DEFINABLE ONE DIMENSIONAL STRUCTURES IN O-MINIMAL THEORIES

ASSAF HASSON*, ALF ONSHUUS, AND YA'ACOV PETERZIL**

Abstract. This is the first of two papers where we prove the Zil'ber trichotomy principle for one dimensional structures definable in o-minimal ones.

Here we prove: Let $\mathcal{N}$ be a definable structure in an o-minimal structure $\mathcal{M}$, with $\dim_{\mathcal{M}}(N) = 1$. If $\mathcal{N}$ is stable then it is necessarily 1-based. Along the way, we develop a fine intersection theory for definable curves in o-minimal structures.

The present work is a first step in an attempt to classify structures interpretable in o-minimal theories. Such structures may be stable (e.g., abelian groups, algebraically closed fields of characteristic zero, compact complex manifolds) or unstable (e.g., ordered vector spaces, expansions of real closed fields, simple compact Lie groups). The ultimate goal would be to treat in some uniform manner all of these examples by exploiting the ambient o-minimal environment.

Definition 0.1. A structure $\mathcal{N}$ is *definable* in an o-minimal structure $\mathcal{M}$ if the universe of $\mathcal{N}$, call it N, is an $\mathcal{M}$-definable set and its atomic relations are certain $\mathcal{M}$-definable subsets of N^k, $k \in \mathbb{N}$. The *$\mathcal{M}$-dimension of $\mathcal{N}$*, $\dim_{\mathcal{M}}(\mathcal{N})$, is just the dimension of the universe N in $\mathcal{M}$.

If $\mathcal{N}$ is definable in an o-minimal structure $\mathcal{M}$, quite a few good properties are induced on $\mathcal{N}$. Here is a partial list:

(i) $\mathcal{N}$ has the non-independence property, NIP.
(ii) $\mathcal{N}$ is super-rosy of finite þ-rank, [16].
(iii) $\mathcal{N}$ eliminates the $\exists^\infty$ quantifier: i.e., given an $\mathcal{N}$-definable family of sets $\mathcal{S}$, there exists $k \in \mathbb{N}$ such that $A \in S$ is finite if and only if $|A| \leqslant k$.
(iv) If N is $\mathcal{N}$-minimal (i.e. N has no $\mathcal{N}$-definable infinite subsets of smaller $\mathcal{M}$-dimension) then $\mathrm{acl}_{\mathcal{N}}$ (the algebraic closure in the sense of $\mathcal{N}$) satisfies the exchange property (see [18]).
(v) If $\mathcal{N}$ is stable then it has finite U-rank, with $\mathrm{U}(\mathcal{N}) \leqslant \dim_{\mathcal{M}}(\mathrm{N})$, the o-minimal dimension of N as an $\mathcal{M}$-definable set. [8].

Recall the following (see [14] for more details):

Definition 0.2. Let $\mathcal{N}$ be any structure.

(1) $\mathcal{N}$ is *geometric* if it satisfies:
   - The algebraic closure operator, $\mathrm{acl}_{\mathcal{N}}$, satisfies the exchange property.
   - $\mathcal{N}$ eliminates the $\exists^\infty$ quantifier (see above).
(2) Any geometric structure admits an intrinsic notion of dimension. A *plane curve* in $\mathcal{N}$ is a definable 1-dimensional $C \subseteq N^2$.

*Supported by the EPSRC grant no. EP C52800X 1.
** Partially supported by the EC FP6 through the Marie Curie Research Training Network MODNET (MRTN-CT-2004-512234).

(3) A definable family $\mathcal{F}$ of plane curves is *normal* if any two curves in the family intersect in at most finitely many points. The family $\mathcal{F}$ is *almost normal* if for every $C \in \mathcal{F}$ there are at most finitely many $C' \in \mathcal{F}$ whose intersection with C is infinite.

As will be explained below, this work is concerned with a project of classifying geometric structures interpretable in o-minimal theories. Before we proceed, some clarification of the scope of the work covered in this paper may be in place. First, we observe that from properties (iii) and (iv) above it follows that if $\mathcal{N}$ is definable in an o-minimal structure $\mathcal{M}$, and in addition N is $\mathcal{N}$-minimal then $\mathcal{N}$ is a geometric structure. In particular, if $\mathcal{N}$ is any structure such that there exists some 1-dimensional definition of $\mathcal{N}$ in an thoery o-minimal theory then $\mathcal{N}$ is a geometric structure, though not all geometric structures interpretable in o-minimal theories arise in this way.

Let us now consider the following setting: *Fix $\mathcal{M}$, a sufficiently saturated o-minimal structure with a dense underlying order*, and assume that $\mathcal{N}$ is a geometric structure definable in $\mathcal{M}$. We use $\mathrm{acl}_{\mathcal{M}}$ and $\mathrm{acl}_{\mathcal{N}}$ to denote that algebraic closure operations in the sense of $\mathcal{M}$ and $\mathcal{N}$, respectively.

Our main interest is to extract algebraic information from the geometric structure of $\mathcal{N}$. The idea is to follow Zilber's division of geometric structures $\mathcal{N}$ into three types:

Degenerate: $\mathrm{acl}_{\mathcal{N}}(A) = \bigcup\{\mathrm{acl}_{\mathcal{N}}(a) : a \in A\}$ for all $A \subset \mathcal{N}$.

Linear: Every almost normal definable family of plane curves in $\mathcal{N}$ is 1-dimensional (in the sense of $\mathcal{N}$), but $\mathcal{N}$ is non-degenerate.

Rich: There exists an almost normal 2-dimensional (in the sense of $\mathcal{N}$) definable family of plane curves.

Clearly, degenerate geometries do not allow the existence of infinite definable (or even type-definable) groups. Therefore there is no point looking for algebraic data in degenerate structures. But the Linear/Rich dichotomy has been subject to much research. The underlying thesis, sometimes known as Zilber's Principle, is that linear geometries should arise from pure linear structures (e.g a vector space with no additional structure) and that in contexts of "topological flavour" rich combinatorial geometries arise from the geometry of interpretable fields.

In the stable case, though not in general true (see [13]), Zilber's Principle has been first proved in [15] for strongly minimal Zariski geometries, and extended later to several related contexts. In the o-minimal world a local version of the principle was proved in [19].

A natural conjecture to make in this context is:

Conjecture *If $\mathcal{N}$ is a geometric structure definable in an o-minimal $\mathcal{M}$ then Zilber's Principle holds for $\mathcal{N}$. In particular, if $\mathcal{N}$ is rich then it interprets a field.*

Note that the above conjecture, if true, will cover some remaining open cases of Zilber's original conjecture, for strongly minimal structures interpretable in algebraically closed fields of characteristic zero (See Rabinovich's work [25] on the main cases of this question).

Our present work can be seen as a a first step towards a proof of the above conjecture. We deal with the special case where $\mathcal{N}$ is definable in $\mathcal{M}$ and $\dim_{\mathcal{M}}(N) =$

1 (see [9] and [20] for the investigation of certain cases of definable, stable, two-dimensional structures). We show:

Theorem 1. *Let $\mathcal{M}$ be an o-minimal structure and let $\mathcal{N}$ be a definable in $\mathcal{M}$ with $\dim_{\mathcal{M}}(N) = 1$. If $\mathcal{N}$ is stable, then it is 1-based (and therefore not rich).*

In a subsequent paper, we use the analysis of 1-types to obtain the following complete statement of the Zil'ber Principle for definable 1-dimensional structures in o-minimal ones:

Theorem 2. *Let $\mathcal{N}$ be a 1-dimensional structure definable in an o-minimal one. Then one of the following holds:*

(1) *$\mathcal{N}$ is degenerate.*
(2) *$\mathcal{N}$ is linear: There exists $X \subseteq N^{eq}$ with $\dim_{\mathcal{M}}(X) = 1$ and such that either X is strongly minimal and locally modular (whose $\mathcal{N}$-definable structure arises from a definable vector space) or X is an o-minimal group interval, whose $\mathcal{N}$-definable structure arises from an ordered vector space.*
(3) *$\mathcal{N}$ is rich and interprets a real closed field.*

In [10] Theorem 2 is generalised to *unidimensional* unstable structures interpretable in dense o-minimal structures. This reduces the above conjecture to Zilber's trichotomy conjecture for strongly minimal sets definable in o-minimal theories.

The results of this paper are true for $\mathcal{N}$ *definable* in arbitrary dense o-minimal structures (and by elimination of imaginaries, we may replace "definable" with "interpretable", if $\mathcal{M}$ expands an o-minimal group). However, in order to keep the exposition cleaner, throughout the main part of the work, we will assume that in fact $\mathcal{M}$ expands an o-minimal field. In Appendix B we generalise the results to arbitrary (dense) o-minimal structures.

The structure of the paper Section 1 is quite technical. In it we develop a fine theory of tangency and transversality for curves in definable families. An important ingredient in the whole argument turns out to be the theory of limit sets as developed by v.d. Dries in [6] and we review it here and in Appendix A. In Section 2 we prove that every stable definable 1-dimensional structure is necessarily 1-based and therefore cannot be rich.

Acknowledgement. We wish to thank J. Ramakrishnan for pointing out a gap in the original proof. We would also wish to thank the anonymous referee for his/her thorough reading of the paper and many useful suggestions, which allowed us to substantially change the exposition and thus, we believe, improve it significantly.

1. Some elements of intersection theory

In this section we develop elements of intersection theory which will be used in the proof of Theorem 2. Some parts of this theory were already developed in [19] and we will use several results and proofs from that paper. However, the setting there allowed us to "trim" the original family of curves as the proof progressed, by working locally in regions where the original family is "well-behaved". In the current setting this is impossible (because the ordering is not assumed to be definable in $\mathcal{N}$) and therefore one needs to develop a finer intersection theory. We mainly concentrate on counting the number of intersection points of plane curves.

Although the treatment of the strongly minimal case will focus on reaching a contradiction (from the assumption that the structure is one dimensional and not

1-based) the results proved in this section are true in any o-minimal structure with a family of curves as given below. Some of the results are stated under the assumption that the o-minimal structure expands a field. In Appendix B we will show how to avoid this assumption when we come to prove our main theorem.

We fix an o-minimal structure $\mathcal{M}$ expanding a dense linear ordering. Some statements will depend on $\mathcal{M}$ expanding a real closed field, but this will be explicitly mentioned. The notions of dimension, dependence and genericity here are all with respect to the structure $\mathcal{M}$.

1.1. **Intersections of two curves.** We use the term *curve* to denote any $\mathcal{M}$-definable one-dimensional set. A *plane curve* is a curve in M^2.

