Locally definable groups and lattices

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Ravello 2013

Locally definable groups (also knows as $\bigvee$-definable, Ind-definable)

Assume $\mathcal{M}$ is an $\aleph_1$-saturated structure with $\mathcal{M} = \mathcal{M}^{eq}$.

Definition

A locally definable group $\langle \mathcal{G}, \cdot \rangle$ is a countable directed union of definable sets $\mathcal{G} = \bigcup_n X_n \subseteq S$, for some fixed sort S, such that for every m, n, the restriction of multiplication to $X_m \times X_n$ and the restriction of $()^{-1}$ to X_m are definable.

A special case: locally definable, *definably generated groups*

The group $\mathcal{G}$ is generated by a definable symmetric subset $X \subseteq \mathcal{G}$.

$$\mathcal{G} = \bigcup_n \overbrace{X \cdots X}^{n\text{-times}}$$

Examples

• Any countable group G can be realized as a locally definable group in any structure. If G is a finitely generated group then it is also definably generated.

• The commutator subgroup $[G, G]$ of a definable group G is a definably generated subgroup.

In an o-minimal structure, let G be a definable group.

• The universal cover of G is a definably generated group.

• The group of all definable automorphisms of G can be realized as a locally definable group (if $\mathcal{L}$ is countable)

Some explicit examples

• The subgroup of “finite” elements in a non-archimedean real closed field is definably generated by the unit interval: $\mathcal{G} = \bigcup_n (-n, n)$.

• In a non-archimedean abelian group $\langle G, <, +\rangle$, let $a_{n+1} >> a_n > 0$. Then the group $\mathcal{G} = \bigcup_n (-a_n, a_n)$ is locally definable but **not** definably generated.

• Let T be a two dimensional compact real torus and let $X \subseteq T$ be a 1-dimensional line segment of irrational slope. The group $\langle X\rangle$ is a definably generated dense subgroup of T (but not dense in a saturated structure).

Compatible subsets

Definition

Let $\mathcal{G} = \bigcup_n X_n$ be a locally definable group. A subset $\mathcal{X} \subseteq \mathcal{G}$ is called *compatible in* $\mathcal{G}$ if for every definable $Y \subseteq G$, the set $\mathcal{X} \cap Y$ is definable. Equivalently, every $\mathcal{X} \cap X_n$ is definable.

Examples

• Inside the group of "finite" elements $\mathcal{G} = \bigcup_n(-n, n) \subseteq (R, +)$, the group $\mathbb{Z}$ is compatible.

• Inside the group $\mathcal{G} = \bigcup_n(-a_n, a_n)$, $a_{n+1} >> a_n$, there are no compatible 1-generated subgroups.

Definable vs. *locally* definable quotients

Definition

For $\mathcal{H} \subseteq \mathcal{G}$ locally definable groups, we say that the set $\mathcal{G}/\mathcal{H}$ is **(locally) definable** if there exists a (locally) definable set X and a locally definable surjective $\phi : \mathcal{G} \to X$, with $\phi(g_1) = \phi(g_2)$ iff $g_1\mathcal{H} = g_2\mathcal{H}$.

Example

$\langle R, <, + \rangle$ an ordered, divisible, abelian group, $a, b > 0$. let $\mathcal{G}$ be the subgroup of $(R^2, +)$ generated by the rectangle $(-a, a) \times (-b, b)$.

- The group $\mathcal{G}/\mathbb{Z}a$ is locally definable,
- The group $\mathcal{G}/(\mathbb{Z}a \oplus \mathbb{Z}b)$ is definable.

Fact (in o-minimal structures)

If $\mathcal{H} \subseteq \mathcal{G}$ is compatible then $\mathcal{G}/\mathcal{H}$ is locally definable.

Lattice in locally definable groups

Recall (classical setting)

For G a locally compact group, **a lattice in** G is a subgroup $L \subseteq G$ such that (i) L is discrete and (ii) the G-space G/L has **finite** left G-invariant Haar measure.

Definition (model theoretic setting)

Let $\mathcal{G}$ be locally definable, **a lattice in** $\mathcal{G}$ is a subgroup $\Gamma \subseteq \mathcal{G}$ such that
(i) For every definable $X \subseteq \mathcal{G}$, the set $\Gamma \cap X$ is finite (Γ is ~~locally finite~~)).
(ii) $\mathcal{G}/\Gamma$ is a definable set.

