{R_X \in \mathcal{L}_D\}, \{\lambda \in \Lambda\} \rangle$$

(which, recall, is an elementary extension of $\mathcal{M}$).

It is clearly sufficient to consider finitely many predicates from $\mathcal{L}_D$ so, there exists $a \in dcl_{\mathcal{N}}(\emptyset) \cap D$, such that all those R_X's are contained in $I = [-a, a]$ for some $a \in D$.

For $a \in D$, let $I = [-a, a]$ and let $\mathcal{P}_a$ be the collection of all 0-definable subsets of I^n, as n varies, in the o-minimal structure $\mathcal{N}$.

Claim 6.3. *$\mathcal{P}_a$ satisfies assumption (i) and (ii) of Proposition 6.2 (see below), with respect to the ordered vector space structure* $\widehat{V} = \langle \widehat{N}, <, +, 1, \{\widehat{\lambda} \in \widehat{\Lambda}\} \rangle$.

Proof. (i) Every a-definable subset of I^n in $\widehat{V}$ is already in $\mathcal{P}_a$:

The problem is that $\widehat{V}$ has linear functions which do not exist in $\mathcal{N}$. However, by quantifier elimination in ordered vector spaces, every a-definable subset of $\widehat{N}^n$ in the ordered vector space $\widehat{V}$, is a boolean combination of solutions to:

(2)

$$\widehat{\lambda}_1(x_1) + \cdots + \widehat{\lambda}_k(x_k) + \widehat{\lambda}_{k+1}(a) = 0 \ ; \ \widehat{\lambda}_1(x_1) + \cdots + \widehat{\lambda}_k(x_k) + \widehat{\lambda}_{k+1}(a) > 0,$$

for $\widehat{\lambda}_i \in \widehat{\Lambda}$.

Because I is contained in D, for every $x_i \in I$, $\widehat{\lambda}_i(x_i) = \lambda_i(x_i)$ and therefore these equalities and inequalities are already definable in $\mathcal{N}$ and hence belong to $\mathcal{P}_a$.

(ii) Every $\mathcal{I}$-definable set is in $\mathcal{P}_a$: This is clear from the definition of $\mathcal{P}_a$. End of Claim 6.3.

Now that the assumptions of Proposition 6.2 are established, one may conclude that the structure

$$\widehat{\mathcal{N}}_a = \langle \widehat{N}, <, +, 1, \{P : P \in \mathcal{P}\}, \{\widehat{\lambda} \in \widehat{\Lambda}\} \rangle$$

has Quantifier elimination.

Since every $\widehat{\mathcal{N}}$-formula ϕ involves only finitely may predicates R_X, there exists $a \in D$, for which all those R_X are contained in some $[-a, a]^n$. Because $\widehat{\mathcal{N}}_a$ has QE, there exists a quantifier-free $\widehat{\mathcal{N}}_a$-formula ψ which defines the same set as ϕ, and because every predicate in $\mathcal{P}_a$ is already in $\mathcal{L}_D$, the formula ψ is actually an $\widehat{\mathcal{N}}$-formula. It follows that $\widehat{\mathcal{N}}$ eliminates quantifiers.

Let us see why $\widehat{\mathcal{N}}$ is o-minimal. Again, it is sufficient to show:

Claim 6.4. *For every $a \in D$, $\widehat{\mathcal{N}}_a$ is o-minimal.*

Proof. We still use $I = [-a, a]$. By quantifier elimination, every 0-definable set in $\widehat{\mathcal{N}}_a$ is a boolean combination of terms inequalities in the ordered vector space structure, and formulas of the form

$$(t_1(x_1, \dots, x_n), \dots, t_k(x_1, \dots, x_n)) \in X,$$

for some $\mathcal{I}$-definable $X \subseteq I^k$ and $t_1, \dots, t_k$ terms in the ordered vector space language. It is clearly sufficient to handle this last type of formulas, which gives rise to 1-variable formulas:

$$(\widehat{\lambda}_1(x) + a_1, \dots, \widehat{\lambda}_k(x) + a_k) \in X,$$

for $a_1, \dots, a_k \in N$. It may be assumed that none of the $\widehat{\lambda}_i$ is 0. Because $\widehat{\lambda}(x) + a = \widehat{\lambda}(x + \widehat{\lambda}^{-1}(a))$, every such formula defines a set of the form:

$$B = \{x \in \widehat{N} : (\widehat{\lambda}_1(x + b_1), \dots, \widehat{\lambda}_k(x + b_k)) \in X\},$$

for $b_1, \dots, b_k \in \widehat{N}$. Now let

$$A = \{(x_1, \dots, x_k) \in \widehat{N}^k : (\widehat{\lambda}_1(x_1), \dots, \widehat{\lambda}_k(x_k)) \in X\}.$$