Definition 1.1. Given an $\mathcal{M}$-definable plane curve, X, and $p \in X$, we say that X *is C^0 at p* if X is locally, around p, $\mathcal{M}$-homeomorphic to an open interval (we also say that p is a C^0-point of X).

It is not difficult to see that in this case X divides M^2, sufficiently close to p, into two definably connected components, W_1, W_2. Namely, for every sufficiently small rectangular neighbourhood W of p, the set $W \setminus X$ has two definably connected components, one having the same germ at p as W_1 and the other as W_2. We call such W_1, W_2 *local components of $M^2 \setminus X$ at p* (for more on local components, see [21] Section 2.3).

Definition 1.2. If X and Y are two $\mathcal{M}$-definable plane curves and p is a C^0-point on both, we say that X and Y *touch each other at p* (see Figure 1 below) if there are W_1, W_2, local components of $M^2 \setminus X$, such that either $Y \cap U \cap W_1 = \emptyset$ or $Y \cap U \cap W_2 = \emptyset$.

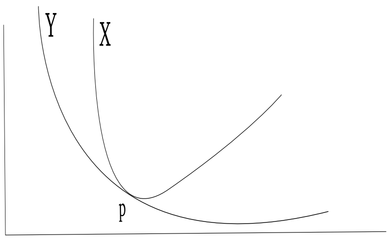


Figure 1:
X touches Y at p from above

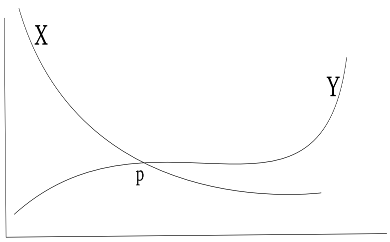


Figure 2:
$X \prec_p^+ Y, Y \prec_p^- X$

It is easy to see that "touching" is a symmetric relation. If $\mathcal{M}$ expands a real closed field and X and Y are C^1-curves then this notion of tangency is stronger than the two curves having the same tangent space at p.

The following definitions come from 2.13 of [19]:

Definition 1.3. Let X and Y be two definable plane curves. Assume that $p = \langle x_0, y_0 \rangle \in X \cap Y$ and that X and Y are graphs of functions, $f(x)$ and $g(x)$, respectively, near p_0. We say that $X \preccurlyeq_p^+ Y$ ($X \prec_p^+ Y$) if there is $x_1 > x_0$ such that $f(x) \leqslant g(x)$ ($f(x) < g(x)$) for all x in the interval (x_0, x_1).

We define $X \preccurlyeq_p^- Y$ ($X \prec_p^- Y$) if there exists $x_1 < x_0$ such that for all $x \in (x_1, x_0)$, we have $f(x) \leqslant g(x)$ ($f(x) < g(x)$). We write $X \preccurlyeq_p Y$ ($X \prec_p Y$) if $Y \preccurlyeq_p^- X$ ($Y \prec_p^- X$) and $X \preccurlyeq_p^+ Y$ ($X \prec_p^+ Y$).

If X and Y are graphs of functions $f(x)$, $g(x)$, C^1 with respect to a definable real closed field, and $f(x_0) = g(x_0)$, $f'(x_0) < g'(x_0)$ then we have $X \prec_p Y$ (see Figure 2)

Notice that, in the setting above, X and Y touch each other at p if and only one of the following holds:
(i) $X \preccurlyeq_p^- Y$ and $X \preccurlyeq_p^+ Y$ (namely, X touches Y "from below") , or
(ii) $Y \preccurlyeq_p^- X$ and $Y \preccurlyeq_p^+ X$ (X touches Y "from above").

1.2. **Normal families of curves.**

Definition 1.4. Recall that a definable family of curves $\mathcal{F} = \{C_q : q \in Q\}$ in M^n is called *normal* if for every $q_1, q_2 \in Q$, the curves C_{q_1} and C_{q_2} intersect in at most finitely many points. We say that *a normal $\mathcal{F}$ has dimension k* if $\dim Q = k$.

A family $\mathcal{F}$ as above is called *almost normal* if for every $q_1 \in Q$ there are at most finitely many $q_2 \in Q$ such that $\mathcal{C}_{q_1} \cap \mathcal{C}_{q_2}$ is infinite. The dimension of $\mathcal{F}$ is by definition $\dim(Q)$.

Note that if $\mathcal{F}$ is almost normal and $q \in Q$ is generic then there exists a neighbourhood $U \ni q$ such that $C_q \cap C_{q'}$ is finite for all $q' \in U$. If we take U definable over generic parameters, then we get by the genericity of q, an open $V \ni q$ such that for all $q_1, q_2 \in V$, $C_{q_1} \cap C_{q_2}$ is finite. Therefore, $\mathcal{F}_V := \{C_q : q \in V\}$ is normal. Since the results of this section are all local in nature, using the above observation, they are all true for almost normal families (with the same proofs). For the sake of clarity, the statements are only given for normal families.

Lemma 1.5. *Let $\mathcal{F} = \{C_q : q \in Q\}$ be a definable, k-dimensional normal family of curves in $P \subseteq M^n$, with $\dim P = 2$ and $k > 1$. Assume that $\mathcal{F}$ is definable over A.*
(i) For every $\langle p, q\rangle \in P \times Q$, if $p \in C_q$ and $\dim(p, q/A) = k + 1$ then p and q are generic over A in P and Q, respectively, and $\dim(p/qA) = 1$.
(ii) If $P \subseteq M^2$, $p = \langle x_0, y_0\rangle$ and $\dim(p, q) = k + 1$ then there exist open intervals I and J, around x_0 and y_0, respectively, such that $C_q \cap I \times J$ is the graph of a continuous, strictly monotone function.

Proof. (i) We assume that $p \in C_q$ and $\dim(p, q/A) = k+1$. Because for every $q \in Q$ we have $\dim(C_q) = 1$, it follows that $\dim(q/A) = k$, and hence $\dim(p/qA) = 1$. Next, we claim that $\dim(q/pA) = k-1$: Indeed, if not then we must have $\dim(q/pA) = k$, namely q generic in Q over pA. It follows that there exists a definable open $U \ni q$ such that for every $q' \in Q' \cap U$, we have $p \in C_{q'}$. Choosing the parameters defining U to be sufficiently independent we have p still generic in C_q over all these parameters and therefore there exists an infinite subset of C_q which is contained in $C_{q'}$, for every $q' \in Q' \cap U$. This contradicts the normality of $\mathcal{F}$. Therefore $\dim(q/pA) = k - 1$.

By the dimension formula, we have $\dim(p/A) = 2$.

(ii) Because p is generic in C_q, the curve C_q is the graph of a continuous function f_q near p (either in x or in y). If this function were locally constant, say as a function of x, then, by the genericity of q we would get a k-dimensional set of f_q locally constant near x_0. Because $k > 1$ we would get an infinite set of q' such that $f_{q'}$ all agree on a whole interval, contradicting normality. □

When $\mathcal{F}$ is a fixed family of curves parameterised by Q we will sometimes write $q_1 \preccurlyeq_p q_2$ instead of $C_{q_1} \preccurlyeq_p C_{q_2}$.

1.3. **Nice families of curves.** Before proceeding to the next lemma we prove a definable version of Ramsey's Theorem for definable binary relations in o-minimal structures:

Fact 1.6. *Let R be a definable binary relation on an interval $I \subseteq M$. Assume that some generic $x_0 \in I$ belongs to the closure of $\{y \neq x_0 : R(x_0, y)\}$.*

Then, there exists an interval $J \ni x_0$ such that for every $x < y$ in J, we have $R(x, y)$, or for every $y < x \in J$ we have $R(x, y)$. In particular, if R is symmetric then there exists an interval $J \ni x_0$ such that for every $x \neq y$ in J we have $R(x, y)$.

Proof. Without loss of generality $I = [a, b]$. For every $x \in I$, let

$$s_1(x) = \inf\{y \in [a, b] : y < x \,\&\, \forall y' \in (y, x)\, R(x, y')\}$$

and let

$$s_2(x) = \sup\{y \in [a, b] : y > x \,\&\, \forall y' \in (x, y)\, R(x, y')\}.$$

Our assumption on x_0 implies that either s_1 or s_2 are defined in some neighbourhood of x_0 and we either have $s_1(x_0) < x_0$ or $s_2(x_0) > x_0$. Assume first that $s_2(x_0) > x_0$. Because x_0 is generic, s_2 is continuous at x_0 and we can therefore find an interval $(c, d) \ni x_0$, such that for every $x \in (c, d)$, $s_2(x) > d$. By the definition of s_2 this in turn implies that for all $x < y$ in (c, d), we have $R(x, y)$.

If $s_1(x_0) < x_0$ then we get the other clause holds. If R is symmetric then we clearly have $R(y, x)$ as well. □

Given a definable family of curves $\mathcal{F} = \{C_q : q \in Q\}$ all in P, we are going to make extensive use of the dual family $\mathcal{L} = \{\ell_p : p \in P\}$, where

$$\ell_p = \{q \in Q : p \in C_q\}.$$

We will later formulate conditions under which ℓ_p is a 1-dimensional set. However, it is easy to see that if $\dim(P) = \dim_{\mathcal{M}}(Q) = 2$ and $\mathcal{F}$ is a normal family then for generic $p \in P$, we have $\dim(\ell_p) \leqslant 1$.

Lemma 1.7. *Let $\mathcal{F} = \{C_q : q \in Q\}$ be a $\emptyset$-definable two-dimensional normal family of curves in $P \subseteq M^2$. Assume that $p_0 \in C_{q_0}$ and that $\dim(p_0, q_0/\emptyset) = 3$. Then there exist a neighbourhood $U \times W \subseteq M^2 \times Q$ of $\langle p_0, q_0 \rangle$ such that for all $q_1, q_2 \in W$ the curves C_{q_1} and C_{q_2} intersect at most once in U and they do not touch each other at their point of intersection.*

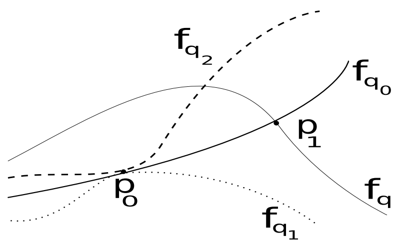
Proof. Assume that p_0 and q_0 are as above. By Lemma 1.5 (i), p_0 and q_0 are generic in P and Q, respectively. We may therefore assume that $Q \subseteq M^2$. Then, by possibly shrinking P and Q, we may also assume that for every $q \in Q$ the curve $C_q \subseteq P$ is the graph of a continuous partial function f_q from M into M and that f_q vary continuously with q.