Example

If $\mathcal{G}$ is **definable** then the only lattices are the finite subgroups (including the trivial group).
If $\mathcal{G} = \bigcup_{k\in\mathbb{N}}(-k,k)^n \subseteq R^n$ then $\Gamma = \mathbb{Z}^k$ is a lattice.

Fact

Let $\mathcal{G}$ be locally definable. For $\Gamma \subseteq \mathcal{G}$ ~~locally finite~~,
$\mathcal{G}/\Gamma$ is definable iff $\exists$ a definable $Y \subseteq \mathcal{G}$ such that $\Gamma \cdot Y = \mathcal{G}$.

The set Y is "a fundamental set" for Γ.

Proof

$\Rightarrow$:

If $\phi : \mathcal{G} \to X$ is locally definable with X definable and $\Gamma = ker\phi$, then by compactness there exists a definable $Y \subseteq \mathcal{G}$ with $\phi(Y) = X$. Since $ker\phi = \Gamma$, we have $\Gamma \cdot Y = \mathcal{G}$.

$\Leftarrow$:

Assume $\Gamma \cdot Y = \mathcal{G}$. The group Γ is ~~locally finite~~ so $Y^{-1}Y \cap \Gamma$ is finite.

$\Rightarrow$ the relation "$y_1\Gamma = y_2\Gamma$" is definable for $y_1, y_2 \in Y$.

$\Rightarrow$ the set $X = Y/\Gamma$ is definable, and equals $\mathcal{G}/\Gamma$.

$\Rightarrow$ the natural quotient map $\phi : \mathcal{G} \to X$ is locally definable.

From now on $\mathcal{M}$ is o-minimal (See Baro-Otero on the topology of locally definable spaces)

Lattices are finitely generated

If $\mathcal{G}$ is connected (no clopen compatible subset) and Γ is a lattice in $\mathcal{G}$ then Γ is a finitely generated group.

Proof

Let $Y \subseteq \mathcal{G}$ be definable fundamental set, $\Gamma \cdot Y = \mathcal{G}$. The set Y has finitely many "neighbors". Namely, the following set is finite:

$$A = \{\gamma \in \Gamma : \gamma \overline{Y} \cap \overline{Y} \neq \varnothing\} = (\overline{Y})(\overline{Y})^{-1} \cap \Gamma$$

W.l.o, $e \in Y$. We now show that A generates Γ:
Given $\gamma_0 \in \Gamma$, there is a definable path $\sigma \subseteq \mathcal{G}$ connecting e and γ_0.
The path σ passes through $Y, \gamma_1 Y, \ldots, \gamma_k Y = \gamma_0 Y$, with each $\gamma_{i+1} Y$ a neighbor of $\gamma_i Y$, so $\gamma_{i+1}^{-1}\gamma_i \in A$. Hence, $\gamma_0 \in \langle A \rangle$.

Still in the o-minimal setting

We can replace ~~locally finite~~ with compatible, 0-dimensional.

Main Question

Which locally definable groups contain a lattice?

Classical setting

(Borel) Every connected semisimple Lie group contains a lattice.

E.g. $SL(2,\mathbb{Z})$ is a lattice in $SL(2,\mathbf{R})$ and the quotient is $S^3 -$ trefoil knot.

But the solvable group $\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \neq 0, b \in \mathbb{R}\}$ does not contain a lattice.

O-minimal setting, an immediate obstacle

The group $\bigcup_n (-a_n, a_n) \subseteq (R,+)$, $a_{n+1} >> a_n$, does not contain a lattice.

Main Question

A necessary condition

If $\mathcal{G}$ is a locally definable and connected (no compatible clopen subset with respect to the group topology) and $\mathcal{G}$ contains a lattice then it must be definably generated.

Modified Question

Which connected **definably generated** groups contain a lattice?

Lattices and generic sets

Definition

A definable set $Y \subseteq \mathcal{G}$ is **left generic in** $\mathcal{G}$ if boundedly many left translates of Y cover $\mathcal{G}$. Equivalently, every definable set in $\mathcal{G}$ can be covered by finitely many left translates of Y.