Because $X \subseteq I^k$ (and I is short) the set A is also contained in some J^k, for some short J, and therefore definable in the o-minimal $\mathcal{N}$ itself. The set B is now the set of all $x \in \widehat{N}$ such that $(x, \dots, x) \in A - (b_1, \dots, b_k)$. This set is also definable in $\mathcal{N}$ and therefore it is a finite union of intervals.

The structure $\widehat{\mathcal{N}}_a$, and therefore $\widehat{\mathcal{N}}$, is o-minimal. □

For every $\widehat{\lambda} \in \widehat{\Lambda}$, $\widehat{\lambda}(D) \subseteq D$, hence the set $D \subseteq N$ is an $\widehat{\mathcal{L}}$- substructure of $\widehat{\mathcal{N}}$, which is denoted by $\widehat{\mathcal{D}}$. It is left to see that $\widehat{\mathcal{D}}$ is an elementary substructure of $\widehat{\mathcal{N}}$:

This is a repetition of the proof of Theorem 1.2 from [17]. By o-minimality, it is sufficient to prove that $dcl_{\widehat{\mathcal{N}}}(D) = D$. Equivalently, it will be shown that for every $a \in N \setminus D$, there exists an automorphism σ of $\widehat{\mathcal{N}}$, fixing D point-wise, such that $\sigma(a) \neq a$.

Fix $a \in \widehat{N} \setminus D$. Because D is a $\widehat{\Lambda}$-subspace of N, it has a (non-definable) complement D^c in $\widehat{N}$ such that $N = D^c \oplus D$ (ordered lexicographically) as an ordered vector space. If one now takes $\sigma(d) = d$ for every $d \in D$, and $\sigma(y) = 2y$ for every $y \in D^c$ then σ is an automorphism of the ordered vector space V whose fixed elements are exactly the elements of D. Because every other atomic relation of $\widehat{\mathcal{N}}$ is contained in D^n for some n, σ is clearly an automorphism of $\widehat{\mathcal{N}}$ fixing D point-wise and moving a. It follows that $dcl_{\widehat{\mathcal{N}}}(D) = D$ and therefore

$\widehat{\mathcal{D}}$ is an elementary substructure of $\widehat{\mathcal{N}}$. This ends the proof of Theorem 6.1 □

Remark 6.5. *Proposition 6.2 above is exactly Proposition 5.1 from* [13]*. However, it was pointed out by Belegradek,* [1]*, that the proof of that proposition contained a serious gap. The gap was then fixed by Belegradek himself, using an idea of Hrushovski, to yield a similar, but slightly different result. The two results are discussed in Appendix,*

7. Definable groups in semi-bounded structures

There are several papers on definable sets and groups which are definable in o-minimal expansions of ordered groups (rather than real closed fields). The main difficulty there is the lack of a triangulation theorem and therefore the development of the basic topological tools is much more difficult. In [2] and [7] sheaf Cohomology for such structures has started to emerge. In [8] the authors use this Cohomology to give an upper bound for the number of torsion points in abelian definable groups. In [6] other properties of groups in the semi-bounded setting are developed.

Recall that by Pillay's Theorem, [20], every definable group admits a finite definable atlas making it into a topological group. Namely, there exist finitely many definable open subsets of M^n, $\{U_i : i = 1,\ldots,k\}$, together with definable injections $\phi_i : U_i \to G$, such that $G = \bigcup_{i=1}^k \phi_i(U_i)$, the transition maps are continuous, and such that the group operations on G are continuous when read through the charts. The topology induced on G by the atlas is called the group topology of G.

Here is a simple observation:

Lemma 7.1. *If G is a definably compact group in a semi-bounded structure then its universe and all charts must be a bounded set.*

Proof. Because $\mathcal{M}$ is semi-bounded, if G is not bounded then one of its charts U_i is not bounded either. Hence, there exists a definable injection $\sigma : (a, \infty) \to U_i$ whose image is unbounded. Because G is definably compact the curve $\phi_i \circ \sigma(t)$ map has a limit point g in G (in the group topology) as t tends to ∞. This limit point belongs to another chart U_j but now it is easy to obtain a definable injection from an unbounded interval to a bounded set. Contradiction. □

7.1. Definable groups in short models.

Definition 7.2. Let $\mathcal{M}$ be an o-minimal semi-bounded structure which is not linear. $\mathcal{M}$ is called *short* if every element in M is a short element.