Claim 1 For all $q \neq q_0$ sufficiently close to q_0 the curve C_q does not touch C_{q_0} at p_0.

Consider the set ℓ_{p_0}. Because p_0 is generic in P and $\dim(q_0/p_0) = 1$ (Lemma 1.5 (i)) the set ℓ_{p_0} is one-dimensional and q_0 is generic in ℓ_{p_0} over p_0. Hence, we may assume for the following that ℓ_{p_0} is an interval in M. Consider the symmetric relation on ℓ_{p_0}: $R(q_1, q_2)$ iff C_{q_1} touches C_{q_2} at p_0.

If the claim fails then the relation R satisfies the assumption of Fact 1.6 (with q_0 for x_0). Hence, there exists a one-dimensional set $Q_{p_0} \subseteq \ell_{p_0}$ containing q_0 and definable over p_0 and other independent parameters, such that $C_{q'}$ touches C_{q_0} at p_0 for every $q' \in Q_{p_0}$. Now, by varying p_0, we may find an open set U containing p_0 and a relatively open $Q_1 \subseteq Q$ such that for every $p \in U$ and every $q_1, q_2 \in Q_1$, if $p \in C_{q_1} \cap C_{q_2}$ then C_{q_1} and C_{q_2} touch each other at p. We may assume that U and Q_1 are 0-definable (by choosing them to be definable over sufficiently generic and independent parameters). Let us show that this contradicts the fact that Q is two-dimensional and $\mathcal{F}$ is normal:

Let $p_0 = \langle x_0, y_0 \rangle$ and $Q_0 = \{q \in Q_1 : f_q(x_0) = y_0\}$. Choose $x_1 > x_0$ generic in $\operatorname{dom} f_{q_0}$ (over all mentioned parameters) and $y_1 = f_{q_0}(x_1)$ such that $p_1 := \langle x_1, y_1 \rangle \in U$. Since y_1 is generic in M over $p_0 q_0$, there are $q_1, q_2 \in \ell_{p_0} \cap Q_0$ such that $f_{q_1}(x_1) < y_1 < f_{q_2}(x_1)$. If we now take any $q \in Q_1$ such that $f_q(x_0) \neq y_0$ then necessarily $f_q(x_1) \neq y_1$, for otherwise the graph of f_q will have to cross (and therefore not touch) either C_{q_1} or C_{q_2} in the interval (x_0, x_1) (see figure below). By the normality of $\mathcal{F}$ and the genericity of $\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle$ there are at most finitely many $q \in Q_1$ such that both points are in C_q. So there are at most finitely many $q \in Q_1$ such that $\langle x_1, y_1 \rangle \in C_{q_1}$, in contradiction to the fact that $\mathcal{F}$ is two-dimensional. End of Claim 1.



f_q crosses f_{q_2} between p_0 and p_1.

Claim 2 There are definable open $U \subseteq P$ and $W \subseteq Q$ such that for every $q_1 \neq q_2 \in W$ and for every $p \in C_{q_1} \cap C_{q_2} \cap U$, the curves C_{q_1} and C_{q_2} do not touch each other at p.

Indeed, by Claim 1 there is a relatively open $W' \subseteq Q$ such that for every $q \neq q_0$ in W', if $p_0 \in C_q \cap C_{q_0}$ then C_q and C_{q_0} do not touch each other at p_0. By choosing the parameters defining W' sufficiently independent, we may assume that W' is 0-definable (adding the generic parameters to the language). By the genericity of q_0 in Q_{p_0}, we may find an open $W \subseteq Q$, $q_0 \in W \subseteq W'$, such that for every $q', q'' \in W$, if $p_0 \in C_{q'} \cap C_{q''}$ then the two curves do not touch each other at p_0. Again, we may assume that W is 0-definable. Finally, we may use the genericity of p_0 to obtain the desired U. End of Claim 2.

Claim 3 There exist open neighbourhoods $U \ni p_0$, $W \ni q_0$ such that for all $q_1, q_2 \in W$, the curves C_{q_1}, C_{q_2} intersect at most once in U.

The argument below is extracted from Lemma 4.3 in [19], so we will be brief. We still assume that $Q \subseteq M^2$. By Claim 2 (after possibly shrinking P and Q) we may also assume that for all $p \in P$ and $q_1, q_2 \in Q$, if C_{q_1}, C_{q_2} intersect in $p \in U$ then either $q_1 \preccurlyeq_p q_2$ or $q_2 \preccurlyeq_p q_1$.

We may find open definable neighbourhoods W of q_0 and U of p_0 such that for every $p \in U$ and for every $q_1 = \langle a_1, b_1 \rangle, q_2 = \langle a_2, b_2 \rangle \in W$ with $p \in C_{q_1} \cap C_{q_2}$, we have, without loss of generality: $q_1 \preccurlyeq_p^+ q_2$ if and only if $a_1 \leqslant a_2$. In particular, the $\preccurlyeq_p^+$-ordering depends only on q_1, q_2 and not on p.

We claim that these U and W are the desired neighbourhoods. Indeed, if C_{q_1} and C_{q_2} intersect more than once in U, for $q_1, q_2 \in W$, then take two consecutive points of intersections p_1 and p_2. Notice that if $q_1 \preccurlyeq_{p_1}^+ q_2$ then, because we assumed that C_{q_1} and C_{q_2} do not touch each other, we necessarily have $q_2 \preccurlyeq_{p_2}^+ q_1$. But this contradicts the fact that the $\preccurlyeq_p^+$-ordering depends only on q_1 and q_2 (and not on p). □

The following is a variation of a similar definition from [19].

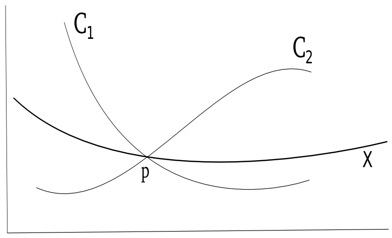
Definition 1.8. Let $\mathcal{F} = \{C_q : q \in Q\}$ be a definable normal family of plane curves, all contained in $U \subseteq M^2$, where Q is an open subset of M^2. We say that $\mathcal{F}$ is *a nice family* if the following hold:
(i) For every $q_1, q_2 \in Q$, C_{q_1} and C_{q_2} intersect at most once in U.
(ii) There exists an interval $I \subseteq M$ such that for every $q \in Q$, the curve C_q is the graph of a function $f_q : I \to M$.
(iii) The partial function $F(a, b, x)$ which sends $\langle a, b, x\rangle \in Q \times I$ to $f_{\langle a,b\rangle}(x)$ is continuous in all variables and strictly monotone in each of its variables. Moreover, in each of the variables, F has the same monotonicity behaviour, as the other two variables vary. E.g., if $F(a_1, b_1, -)$ is strictly increasing in the last variable, for some $\langle a_1, b_1\rangle \in W$ then for all $\langle a, b\rangle \in W$, the function $F(a, b, -)$ is strictly increasing.

It is not hard to see that if $\mathcal{F}$ is a nice family then every dual curve ℓ_p, $p \in M^2$, has dimension at most one, and for p sufficiently generic, $\dim(\ell_p) = 1$.

Corollary 1.9. *Let $\mathcal{F} = \{C_q : q \in Q\}$ be a 0-definable normal family of plane curves, where Q is a two-dimensional subset of M^2. Assume that $p_0 \in C_{q_0}$, with $\dim(p_0, q_0/\emptyset) = 3$. Then there exists an open neighbourhood $U \times W$ of $\langle p_0, q_0\rangle$ such that the family $\mathcal{F}_{U,W} = \{C_q \cap U : q \in W\}$ is nice.*

Proof. Clause (i) of the definition of a nice family follows from Lemma 1.7, while the other clauses follow from genericity of p_0, q_0. □

Definition 1.10. Given a nice family of plane curves $\mathcal{F} = \{C_q : q \in Q\}$ and a definable curve $X \subseteq M^2$ we say that X *is $\mathcal{F}$-bounded at* p if there are $q_1, q_2 \in Q$ such that $p \in C_{q_1}, C_{q_2}$ and $C_{q_1} \preccurlyeq_p X \preccurlyeq_p C_{q_2}$ (see figure).



X is bounded at p

The following is just Theorem 10.5 from [19]. The proof there uses no more than our definition of a nice family.

Fact 1.11. *Let $\mathcal{F}$ be a nice family of curves in $U \subseteq M^2$ and let $X \subseteq U$ be a definable curve that is $\mathcal{F}$-bounded at every point. If p is generic in X then there exists a unique curve from $\mathcal{F}$ which touches X at p.*

In particular, if $X = C_q$ then the only curve from $\mathcal{F}$ that touches C_q at p is C_q itself.

Corollary 1.12. *Let $\mathcal{F}$ be a two-dimensional normal family of plane curves. Let $X \subseteq M^2$ be an A-definable curve, p generic in X (over A) and also generic in M^2 (over $\emptyset$). Then there are at most finitely many curves from $\mathcal{F}$ touching X at p.*

Proof. If there were infinitely many touching curves in $\mathcal{F}$ we could find $q \in Q$ such that $\dim(q/pA) = 1$, $\dim(p, q/\emptyset) = 3$ and C_q touches X at p. By Lemma 1.9, there exists a neighbourhood $U \times W$ of $\langle p, q\rangle$ such that $\mathcal{F}_{U,W} = \{\langle p, q\rangle \in U \times W : p \in C_q\}$ is nice. Furthermore, we can choose U to be definable over independent parameters. It follows from 1.11 that $q \in \mathrm{dcl}_{\mathcal{M}}(Ap)$, contradiction. □

Definition 1.13. Let $\mathcal{F}$ be a normal family of curves and X a definable curve in M^2. For p a C^0-point of X, let us denote by $\tau_{\mathcal{F}}(X, p)$ the set of all $q \in Q$ such that C_q touches X at p.

We let

$$\tau_{\mathcal{F}}(X) = \bigcup\{\tau_{\mathcal{F}}(X, p) : p \text{ a } C^0\text{-point of } X\} \subseteq Q.$$

We say that X is *$\mathcal{F}$-linear near p* if there exist a neighbourhood U of p and $q \in Q$ such that for every $p' \in U \cap X$, we have $\tau_{\mathcal{F}}(X, p') = \{q\}$.

When the context is clear we suppress the subscript $\mathcal{F}$ and just use the notation $\tau(X, p)$ and $\tau(X)$.