Lattice $\Rightarrow$ generic set

If $\mathcal{G}$ contains a lattice then it contains a definable fundamental set Y, $\Gamma \cdot Y = \mathcal{G}$. So $\mathcal{G}$ contains a definable generic set.

Is the converse also true?

Does the existence of a generic set in $\mathcal{G}$ imply the existence of lattice?

Abelian groups

The last question has a positive answer in the abelian case.

Theorem (Elefetheriou-P)

Assume that $\mathcal{G}$ is connected, definably generated and abelian. If $\mathcal{G}$ contains a definable generic set then it contains a lattice, isomorphic to $\mathbb{Z}^k$ with $k \leqslant \dim \mathcal{G}$.

About the proof

If $\mathcal{G}$ contains a generic set then there exists a minimal type-definable normal subgroup of bounded index $\mathcal{G}^{00}$.
The group $\mathcal{G}/\mathcal{G}^{00}$, with the Logic topology, is a real abelian Lie group of dimension at most $\dim \mathcal{G}$.
If $\mathcal{G}/\mathcal{G}^{00}$ is compact then $\mathcal{G}$ is already definable.
Otherwise, $\mathcal{G}/\mathcal{G}^{00}$ contains a standard lattice and one can use it to obtain a lattice in $\mathcal{G}$.

Definably generated groups and generic sets

Conjecture

Every connected, definably generated (abelian) group contains a definable generic set.

Note

A definably generated $\mathcal{G}$ contains a generic set if and only if it has a definable generating "approximate subgroup". i.e. a symmetric set Y and a finite set F, such that $YY \subseteq F \cdot Y$.

The rank of a lattice

As we saw, if an abelian group $\mathcal{G}$ contains a lattice then it can be chosen to be $\cong \mathbb{Z}^k$, with $k \leqslant \dim \mathcal{G}$. Here we have a much stronger result:

Theorem (Berarucci-Edmundo-Mamino)

If $\mathcal{G}$ is a connected, locally definable (not necessarily definably generated!) abelian group and $\Gamma \subseteq \mathcal{G}$ is compatible and 0-dimensional then $\mathit{rank}(\Gamma) \leqslant \dim \mathcal{G}$.
It follows that for every n, the n-torsion group $\mathcal{G}[n]$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^s$, for $s \leqslant \dim \mathcal{G}$.

The proof uses algebraic topology for locally definable groups.

Equivalences of the conjecture for abelian groups

Summarizing the results:

Theorem (EP)

Let $\mathcal{G}$ be a connected, definably generated abelian group.
The following are equivalent:

1. $\mathcal{G}$ contains a lattice.
2. $\mathcal{G}$ contains a definable generic set.
3. The group $\mathcal{G}^{00}$ exists.

By the result of B-E-M, in order to prove that the above are all true, it is enough to prove:

Let $\mathcal{G}$ be connected, definably generated, abelian. Then either $\mathcal{G}$ is **definable** or there exists $g \in \mathcal{G}$ such that $\langle g \rangle$ is infinite and compatible in $\mathcal{G}$.

A simple case: Subgroups of $(R^n, +)$

Theorem (E-P)

Let R be a real closed field. If $\mathcal{G}$ is a definably generated subgroup of $(R^n, +)$, definable in an o-minimal expansion of R, then $\mathcal{G}$ contains a lattice, and all the above properties hold.

The main tool is the following connection to convexity:

Main lemma

Let $X \subseteq R^n$ be a definable symmetric set containing 0. Then there is an n such that

$$\overbrace{X + \cdots + X}^{n \text{ times}}$$

contains the R-convex hull of X.

Lattices, divisibility and convexity

So far we left out another open question about locally definable groups: Recall: Every abelian, connected **definable** group in an o-minimal structure is a divisible group.

Question

Is every connected, locally definable abelian group necessarily divisible? (Conjecture: YES)

Theorem (B-E-M)

Let $\mathcal{G}$ be a connected, definably generated abelian group. Then $\mathcal{G}$ contains a lattice if and only if:
(i) $\mathcal{G}$ is divisible, and
(ii) for every definable $X \subseteq \mathcal{G}$ there exists a definable $Y \subseteq \mathcal{G}$, such that Y contains the "$\mathcal{G}$-convex hull of X".