It follows that if $\mathcal{M}$ is a short model then every definably compact group in $\mathcal{M}$ is definable in some o-minimal expansion of a real closed field. Indeed, all the charts of G must be bounded so there exists an interval I such that all charts are contained in I^n for some n. Because $\mathcal{M}$ is short I admits a definable real closed field.

This in turn implies, using the (heavy) theorem of Edmundo and Otero [9]:

Corollary 7.3. *If G is a definably compact, definably connected abelian n-dimensional group in a short model then for every $k \in \mathbb{N}$,*

$$Tor_k(G) = (\mathbb{Z}/k\mathbb{Z})^n.$$

7.2. **Uniformity in parameters.** Because not every definable group in o-minimal expansion of group can necessarily be embedded, as a topological group, in M^n (Eleftheriou has recently found an example of a non-embeddeble semi-linear group), there is some subtlety in showing that definable connectedness and definable compactness, with respect to the group topology, are definable properties in parameters.

In this section $\mathcal{M}$ can be any o-minimal expansion of a group.

Lemma 7.4. *Let $\mathcal{M}$ be semi-bounded and let $\{G_s : s \in S\}$ be a uniformly definable family of abelian groups. Then:*
(i) The set of s for which G_s is definably compact is definable.
(ii) The set of s for which G_s is definably connected is definable.

Proof. (i) Without loss of generality every G_s has the same dimension n. Again, by Pillay's theorem, [20], there exists, uniformly in s, a definable family of open subsets of M^n, $\{U_{i,s} : s \in S, i = 1, \dots, k\}$, together with a definable family of bijections $\phi_{i,s} : U_{i,s} \to G_s$, such that $G_s = \bigcup_{i=1}^k \phi_{i,s}(U_{i,s})$ for every $s \in S$, the transition maps are continuous, and such that the group operations on G_s are continuous when read through the charts. By 7.1, it may be assumed that each $U_{i,s}$ is a bounded subset of M^n (the set of $s \in S$ for which all charts $U_{i,s}$ are bounded is clearly definable). Note that by definition of the group topology, each $\phi_{i,s}$ is a homeomorphism between $U_{i,s}$ (in the M^n topology) and its image (in the group topology).

For every $\epsilon > 0$ in M and a definable open $U \subseteq M^n$, let U^ϵ be the set of all elements in U whose distance (using the maximum norm) from the boundary of U is greater than ϵ. This is easily seen to be an open set as well. The following claim is based on an observation of Eleftheriou:

Claim 7.5. *Let G be definable in an o-minimal expansion of a group. Assume that all charts $\{U_i : 1 \leq i \leq k\}$ in the atlas of G are bounded.*

Then G is definably compact if and only if there exists an $\epsilon > 0$ such that

$$G = \bigcup_{i=1}^{k} \phi_i(U_i^\epsilon).$$

Proof. If G is definably compact then the negation of the condition yields a definable curve $\gamma : (0, a) \to G$, such that for every t,

$$\gamma(t) \in G \setminus \bigcup_{i=1}^{k} \phi_i(U_i^t).$$

If $g \in G$ is the limit of $\gamma(t)$ as t tends to 0 (which exists by definable compactness) then for some $i = 1, \ldots, k$, $\phi_i^{-1}(g) \in U_i$, therefore for all sufficiently small $\epsilon > 0$, $\phi_i(g) \in U_i^\epsilon$. This easily leads to a contradiction.

For the converse, if there exists an ϵ as above, then any definable curve γ in G will be eventually contained in one of the $\phi_i(U_i^\epsilon)$, and because U_i^ϵ is bounded the curve $\phi_i^{-1}(\gamma(t))$ has a limit in $x \in M^n$, which must be in U_i. The element $\phi_i(x) \in G$ is the limit of $\gamma(t)$. $\square$*Claim*

Returning now to the lemma, clearly, the ϵ-condition in the above claim is first-order, therefore once all the charts of every G_s are bounded, the set of $s \in S$ for which G_s is definably compact is definable.