Observe that if X is $\mathcal{F}$-linear then there exists a curve $C_q \in \mathcal{F}$ touching X at every point in $U \cap X$ in some open set U and therefore $U \cap X = U \cap C_q$. Thus, if $\mathcal{F}$ is a nice family then X is $\mathcal{F}$-linear near a generic p if and only if X coincides, near p, with some C_q in $\mathcal{F}$. If $\mathcal{F}$ is not assumed to be nice then we just have the "only if" direction. Namely, it is possible that $C_q \in \mathcal{F}$ has curves from $\mathcal{F}$ other than itself touching it at p, in which case $X = C_q$ will not be $\mathcal{F}$-linear.

Assume that $\mathcal{F}$ is a nice family of curves, X is $\mathcal{F}$-bounded and not $\mathcal{F}$-linear near any of its points. By 1.11, for every generic $p \in X$ the set $\tau(X, p)$ contains a single element (which is, as X is not $\mathcal{F}$-linear, inter-definable with p over A). It follows that for all but finitely many $p \in X$, we have $|\tau(X, p)| = 1$ and for all other $p \in X$, $\dim(\tau(X, p)) \leqslant \dim \ell_p \leqslant 1$ (see above discussion). Hence, $\tau(X)$ is a one-dimensional set.

By Corollary 1.12 this would still be true for normal $\mathcal{F}$, provided that every ℓ_p, $p \in X$, is at most one-dimensional. Furthermore, under the same assumption, if $p \in X$ is generic and $q \in \tau(X, p)$ then, by 1.12, the element q is generic in $\tau(X)$ over A.

Infinitesimals It is useful for much of what is coming next to introduce the notion of infinitesimals: Given $p \in M^n$, we denote by ν_p the *infinitesimal neighbourhood of* p, defined as follows: Given $\mathcal{M}^*$ an $|M|^+$-saturated elementary extension of $\mathcal{M}$, ν_p is the intersection of all $\mathcal{M}$-definable open neighbourhoods of p in $\mathcal{M}^*$. Notice that for $a = \langle a_1, \ldots, a_n\rangle \in M^n$, we have $\nu_a = \nu_{a_1} \times \cdots \times \nu_{a_n}$.

Even though the set ν_p is not definable, most of the statements involving ν_p appearing in this paper can be restated in a first order manner. For example, the statement "For every $q \in \nu_{q_0}$ the set $X \cap C_q \cap \nu_{p_0}$ is nonempty" in the statement of the next lemma is equivalent to "There exist neighbourhoods U of p_0 and W of q_0 such that for all $q \in W$, the set $X \cap C_q \cap U$ is nonempty".

Lemma 1.14. *Let $\mathcal{F}$ be a nice family of plane curves in $U \subseteq M^2$, $X \subseteq U$ an A-definable curve that is $\mathcal{F}$-bounded everywhere. Assume that $p_0 \in X$ is generic over A, that X is not $\mathcal{F}$-linear near p_0 and that C_{q_0} touches X at p_0. Then for all sufficiently small local components W_1, W_2 of $M^2 \setminus \tau(X)$ at q_0, there is $i \in \{1, 2\}$, such that the following are true:*

(1) *Given any $q \in W_i \cap \nu_{q_0}$, the set $C_q \cap X \cap \nu_{p_0}$ is empty.*
(2) *Given any $q \in W_{3-i} \cap \nu_{q_0}$, the set $C_q \cap X \cap \nu_{p_0}$ contains at least two points p_1, p_2. Moreover, p_1, p_2 can be chosen so that $X \preceq_{p_1} C_q$ and $C_q \preceq_{p_2} X$.*

Proof. Write $p_0 = \langle x_0, y_0 \rangle$, $q_0 = \langle a_0, b_0 \rangle$. As discussed above, $q_0 \in \tau(X)$ is generic over A so in particular it is a C^0 point of $\tau(X)$, and $\tau(X)$ is the graph of a definable function (with respect to one of the coordinates, say the first one). Consider the continuous function $f_{\langle a,b \rangle}(x) = F(a, b, x)$, as given by the definition of nice families. Without loss of generality the domain of F is a product of intervals $I_1 \times I_2 \times I_3$ and its range is contained in the interval I_4. Since p_0 is generic in X, the set X itself is also, locally near p_0, the graph of a function, and without loss of generality we assume it is a function of x, call it $g(x)$. (If X is not a function of x, then, by genericity it is locally of the form $x = c$; in this case, notice that since $\mathcal{F}$ is nice, we can interchange the roles of x and y in $\mathcal{F}$ and consider it as a family of functions of y).

There are several cases to consider, but we will handle only one of them (the rest can be handled in a similar way). We assume that C_{q_0} touches X from above at p_0. Namely, for all x near x_0, we have $f_{q_0}(x) \geqslant g(x)$. Because p_0 is generic in X and C_{q_0} is the unique touching curve from $\mathcal{F}$ at p_0, there is a definable neighbourhood U of p_0 such that for all $p \in U \cap X$, if $C_q \in \mathcal{F}$ is the unique curve touching X from above at p, then $C_q \cap U$ lies above $X \cap U$. By shrinking either U or the above intervals, we may assume that $U = I_3 \times I_4$.

Let us also assume (by the definition of nice families) that $F(a, -, x)$ is strictly increasing in the second variable in I_2, for all $\langle a, x \rangle \in I_1 \times I_3$.

We let $W = I_1 \times I_2$ and by shrinking it, if needed, we may assume that $W \cap \tau(X) \subseteq \tau(U \cap X)$.

It follows that for every $\langle a, b \rangle \in W \cap \tau(X)$, and every $\langle a, b_1 \rangle \in W$, with $b_1 > b$, we have for every $x \in I_3$,

$$F(a, b_1, x) > F(a, b, x) \geqslant g(x),$$

(the last inequality following from our assumptions that the curves touching X in U lie above $X \cap U$). In particular, the curve C_{a,b_1} lies strictly above the curve X inside of U, and therefore the two curves do not intersect in ν_{p_0}.

So, in the case we just considered, the local component W_1 of $M^2 \setminus \tau(X)$ which lies above the curve $\tau(X)$ satisfies: For all $q_1 \in W_1 \cap \nu_{q_0}$, $C_{q_1} \cap X \cap \nu_{p_0} = \emptyset$ (see Figures 3,4, with C_{q_1} for C_{ab_1}).

Let us consider the local component W_2 which lies below $\tau(X)$. We need to show that for every $q \in \nu_{p_0} \cap W_2$, there are $p_1 \neq p_2$, both in $C_q \cap X \cap \nu_{p_0}$, such that $X \preceq_{p_1} C_q$ and $C_q \preceq_{p_2} X$.

By a standard compactness argument, it is sufficient to prove: For every definable neighbourhood U_1 of p_0 there exists a neighbourhood $V \subseteq Q$ of q_0 such that for every $q \in V \cap W_2$, there are $p_1 \neq p_2$ both in $C_q \cap X \cap U_1$ such that $X \preceq_{p_1} C_q$ and $C_q \preceq_{p_2} X$.

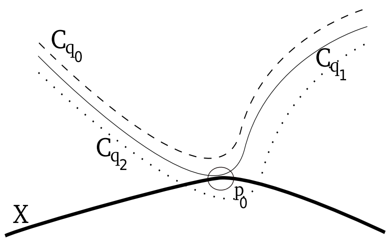


Figure 3:
The P universe

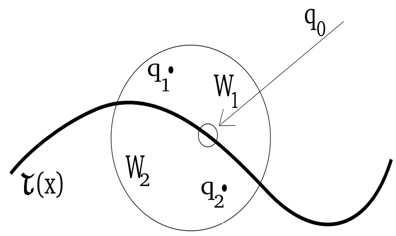


Figure 4:
The Q universe

We fix $U_1 \subseteq U$, $U_1 = J_1 \times J_2$ and fix $z_1, z_2 \in J_1$, $z_1 < x_0 < z_2$, such that $f_{a_0,b_0}(z_1) > g(z_1)$ and $f_{a_0,b_0}(z_2) > g(z_2)$ (this can be done because $C_{a_0,b_0} \cap U$ lies above $X \cap U$; we are using here the non-$\mathcal{F}$-linearity of X). Let J_1' be the interval (z_1, z_2) and $U_1' = J_1' \times J_2$.

We now choose a neighbourhood $W' \subseteq W$ of q_0 so that:

a: $W' \cap \tau(X) \subseteq \tau(U_1' \cap X)$

b: for every $\langle a, b_1 \rangle \in W_2 \cap W'$ there exists $\langle a, b \rangle \in \tau(X) \cap W'$, with $b > b_1$ (this uses the definition of W_2 and continuity of g at a_0), and

c: for every $q \in W'$, $f_q(z_1) > g(z_1)$ and $f_q(z_2) > g(z_2)$ (this uses the continuity of F).

We now pick an arbitrary $\langle a, b_1 \rangle \in W_2 \cap W'$. By (a),(b) above, there exists $b > b_1$, with $\langle a, b \rangle \in \tau(X) \cap W'$ and C_{ab} touching X at some $\langle x, g(x) \rangle \in U_1'$. By the monotonicity assumption on F, we have $F(a, b_1, x) < F(a, b, x) = g(x)$.

Put together with (c) above we have

$$f_{ab_1}(z_1) > g(z_1) \ , f_{ab_1}(x) < g(x) \ \text{ and } f_{ab_1}(z_2) > g(z_2).$$

By the intermediate value theorem, there are $x_1 \in (z_1, x)$, $x_2 \in (x, z_2)$ with

$$f_{ab_1}(x_1) = g(x_1) \ \text{ and } f_{ab_1}(x_2) = g(x_2).$$

Moreover, we can choose x_1, x_2 such that, for $p_1 = \langle x_1, g(x_1) \rangle$ and $p_2 = \langle x_2, g(x_2) \rangle$, we have

$$C_{ab_1} \preceq_{p_1} X \text{ and } X \preceq_{p_2} C_{ab_1},$$

as needed. This completes the proof of the theorem. □

Lemma 1.15. *Let $\mathcal{F}$ be a nice family. Let $X \subseteq M^2$ be an A-definable curve, $p_0 \in X$ generic over A. If there is a curve from $\mathcal{F}$ touching X at p_0 then X is $\mathcal{F}$-bounded near p_0.*

Proof. First note that it is sufficient to prove: There are $p_1, p_2 \in \nu_{p_0}$ and $q_1, q_2 \in Q$ such that $C_{q_1} \preceq_{p_1} X$ and $X \preceq_{p_2} C_{q_2}$. Indeed, because the sets

$$\{p \in X : \exists q \in Q \ \ X \preceq_p C_q\} \ , \ \{p \in X : \exists q \in Q \ \ C_q \preceq_p X\}$$

are A-definable, if they contain an element in ν_{p_0} then they contain an A-definable neighbourhood of p_0 in X.

By Lemma 1.14 (2), the two sets indeed intersect ν_{p_0}, as needed. □

Lemma 1.16. *Let $\mathcal{F}$ be a nice family of curves, X an A-definable curve, $p \in X$ generic over A and $q_0 \in Q$. If $p \in C_{q_0} \cap X$ then one and only one of the following holds:*
(i) C_{q_0} touches X at p, or
(ii) For every $q \in \nu_{q_0}$, we have $|C_q \cap X \cap \nu_p| = 1$.

Proof. Notice that by 1.14, clauses (i) and (ii) cannot hold simultaneously. Assume that C_{q_0} does not touch X at p.

Then C_{q_0} intersects both local components of $M^2 \setminus X$ in ν_p. By continuity, $C_{q'}$ intersects both components of $M^2 \setminus X$ in ν_p for all $q' \in \nu_{q_0}$. By the definable connectedness of $C_{q'}$ (or equivalently, the continuity of the function $f_{q'}$), every such curve $C_{q'}$ must intersect X in ν_p at least once.

Assume now, towards contradiction, that there exists $q \in \nu_{q_0}$ such that $C_q \cap X \cap \nu_p$ contains more than one point. We claim that X is $\mathcal{F}$-bounded near p. There are two possibilities to consider:

(i) C_q touches X at one or more of these points of intersection. Every point in $\nu_p \cap X$ is generic in X hence, by Lemma 1.15 above, X is $\mathcal{F}$-bounded in some open subset of ν_p. Because of the genericity of p (over the parameters defining $\mathcal{F}, X$), the curve X is $\mathcal{F}$-bounded in a neighbourhood of p.

(ii) C_q does not touch X at any of their points of intersection. In this case, as we observed before, we can find $p_1, p_2 \in \nu_p$ distinct points of intersection of X with C_q such that $C_q \preccurlyeq_{p_1} X$ and $X \preccurlyeq_{p_2} C_q$. As pointed out in the proof of 1.15, this implies that X is $\mathcal{F}$-bounded in a neighbourhood of p.

Since X is $\mathcal{F}$-bounded, by 1.11 there exists a unique $q_1 \in Q$ such that C_{q_1} touches X at p. Since, by assumption, $q_1 \neq q_0$ and since $q_1 \in \mathrm{dcl}(p, A)$, we have $q_1 \notin \nu_{q_0}$. We may now replace Q by a 0-definable smaller open neighbourhood Q' of q_0, which does not contain q_1 and consider the smaller family $\mathcal{F}' = \{C_{q'} : q' \in Q'\}$.

Repeating the same argument, we see that X is still $\mathcal{F}'$-bounded near p and therefore, again by 1.11, there is still a curve C_{q_2}, with $q_2 \in Q'$, touching X at p. This contradicts the uniqueness of q_1 above. □

The last lemma can be seen as saying that in a nice family of curves, every curve that intersects X at a generic point, either touches X or intersects it transversally.

Lemma 1.17. *Let $\mathcal{F}_0 = \{C_q : q \in Q_0\}$, $\mathcal{F}_1 = \{D_q : q \in Q_1\}$ be two 0-definable two-dimensional nice families of plane curves. Let q_0, q_1 be generic in Q_0, Q_1, respectively, and $p_0 \in C_{q_0} \cap D_{q_1}$ such that $\dim(p_0, q_0/\emptyset) = \dim(p_0, q_1/\emptyset) = 3$. Assume that there exists some A-definable curve X such that p_0 is also generic in X over A. If C_{q_0} and D_{q_1} both touch X at p_0 then C_{q_0} and D_{q_1} touch each other at p_0, and q_0 and q_1 are inter-definable over p_0.*

Proof. Notice that if C_{q_0} and D_{q_1} touch X on opposite sides, then they clearly touch each other as well (see Figure 5). Assume then that the two curves touch X from above. Because of the genericity of p_0 in X, for every $p \in \nu_{p_0}$, there is $q \in \nu_{q_0}$ such that C_q touches X from above at p.

Assume first that X is $\mathcal{F}$-linear near p_0. Because p_0 is generic in X over A, there is a neighbourhood U of p_0 such that $U \cap X = U \cap C_{q_0}$ and then clearly D_{q_1} touches C_{q_0} as well. We can therefore assume from now on that p_0 is the only intersection point of X and C_{q_0} in some neighbourhood of p_0.

We first claim that D_{q_1} is $\mathcal{F}_0$-bounded near p_0. Indeed, let $p_0 = \langle x_0, y_0 \rangle$ and as before, use $f_q(x)$ to denote the function whose graph, within a fixed small neighbourhood of p_0, is C_q. We will let $g_{q_1}(x)$ denote the function whose graph is D_{q_1} and $h(x)$ denote the function whose graph is X. By the above, there are $q', q'' \in Q_0$ such that $C_{q'}$ and $C_{q''}$ touch X at $x' < x_0$ and $x'' > x_0$, respectively, both in ν_{p_0}. Because the three curves $C_{q'}, C_{q''}$ and D_{q_1} lie above X, we have

$$f_{q'}(x') = h(x') < g_{q_1}(x') \ , f_{q'}(x_0) > h(x_0) = g_{q_1}(x_0)$$

and

$$f_{q''}(x_0) > h(x_0) = g_{q_1}(x_0) \ , f_{q''}(x'') = g(x'') < g_{q_1}(x'').$$

It follows that $C_{q'} \preceq_{p_1} D_{q_1}$ and $D_{q_1} \preceq_{p_2} C_{q''}$, for some $p_1, p_1 \in \nu_{p_0}$ (see Figure 6 below). This implies, as was observed in the proof of Lemma 1.15, that D_{q_1} is $\mathcal{F}_0$-bounded near p_0.

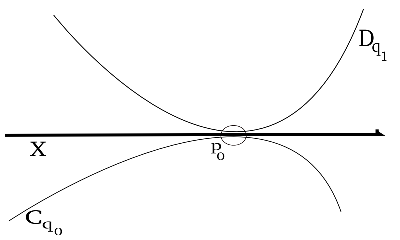


Figure 5:
C_{q_0}, D_{q_1} on opposite sides

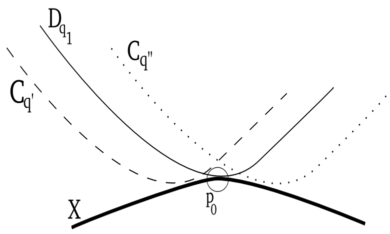


Figure 6:
D_{q_1} is $\mathcal{F}_0$-bounded

It follows, using 1.11, that a unique $C_q \in \mathcal{F}_0$ touches D_{q_1} at p_0. Moreover, even if we shrink Q_0 to a smaller neighbourhood of q_0, the curve D_{q_1} remains $\mathcal{F}_0$-bounded with respect to this new $\mathcal{F}_0$ and thus we can find again a unique $C_{q'}$ in this new family which touches D_{q_1}. It follows that C_{q_0} must be the curve from $\mathcal{F}_0$ which touches D_{q_1}. □

Note that in the presence of a field the above theorem follows more or less immediately from the fact that C_{q_0} and D_{q_1} have the same slope at p_0.

1.4. **The $\mathcal{M}$-rank of a definable curve.** We are going to need the notion of a rank of a curve (not to be confused with its dimension - always 1), which is similar, in the totally transcendental setting, for definable sets of Morley degree one, to the Morley rank of the canonical parameter.

Definition 1.18. Let X, Y be n-dimensional sets definable in some o-minimal structure. We define

$$X \sim Y \iff \dim(X \triangle Y) < n$$

Observe that $\sim$ is an equivalence relation on definable sets of dimension n.

The following can be defined for arbitrary definable sets, but for simplicity we will stick to curves:

Definition 1.19. Let X be a definable curve. We say that X *has $\mathcal{M}$-rank k over A* if there exist an A-definable normal family of curves $\mathcal{X} := \{X_t : t \in T\}$, $\dim(T) = k$, and some $t_0 \in T$ generic over A such that $X \sim X_{t_0}$. We say that X *has $\mathcal{M}$-rank k* if $A = \emptyset$.

For $p \in X$ a non-isolated point, we say that X has *$\mathcal{M}$-rank k over A at p*, if there exists a definable open neighborhood $U \subseteq M^k$ of p such that $X \cap U$ has $\mathcal{M}$-rank k over A.

The following lemma will imply in particular that both notions of $\mathcal{M}$-rank are well defined:

Lemma 1.20. *Let $X \subseteq M^n$ be a definable curve and assume that $X \sim X_{t_0}$ for some A-definable normal family of curves $\mathcal{X} = \{X_t : t \in T\}$ and t_0 generic in T over A. Assume also that $\dim(X \cap Y_{s_0}) = 1$ for another A-definable family of curves (not necessarily normal) $\mathcal{Y} = \{Y_s : s \in S\}$, and $s_0 \in S$ (not necessarily generic). Then $\dim T \leqslant \dim S$.*

Proof. Let

$$R = \{\langle t, s\rangle \in T \times S : \dim(X_s \cap Y_t) = 1\},$$

and let T_1, S_1 be the projections of R on T and S respectively. The set R is A-definable and because $t_0 \in T_1$ we have $\dim T_1 = \dim T$. We claim that the projection of R onto S_1 is everywhere finite-to-one. Indeed, for $s \in S_1$ let $R_s = \{t \in T : \langle t, s\rangle \in R\}$. If R_s were infinite then the family $\{Y_t \cap X_s : t \in R_s\}$ is a definable infinite family of sub-curves of the curve X_s. By o-minimality, there are $t_1 \neq t_2$ such that $\dim(Y_{t_1} \cap Y_{t_2} \cap X_s) = 1$, contradicting the normality of $\mathcal{X}$.

It follows that $\dim S \geqslant \dim S_1 = \dim R \geqslant \dim T_1 = \dim T$. □

Corollary 1.21. *The two notions of $\mathcal{M}$-rank are well-defined. Namely, whenever the $\mathcal{M}$-rank of a curve $X \subseteq M^n$ over a set A exists then it is unique, and whenever the $\mathcal{M}$-rank of X over A at a non-isolated $p \in X$ exists then it is unique (and independent of the chosen neighborhood). In particular, if the $\mathcal{M}$-rank of X over A is k then for every non-isolated $p \in X$, the $\mathcal{M}$-rank of X over A at p is also k.*

Proof. The following statement will cover both cases:

Let $U, V \subseteq M^n$ be definable open sets, with $\dim(U \cap V \cap X) = 1$. Assume that $\mathcal{X} = \{X_t : t \in T\}$ and $\mathcal{Y} = \{Y_s : s \in S\}$ are two normal A-definable families of curves, t_0, s_0 generic over A in T and S, respectively, such that $(X \cap U) \sim X_{t_0}$ and $(X \cap V) \sim Y_{s_0}$. Then $\dim(T) = \dim(S)$.