(ii) We still use the above notation for the atlas of every G_s. We first make the following general observation: *Assume that X is a topological space, with a finite cover $X = \bigcup_{i=1}^{k} U_i$ by open connected U_i's. Let G_X be the graph on $\{1, \ldots, k\}$ with an edge $\{i, j\}$ if and only if $U_i \cap U_j \neq \emptyset$. Then X is connected if and only if G_X is a connected graph.* The same statement is true if X is a definable space and we replace "connected" by "definably connected".

By replacing each $U_{i,s}$ by its definably connected components (this can be done uniformly in s) we may assume that each chart $U_{i,s}$ is definably connected. Now, the above condition on the connectedness of the graph associated to the cover $\{\phi_{i,s}(U_{i,s}) : 1 \le i \le k\}$ is clearly first-order and uniformly definable in s, so the set of $s \in S$ for which G_s is definably connected is definable. $\square$

7.3. **Torsion of definably compact groups.**

Theorem 7.6. *Let G be a definably compact, definably connected, abelian group in an o-minimal expansion $\mathcal{M}$ of an ordered group. Then for every $k \in \mathbb{N}$, we have*

$$Tor_k(G) = (\mathbb{Z}/k\mathbb{Z})^n.$$

Proof. By Eleftheriou-Starchenko [11], the result holds for groups definable in ordered vector spaces over ordered division rings, and hence for all linear expansions of ordered groups. By Edmundo-Otero the result holds in those expansions which are not semi-bounded (see discussion in Section 2.1)

One may therefore assume that $\mathcal{M}$ is semi-bounded. Consider the structure $\widehat{\mathcal{N}}$ as given in Theorem 6.1, and its elementary sub-structure $\widehat{\mathcal{D}}$ (which is a short model).

The group G is definable in the structure $\mathcal{M}$ and therefore in $\widehat{\mathcal{N}}$, possibly over a finite tuple of parameters s. Namely, $G = G_s$ for some 0-definable family $\{G_s : s \in S\}$ of definable groups, in the structure $\widehat{\mathcal{N}}$. By 7.4, one may assume that for every $r \in S(\widehat{\mathcal{D}})$, the group $G_r(\widehat{\mathcal{D}})$ is definably connected, definably compact abelian group.

Because $\widehat{\mathcal{D}}$ is a short model, given $k \in \mathbb{N}$, for every $r \in S(\widehat{\mathcal{D}})$, $Tor_k(G_r(\widehat{\mathcal{D}})) = (\mathbb{Z}/k\mathbb{Z})^n$. This is clearly a first order property of $\widehat{\mathcal{D}}$, hence it is true in $\widehat{\mathcal{N}}$ as well and in particular for $G = G_s$. $\square$

8. Pillay's Conjecture

As is pointed out in [12] (see Remark 4 at the end of Section 8), the presence of an ambient real closed field is used twice in the proof of Pillay's Conjecture:

1. In order to apply Theorem 2.1 from [18] to a definably compact group G one needs to know that closed subsets of G are closed and bounded. This is true if G can be made affine, which in expansions of real closed field can always be achieved, but false in general. The following idea was suggested by Eleftheriou:

Using Claim 7.5, there are finitely many pairs of bounded open sets $V_1 \subseteq U_1, \dots, V_k \subseteq U_k$, subsets of M^n, such that for each i, $Cl(V_i) \subseteq U_i$ (closure taken in M^n) and such that

$$G = \bigcup_i \phi_i(U_i) = \bigcup_i \phi_i(V_i).$$

Given any closed set $X \subseteq G$, each set $\phi_i^{-1}(X) \cap Cl(V_i)$ is closed and bounded in M^n. As is shown in Lemma 3.10 of [11], this is sufficient in order to apply Theorem 2.1 in [18] and prove the required result:

If $X \subseteq G$ is a definable closed set (with respect to the group topology) and $\mathcal{M}_0$ is a small model then the set of $\mathcal{M}_0$-conjugates of X is finitely consistent if and only if X has a point in $\mathcal{M}_0$.

2. The second, and more substantial, missing ingredient in the proof of Pillay's Conjecture is Theorem 7.6, which is now proved in this setting

as well.

It therefore follows that Pillay's conjecture holds in expansions of ordered groups.

9. SOME OPEN QUESTIONS

9.1. **The structure of definable sets.** In [17] and [5], structure theorems for definable sets in semi-bounded structures are given. The conjecture below is a natural strengthening of those results.

Conjecture 1 *If $\mathcal{M}$ is semi-bounded then every definable subset of M^n can be written as a finite union of sets of the form:*

$$C + \{\Sigma_{i=1}^{k}(\lambda_{i,1}(t_i), \cdots, \lambda_{i,n}(t_i)) : t_1 \in I_1, \ldots, t_k \in I_k\},$$

for a definable $C \subseteq D^n$, $\lambda_{i,j} \in \Lambda$ and $I_1, \ldots, I_k$ long (possibly unbounded) intervals.