Note that since p is non-isolated in X, $\dim(X \cap X_{t_0}) = 1$ and $\dim(X \cap Y_{s_0}) = 1$. We can therefore apply Lemma 1.20 twice and conclude that $\dim(T) = \dim(S)$. □

Remark 1.22. (1) If $\mathcal{M}$ is an o-minimal expansion of a field, a typical example of a curve of $\mathcal{M}$-rank n is a generic polynomial of degree $n-1$. Since the family of polynomials of degree $n-1$ is definable and normal, it follows immediately from the definition that such a polynomial is of $\mathcal{M}$-rank at most n. Our choice of a generic polynomial assures that it is indeed of $\mathcal{M}$-rank n and not less.

(2) A curve X need not have $\mathcal{M}$-rank. E.g. if $X \subseteq \mathbb{R}^2$ is the concatenation of a curve X' of $\mathcal{M}$-rank $k > 1$ and a constant curve $X'' = \{y = 0\}$ then there is no definable normal family of curves in which X is a generic member. However, for every non-isolated p in $X' \cap \mathrm{cl}(X'')$, X has $\mathcal{M}$-rank k at p.

(3) Let $X \subseteq M^k$ be a definable curve of $\mathcal{M}$-rank k over A, witnessed by the normal family $\mathcal{X} = \{X_t : t \in T\}$, with $X = X_{t_0}$, $\dim T = k$, and $t_0 \in T$

generic over A. Let U be an A-definable open set with $\dim(U \cap X) = 1$. Then the $\mathcal{M}$-rank of $U \cap X$ over A is still k, as witnessed by the family $\{X_t \cap U : t \in T \,\&\, \dim(U \cap X_t) = 1\}$.

(4) Note that it could well be that a set X has no rank over some set of parameters A but has (positive) rank over some $B \supseteq A$.

As said above, the notion of $\mathcal{M}$-rank is similar to the Morley rank of the canonical parameter of a definable set of Morley degree 1, in the totally transcendental context. Let us make this precise.

Definition 1.23. Let T be a totally transcendental theory, X a definable curve. A *canonical parameter* for X is a set c such that for any automorphism σ (of the monster model), $X \sim \sigma(X)$ iff σ fixes c point-wise

Note that if c and c' are both canonical parameters of some curve X, then $c' \in \mathrm{dcl}(c)$. So by abuse of terminology we will call c *the* canonical parameter of X.

Lemma 1.24. *Let $\mathcal{N}$ be a strongly minimal set definable in $\mathcal{M}$, with $\dim_{\mathcal{M}}(N) = 1$. Let X be a strongly minimal $\mathcal{N}$-definable curve, c the canonical parameter of X and assume that $\mathrm{MR}(c/\emptyset) = k$. Then X has an $\mathcal{M}$-rank which equals to $\dim_{\mathcal{M}}(c/\emptyset)$, and $\dim_{\mathcal{M}}(c/\emptyset) \leqslant k$.*

Proof. By Remark 4.1.4 of [3], if c is the canonical parameter of X there exists some c-definable $Y \sim X$ such that for any automorphism σ, if $\sigma(Y) \sim Y$ then $\sigma Y = Y$. Since the $\mathcal{M}$-rank of X is, by definition, the same as the $\mathcal{M}$-rank of Y we may assume that $X = Y$.

Let $X := C(\bar{x}, c)$. We may assume that $C(\bar{x}, c_1) \neq C(\bar{x}, c_2)$ for all $c_1 \neq c_2$ such that $C(\bar{x}, c_i) \neq \emptyset$. Thus, if $c_1, c_2 \models \mathrm{tp}(c)$ are distinct elements then $C(\bar{x}, c_1) \wedge C(\bar{x}, c_2)$ is finite (because the sets they define are conjugate, and cannot be $\sim$-equivalent unless they are equal, which we assumed was not the case). By compactness there is $\theta \in \mathrm{tp}_{\mathcal{N}}(c)$ such that $\exists \bar{x} C(\bar{x}, c_1)$ and $C(\bar{x}, c_1) \wedge C(\bar{x}, c_2)$ is finite for all distinct $c_1, c_2 \models \theta$. So $C(\bar{x}, y) \wedge \theta(y)$ defines a normal family of curves. Now assume that $\dim_{\mathcal{M}}(c/\emptyset) = k_1 \leq k$ and let $\psi(y)$ witness this. Then $C(\bar{x}, y) \wedge \theta(y) \wedge \psi(y)$ is an $\mathcal{M}$-definable normal family of dimension k_1. Since the $\mathcal{M}$-rank is well defined, this gives the desired conclusion. □

1.5. **Normal families and differentiability.** *Throughout this subsection we assume that $\mathcal{M}$ expands a real closed field.*

Lemma 1.25. *Let $\{X_t : t \in T\}$ be a 0-definable normal family of plane curves, of dimension greater than 1. Let $t_0 \in T$ be generic, $X = X_{t_0}$, and $p_0 = \langle x_0, y_0 \rangle \in X$ generic over t_0. Then X is locally the graph of a strictly monotone C^1-function, $f_{t_0}(x)$ and if d is the derivative of f_{t_0} at x_0 then $\dim(d/p_0) = 1$.*

Proof. The first part follows from genericity together with the fact that $\dim T \geqslant 2$ (otherwise, X could be of the form $x = c$). By fixing a neighbourhood U of p_0 and restricting each X_t to U we may assume that X_t is the graph of a C^1 function f_t. Assume, towards contradiction, that $d \in \mathrm{dcl}_{\mathcal{M}}(p_0)$. Then there exists a definable function D, from a rectangular open $U_0 = I_0 \times J_0 \ni p_0$ into M, $D(p_0) = d$, and there exists a neighbourhood $T_0 \subseteq T$ of t_0, such that for every $t \in T_0$ and $x \in I_0$ we have

$$f_t'(x) = D(x, f_t(x)).$$

By the uniqueness theorem for differential equations in o-minimal structures (Theorem 2.3 in [17]), for every $t \in T_0$, if $f_t(x_0) = f_{t_0}(x_0)$ then f_t and f_{t_0} agree on some interval containing x_0. The normality of $\mathcal{F}$ thus implies that the value of f_t at x_0 determines $t \in T_0$. This contradicts the fact that $\dim(T_0) \geqslant 2$. □

Lemma 1.26. *Assume that $\mathcal{F} = \{C_q : q \in Q\}$ is a $\emptyset$-definable two-dimensional normal family of plane curves. If $p_0 = \langle x_0, y_0\rangle$ is generic in M^2 then for all but finitely many $q \in Q$, if $p_0 \in C_q$, then C_q is the graph of a C^1-function $f_q(x)$ near p_0 and for every such q, $f'_q(x_0)$ and q are inter-definable over p_0.*

Proof. For $p \in M^2$ we still use ℓ_p for a dual curve $\{q \in Q : p \in C_q\}$. The first part of the lemma follows from the fact that if q is generic in ℓ_{p_0} over p_0 then p_0 is generic in C_q over q (by 1.5 (i)).

Assume now $f_q(x)$ is C^1 near x_0 and that $d = f'_q(x_0)$. If there are infinitely many $q' \in \ell_{p_0}$ such that $f'_{q'}(x_0) = d$ then first of all there is $q_0 \in \ell_{p_0}$ such that $\dim(q_0/p_0) = 1$ and $d = f'_{q_0}(x_0)$. Furthermore, because $\dim(\ell_{p_0}) = 1$, we have $d \in \mathrm{dcl}_{\mathcal{M}}(p_0)$. This contradicts Lemma 1.25 with C_{q_0} here taken as X (note that p_0 is generic in C_{q_0} over q_0). □

Before we proceed further, it will be convenient to introduce:

Definition 1.27. Let $C \subseteq M^2$ be a definable curve, $p := \langle x_0, y_0\rangle \in C$ a non-isolated point and assume that there exists an $\mathcal{M}$-definable strictly monotone C^1 (partial) function $f : M \to M$ (with respect to the real closed field R) such that $f(x_0) = y_0$ and C coincides with the graph of f in an open neighbourhood of p. We call $f'(x_0)$ *the slope of C at p.*

We can now establish the connection between the differential notion of tangency and that of touching curves.

Theorem 1.28. *Let $\mathcal{F} = \{C_q : q \in Q\}$ be an $\emptyset$-definable normal family of plane curves of dimension greater than 1. Let X be an A-definable plane curve, p_0 generic in X over A and assume that X has $\mathcal{M}$-rank $\geqslant 2$ at p_0.*

If $p_0 \in C_{q_0}$, for some $q_0 \in Q$, and the slope of C_{q_0} at p_0 equals that of X at p_0 then C_{q_0} and X touch each other at p_0 and $\dim(p_0, q_0/\emptyset) = 3$.

Proof. Because we are proving a local property of X we may assume that the $\mathcal{M}$-rank of X is $k \geqslant 2$. Hence, there is a $\emptyset$-definable normal family of curves $\mathcal{X} := \{X_t : t \in T\}$, t_0 generic T, with $X \sim X_{t_0}$. It follows that $t_0 \subseteq \mathrm{acl}_{\mathcal{M}}(A)$ and therefore p_0 is also generic in X_{t_0} over t_0, so $X \cap U = X_{t_0} \cap U$ for some open $U \ni p_0$. In particular, the slope of X at p_0 is also the slope of X_{t_0} at p_0. Thus, it will suffice to prove the lemma for X_{t_0}. We will therefore assume, without loss of generality, that $X = X_{t_0}$. Denote $p_0 = \langle x_0, y_0\rangle$ and let d be the slope of X and C_{q_0} at p_0.

By 1.5 (i) (applied to the normal family $\mathcal{X}$), p_0 is generic in M^2. It follows from 1.25 that $\dim(d/p_0) = 1$, and from 1.26 it follows that $\dim(q_0/p_0) = 1$, hence $\dim(p_0, q_0/\emptyset) = 3$.

We assume that X is the graph of a C^1-function $y = h(x)$ near x_0. By Theorem 1.9, we may assume that $\mathcal{F}$ is a nice family.

Assume towards contradiction that C_{q_0} does not touch X at p_0. Say, we have $X \prec_{p_0} C_{q_0}$. It follows from continuity of the family that for every $q \in \nu_{q_0}$, there is $x > x_0$, $x \in \nu_{x_0}$, such that $f_q(x) > h(x)$. Since $\dim(d/p_0) = 1$, the point d is an interior point of the set $\{f'_q(x_0) : q \in \ell_{p_0}\}$. In particular, we can find $q \in \ell_{p_0} \cap \nu_{p_0}$

such that $f'_q(x_0) < d$ and therefore $C_q \preccurlyeq_{p_0} X$. These two last facts, together with the continuity of f_q, imply that there is $x' > x_0$, $x' \in \nu_{x_0}$, such that $f_q(x') = h(x')$. In particular, $|C_q \cap X \cap \nu_{p_0}| > 1$, contradicting 1.16. □

1.6. **The Duality Theorem.** Let $\mathcal{F} = \{C_q : q \in Q\}$, $Q \subseteq M^2$, be a nice family of curves in $P \subseteq M^2$, given by:

$$\langle x, y\rangle \in C_{a,b} \text{ if and only if } F(a, b, x) = y.$$

By definition, $\operatorname{dom}(F) = Q\times I$, for some open interval I and the function $F : Q\times I \to M$ is continuous and strictly monotone in each variable separately.

As before, we let $\mathcal{L} = \{\ell_p : p \in P\}$ be the dual family, defined by $q \in \ell_p \Leftrightarrow p \in \mathcal{C}_q$. As we already observed, for every $p \in P$, $\dim(\ell_p) \leqslant 1$. The family $\mathcal{L}$ is given by a dual (partial) function G defined as follows:

$$\langle a, b\rangle \in \ell_{x,y} \text{ iff } G(x, y, a) = b \text{ iff } F(a, b, x) = y \text{ iff } \langle x, y\rangle \in C_{a,b}.$$

It is not hard to see that G is indeed a function, which is continuous and strictly monotone in each variable, on its domain. Furthermore, because

$$q_1, q_2 \in \ell_{p_1} \cap \ell_{p_2} \;\Leftrightarrow\; p_1, p_2 \in C_{q_1} \cap C_{q_2},$$

for $p_1 \neq p_2$ we have $|\ell_{p_1} \cap \ell_{p_2}| \leqslant 1$.

Putting it all together, we see that the family $\mathcal{L}$ is nice once we restrict the last variable (a) to a fixed interval J. Namely,

Claim 1.29. *Let $\mathcal{F}$ be a nice family, $\mathcal{L}$ its dual, as above.*

For every $\langle x_0, y_0\rangle \in C_{a,b}$, there exist open neighbourhoods $U \ni \langle x_0, y_0\rangle$ and $W \ni \langle a, b\rangle$ such that the family $\{\ell_p \cap W : p \in U\}$ is nice.

For a definable curve $Y \subseteq Q$, we use the family $\mathcal{L}$ in the same way we used $\mathcal{F}$ for curves in P. Again, we suppress the subscript from our notation and use $\tau(Y)$ for $\tau_{\mathcal{L}}(Y)$. In particular, for $X \subseteq P$, we let $\tau(\tau(X))$ denote $\tau_{\mathcal{L}}(\tau_{\mathcal{F}}(X))$.

We can now establish the following duality:

Theorem 1.30. *Assume that $\mathcal{F}$ is a nice family of curves. Let $X \subseteq M^2$ be an A-definable curve and let $p_0 \in X$ be generic over A, such that X is not $\mathcal{F}$-linear near p_0. Then*

1. *C_{q_0} touches X at p_0 if and only if ℓ_{p_0} touches $\tau(X)$ at q_0.*
2. *There exists a definable neighbourhood $U \subseteq P$ of p_0, such that $\tau(\tau(X))\cap U = X \cap U$.*

Before proving the theorem, here is an example:

Example: Let $\mathcal{F}$ be the family of all affine lines in $\mathbb{R}^2$, $C_{a,b} = \{y = ax+b\}$. It is easy to see that the dual family $\mathcal{L}$ is also the family of affine lines. Take $X = \{y = x^2\}$. Then $\tau(X) = \{\langle u, v\rangle : v = -u^2/4\}$. Given $p = \langle x_0, x_0^2\rangle \in X$, the $\mathcal{F}$-tangent curve at p is C_q, where $q = \langle 2x_0, -x_0^2\rangle$. We also have $\ell_p = \{\langle u, v\rangle : v = -x_0u + x_0^2\}$.

The slope of ℓ_p equals everywhere to $-x_0$, which is exactly the slope of $\tau(X)$ at $q = \langle 2x_0, -x_0^2\rangle$. Since q belongs to ℓ_p it follows that ℓ_p is tangent to $\tau(X)$ at q.

Proof of theorem. By Claim 1.29, we may assume that $\mathcal{L}$ is also nice (observe that the statement of (1) does not change if we replace P and Q with small open neighbourhoods of p_0 and q_0 respectively).

(1) Assume first that C_{q_0} touches X at p_0. Note that the non-linearity assumption, together with 1.11 imply that p_0 and q_0 are inter-definable over the parameters defining X and therefore q_0 is generic in $\tau(X)$ as well.

Claim The curve ℓ_{p_0} is not equal to the curve $\tau(X)$ in a neighbourhood of q_0.

Indeed, notice that if ℓ_{p_0} equals to $\tau(X)$ in a neighbourhood of q_0 then the same is true for all p in a relatively open $X_1 \subseteq X$ containing p_0. But then all of these ℓ_p intersect each other at infinitely many points, contradicting the normality of $\mathcal{L}$.

Let W_1, W_2 be local components of $M^2 \setminus \tau(X)$ at q_0. Without loss of generality, by Lemma 1.14, for every $q \in W_1 \cap \nu_{q_0}$, we have $C_q \cap X \cap \nu_{p_0} = \emptyset$. However, for every $q \in \ell_{p_0}$ we clearly have $p_0 \in C_q \cap X \cap \nu_{p_0}$. Hence, near q_0, the entire curve ℓ_{p_0} is contained in W_2, so ℓ_{p_0} touches $\tau(X)$ at q_0.

This implies in particular that $p_0 \in \tau(\tau(X))$. Because $\tau(X)$ is defined over A there exists a neighbourhood U of p_0 such that $\tau(\tau(X)) \cap U = X \cap U$. This already gives us (2).

Returning to (1), assume now that ℓ_{p_0} touches $\tau(X)$ at q_0. The claim proved above implies that $\tau(X)$ is not $\mathcal{L}$-linear near q_0. Hence we can apply the result we already proved, (with $\tau(X)$ and $\mathcal{L}$, instead of X and $\mathcal{F}$) to conclude that C_{q_0} touches $\tau(\tau(X))$ at p_0. However, as we showed, $\tau(\tau(X)) = X$ near p_0 so C_{q_0} touches X at p_0. □

Notice that the Duality Theorem implies in particular, in the notation of the theorem, that if X is $\mathcal{F}$-bounded near p_0 then $\tau(X)$ is $\mathcal{L}$-bounded near q_0 (see Lemma 1.15).

Remark After discovering the above duality we found out that it has a classical analogue in projective geometry. In that setting a duality exists between an algebraic variety and the variety of all its tangent hyper-planes. See, e.g., [2].

We need the following corollary;

Corollary 1.31. *Let $\mathcal{F} = \{C_q : q \in Q\}$ be a 0-definable two dimensional normal family of curves in P, where $P, Q \subseteq M^2$. Assume that q_0 is generic in Q, and $X \subseteq M^2$ is an A-definable curve. Assume that p_0, p_1 are generic elements of X and that X is not $\mathcal{F}$-linear near any of them. If $\dim(p_0, q_0/\emptyset) = 3$ and C_{q_0} touches X at both p_0 and p_1 then $p_1 \in \mathrm{dcl}_{\mathcal{M}}(p_0, q_0)$ (although X is A-definable!).*

Proof. We may clearly assume that $p_1 \notin \mathrm{dcl}_{\mathcal{M}}(q_0)$, so $\dim(p_1, q_0/\emptyset) = 3$. By 1.9, there are relative open $P_0, P_1 \subseteq P$, $p_0 \in P_0$, $p_1 \in P_1$, and relative open $Q_0, Q_1 \subseteq Q$, with $q_0 \in Q_0$, such that $\mathcal{F}_0 = \{C_q \cap P_0 : q \in Q_0\}$ and $\mathcal{F}_1 = \{C_q \cap P_1 : q \in Q_1\}$ are nice families. Moreover, we may assume that for $i = 0, 1$, $X \cap P_i$ is nowhere $\mathcal{F}_i$-linear. By choosing $\mathcal{F}_0$ and $\mathcal{F}_1$ properly we may assume that they are both defined over parameters generic over all the data, so absorbing all those parameters into the language we may assume they are 0-definable.

By 1.30 the curves ℓ_{p_0} and ℓ_{p_1} touch $\tau(X)$ at q_0. We now apply 1.17 to the families $\mathcal{L}_0$ and $\mathcal{L}_1$ (the dual families of $\mathcal{F}_0$ and $\mathcal{F}_1$), the points p_0, p_1 and $\tau(X)$, and conclude that ℓ_{p_0} and ℓ_{p_1} touch each other at q_0 and $p_1 \in \mathrm{dcl}_{\mathcal{M}}(q_0, p_0)$. □

Remark The assumption $\dim(p_o, q_0/\emptyset) = 3$ in 1.31 is indeed necessary, as the following example shows.

Let $C = \{\langle x, y\rangle \in \mathbb{R}^2 : y = \sqrt{|x|}\}$ and for $\langle a, b\rangle \in \mathbb{R}^2$ let $C_{a,b} = C + \langle a, b\rangle$. The family $\{C_{a,b} : \langle a, b\rangle \in \mathbb{R}^2\}$ is normal and each $C_{a,b}$ has a cusp at $\langle a, b\rangle$. If $X \subseteq \mathbb{R}^2$ is a C^1-curve then for any $p_0 = \langle a, b\rangle \in X$, the curve $C_{a,b}$ touches X at p_0 (and there are possibly other curves from $\mathcal{F}$ which touch X at p_0). Let $q_0 = \langle a, b\rangle$ and assume that $\mathcal{C}_{q_0}$ touches X at another point p_1. Because in this case $q_0 = p_0$ we cannot expect that $p_1 \in \mathrm{dcl}(p_0, q_0) = \mathrm{dcl}(p_0)$, if the curve X has sufficiently large rank.

Finally, we have:

Lemma 1.32. *Assume that $\mathcal{F}$ is a $\emptyset$-definable nice family of plane curves, parameterised by $Q \subseteq M^2$. Let X be a nowhere $\mathcal{F}$-linear, t_0-definable plane curve, let p_0 be generic in X over t_0 and assume that C_{q_0} touches X at p_0.*

If the $\mathcal{M}$-rank of X at p_0 is $k \geqslant 1$ then there is B such that the rank of $\tau(X)$ over B at q_0 is also k.