9.2. **Definable groups in semi-bounded structures.** It was shown by Edmundo, Eleftheriou, [6], that every definable group in a semi-bounded structure has a definable normal subgroup which is definably isomorphic to $\langle M^n, +\rangle$, such that the quotient is definably isomorphic to a bounded group (namely, a group whose universe is a bounded set in M^n). Because of the above conjectured structure theorem and because definable functions are linear outside short intervals, the following conjecture seems reasonable:

Conjecture 2 *Let G a definable abelian group in a semi-bounded structure. Then there exist a definable group $B \subseteq D^n$ (i.e. B is definable in some o-minimal expansion of a real closed field), a semi-linear group A and a definable extension*

$$0 \to A \to G_0 \to B \to 0,$$

such that G is isomorphic to a definable quotient U/L, for U a $\bigvee$-definable open subset of G_0 and L a finitely generated lattice in U.

The conjecture, if true, will allow to analyze every definable group in an o-minimal expansion of ordered groups in terms of semi-linear groups and groups definable in expansions of real closed fields.

9.3. ***A general transfer principle.*** The arguments used to prove Theorem 7.6 can clearly be used to transfer other results from o-minimal expansions of real closed fields to o-minimal expansions of groups. This suggests a possible general transfer principle between o-minimal expansions of fields and of groups. The following conjecture is modeled after another transfer principle, suggested by L. van den Dries in [4] (and proved false in the original setting):

Let $\phi(R_1, \ldots, R_n, f_1, \ldots, f_k)$ be a sentence in a language $\mathcal{L}$ expanding the language of ordered sets, with $R_1, \ldots, R_n, f_1, \ldots, f_k$ all relation and function symbols that are different than $<$.

Conjecture 3 *Assume that $\phi(R_1, \ldots, R_n, f_1, \ldots, f_k)$ holds in every o-minimal $\mathcal{L}$-expansion of a real closed field, where $<$ is interpreted as the natural ordering of the field.*

Then $\phi(R_1, \ldots, R_n, f_1, \ldots, f_k)$ holds in every o-minimal $\mathcal{L}$-expansion of an ordered group that is not linear, where $<$ is interpreted as the natural ordering.

The arguments presented here show that it is enough to prove the above for short models.

10. Appendix

I now return to Proposition 6.2 (Proposition 5.1 from [13]). As was pointed out in [1], the proof for that theorem contained an error. The error was fixed in Belegradek's paper, using an idea of Hrushovski. However, the new result (Fact 0.1 in [1]), reads as follows:

Fact 10.1. *Let V be an ordered vector space over an ordered division ring D, a a nonnegative element in V and*

$$\mathcal{V} = \langle V, <, +, \{\lambda : \lambda \in D\}; \{P : P \in \mathcal{P}\}\rangle$$

an expansion of V by a collection $\mathcal{P}$ of relations on $I = [-a, a]$. Supposes that every relation on $[-a, a]$ which is a-definable in $\mathcal{V}$ belongs to $\mathcal{P}$. Then the structure $\mathcal{V}$ admits elimination of quantifiers.

To see that Fact 10.1 implies Proposition 6.2 it is left to prove (under the assumptions of 6.2):

Every a-definable subset of I^n in the structure $\mathcal{V}$ is already 0-definable in $\mathcal{I} = \langle I, \{P : P \in \mathcal{P}\}\rangle$.

Proof. We may assume that $\mathcal{V}$ is sufficiently saturated. It is sufficient to prove that every automorphism of $\mathcal{I}$ can be extended to an automorphism of $\mathcal{V}$ which fixes a. Let $\sigma : I \to I$ be such an $\mathcal{I}$-automorphism.

By assumption (i) of 6.2, the relation $< |I^2$ is in $\mathcal{P}$, and therefore σ is order preserving. Let W be the D-linear span of I in V. It is a convex subspace of V and, again by (i) of 6.2, σ can be extended to an ordered-vector-space automorphism of W, which necessarily fixes a. It is not hard to see that σ can now be extended further to an ordered-vector-space automorphism of V, call it σ again. Since $\sigma|I$ preserves all relations from $\mathcal{P}$, it is an automorphism of $\mathcal{V}$, with $\sigma(a) = a$. □

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