Proof. Because q_0 is generic in $Y = \tau(X)$ over t_0 and $\mathcal{L}$ is a nice family, there are neighbourhoods U of p_0 and W of q_0 such that $p \mapsto \tau(X, p)$ induces a bijection of $X \cap U$ with $\tau(X) \cap W$, and moreover, by the Duality Theorem $\tau(\tau(X \cap U)) = X \cap U$.

Because we are proving a local property of X, we may assume that X has rank k. Hence, there is an 0-definable normal family $\mathcal{X} := \{X_t : t \in T\}$ with $\dim(T) = k$ and $t_0 \in T$ generic such that $X \sim X_{t_0}$. Consider the family $\mathcal{Y} = \{\tau(X_t \cap U) : t \in T\}$. We may assume that for all $t \in T$ we have $\tau(\tau(X_t \cap U)) = X_t \cap U$. We may also assume that U is defined over a parameter set B independent from t_0 over $\emptyset$.

It suffices to show that $\mathcal{Y}$ is normal, for if so then $\tau(X) \cap W \sim \tau(X_{t_0} \cap U)$ and this last curve belongs to $\mathcal{Y}$.

If $\mathcal{Y}$ is not normal then there are $t_1, t_2 \in T$ and an open set $W_1 \subseteq W$ such that $\tau(X_{t_1} \cap U) \cap W_1 = \tau(X_{t_2} \cap U) \cap W_1$. If we now apply τ to those sets then, by the Duality Theorem we will conclude, for some $U_1 \subseteq U$, that $X_{t_1} \cap U_1 = X_{t_2} \cap U_1$, contradiction. □

1.7. **Limit sets and a theorem of v.d. Dries.** Somewhat surprisingly, the notion of limit sets plays a crucial role in our subsequent counting of intersection points of curves. Several different notions of limit sets were studied extensively by v.d. Dries in [5]. The definition below, which is suitable for an arbitrary o-minimal structure (not necessarily over the reals) resembles most that of the Hausdorff limit, but is not restrictred to families of compact, or even bounded sets.

Definition 1.33. Consider a definable set $S \subseteq M^k \times M^n$. For $z \in Z_1 = \pi_1(S)$, the projection of S on the first k-coordinates, let $S_z = \{y \in M^n : \langle z, y\rangle \in S\}$. Let $\mathcal{S} = \{S_z : z \in Z_1\}$ be the corresponding family of subsets of M^n. Given $a < b$ in $M \cup \{\pm\infty\}$, and given a definable map $\gamma : (a, b) \to Z_1$, we let $\mathcal{S}(\gamma, a)$ be the set of all $y \in M^n$ such that for every open neighbourhood V of y and for every $\epsilon \in (a, b)$, there is $t \in (a, \epsilon)$ such that $(S_{\gamma(t)} \cap V) \neq \emptyset$.

$\mathcal{S}(\gamma, a)$ is called *the limit of $\mathcal{S}$ along $\gamma(t)$, as t tends to a.* We similarly define $S(\gamma, b)$.

If $\mathcal{M}$ expands a field then, because all open interval are definably bijective with each other, it is sufficient to work with limit sets along γ at $+\infty$.

The following claim is easy to verify.

Claim 1.34. *For $\mathcal{S}$ as above and $\gamma : (a, b) \to M$, the set $S(\gamma, a)$ is definable, and the following are equivalent:*
(i) $y \in S(\gamma, a)$.

(ii) For every $t > a$ in ν_a, $S_{\gamma(t)} \cap \nu_y \neq \emptyset$.
(iii) There exists $t > a$ in ν_a such that $S_{\gamma(t)} \cap \nu_y \neq \emptyset$.

Here are some easy facts about limit sets:

(1) The set $S(\gamma, a)$ is closed in M^n.
(2) If every S_z has dimension r then the dimension of every limit set as above is at most r.
(3) If $\lim_{t\to a} \gamma(t) = z_0 \in M^k$ then
$$\mathcal{S}(\gamma, a) \subseteq \pi_2(\mathrm{cl}(S) \cap (\{z_0\} \times M^n))$$
(where π_2 is the projection onto M^n). If in addition, z_0 is a generic point of Z_1 over the parameters defining $\mathcal{S}$, then the set on the right equals $cl(S_{z_0})$, and we have $\mathcal{S}(\gamma, a) = cl(S_{z_0})$, independently of the particular γ we choose (see proof below).
(4) Assume that $X \subseteq Z_1$ is a definable one-dimensional set, $p \in \mathrm{cl}(X)$. Then there are finitely many ways to approach p in X (depending on the number of local components of $X \setminus \{p\}$). We write $\mathcal{S}(X, p)$ for the union of the finitely many possible limits sets $\mathcal{S}(\gamma, a)$ as $\gamma(t)$ tends to p in X, and call this set *the limit set of $\mathcal{S}$ along X at p.* Note that if $p \in X$ then, by letting $\gamma(t) \equiv p$ the constant map, we see that $S_p \subseteq \mathcal{S}(X, p)$.
If p is a generic point in X then $\mathcal{S}(X, p)$ is just $\mathrm{cl}(S_p)$.

Let's explain the statement about generic points in clause (3) above: Take z_0 generic in X. We easily see that $\mathcal{S}(\gamma, a) \subseteq cl(S_{z_0})$, so we need to show the opposite inclusion. Take $b \in cl(S_{z_0})$ and $U \subseteq M^n$ a definable open neighborhood of b, which, without loss of generality, is defined over parameters independent from all others. Because $S_{z_0} \cap U \neq \emptyset$, there exists a neighborhood V of z_0 such that for all $z \in V$, $S_z \cap U \neq \emptyset$. In particular, for all x sufficiently close to a, $S_{\gamma(x)} \cap U \neq \emptyset$, and therefore $b \in \mathcal{S}(\gamma, a)$. Hence, $\mathcal{S}(\gamma, a) = cl(S_{z_0})$.

The following theorem and proof are due to van den Dries. An unpublished proof of which appears in notes (see [6]). It is similar to the proof of the analogous theorem in §9 of [5]):

Theorem 1.35. *Assume that $\mathcal{M}$ expands a real closed field. Given a definable family $\mathcal{S}$ as above, let*
$$\tilde{\mathcal{S}} = \{\mathcal{S}(\gamma, a) | \gamma : (a, b) \to Z_1 \textit{ definable } \}.$$
Then $\tilde{\mathcal{S}}$ is definable. Namely, there exists a definable family of subsets of M^n, $\mathcal{T} = \{T_x : x \in X\}$ such that every T_x equals $S(\gamma, a)$ for some pair (γ, a) and vice-versa. In addition, $\dim(\tilde{\mathcal{S}}) \leqslant \dim(Z_1)$. (By $\dim(\tilde{\mathcal{S}})$ we mean the smallest possible dimension for a set of parameters for $\tilde{\mathcal{S}}$).

If furthermore all the sets in $\mathcal{S}$ are closed then $\dim(\tilde{\mathcal{S}} \setminus \mathcal{S}) < \dim Z_1$

In this paper we will be using only the first part of the theorem. Since the proof does not appear elsewhere, we present it in Appendix A, with the author's permission.

1.8. **Special points.** *We assume in this section that $\mathcal{M}$ expands an ordered group. We also assume that $\mathcal{F} = \{C_q : q \in Q\}$ is a 0-definable two-dimensional normal family of curves in $P \subseteq M^k$, $\dim P = 2$ (but P is not necessarily contained in the plane), such that its dual family $\mathcal{L}$ is also a normal family of curves.*

A few words of explanations are in place regarding these assumptions.

(1) If $\mathcal{F}$ is an (almost) normal two-dimensional family of curves, its dual family need not consist only of curves, i.e. for $p \in P$ the set ℓ_p of $\mathcal{F}$-curves through p may be 2-dimensional. It is easy to check that by normality the set $\hat{P} := \{p \in P : \dim \ell_p = 2\}$ is finite. Restricting to $P \setminus \hat{P}$, the dual to $\mathcal{F}$ is also a family of curves.
(2) If $\mathcal{F}$ is a normal two-dimensional family of curves whose dual is a family of curves as well, it is easy to check that the dual is almost normal. However, it need not be normal. In the application in Section 2 the family $\mathcal{F}$ will be definable in a strongly minimal structure, with each curve strongly minimal. We leave it as simple exercise to check that in that case replacing P with P/E for an appropriate definable equivalence relation E both $\mathcal{F}$ and its dual are normal. For example, setting $E(q_1, q_2) \iff |C_{q_1} \vartriangle C_{q_2}| < \infty$ and $x \in C_q$ for $q \in P/E$ if and only if $x \in C_{q'}$ for all generic q' such that $q'/E = q$ will produce a (definable) normal family. A little more fiddling will assure that its dual is also normal. See also Fact 2.5.
(3) All results in this subsection still hold under the weaker assumption that $\mathcal{F}$ is only almost normal (whose dual is a family of curves), in which case its dual is easily checked to be almost normal as well.

The next definition, which is quite technical, is motivated by the following discussion: Our ultimate goal is to prove, for a one-dimensional definable $\mathcal{N}$, that the existence of an almost normal two-dimensional $\mathcal{N}$-definable family $\mathcal{F}$ of plane curves contradicts the stability of $\mathcal{N}$. The idea is to start with a curve X, and a curve C_{q_0} from $\mathcal{F}$ touching X at a generic point, p_0. By Lemma 1.14, there are generic curves in $\mathcal{F}$ arbitrarily close to C_{q_0} which either intersect X nowhere near p_0 or intersect it twice there. If there were no other intersection points of these curves with X, this gap in the number of intersection points with X can be seen to contradict stability (assuming that the parameter set of $\mathcal{F}$ has Morley degree one).

However, the problem is that the curves near C_{q_0} might intersect X, or even $\mathrm{cl}(X)$, elsewhere and in order to control the total number of intersection points we need to investigate what happens near those other points. Such points will be called *special for* (q_0, X).

Definition 1.36. Let $X \subseteq P$ be a definable curve. Given $q \in Q$, a point $p \in M^k$ is called *special for* (q, X) if there exist $q' \in \nu_q$, and $p' \in \nu_p$ such that $p' \in X \cap C_{q'}$ (equivalently, for every neighbourhoods U, V of q, p, respectively, there exist $q' \in U, p' \in V$, such that $p' \in X \cap C_{q'}$).

A similar definition can be given even if some of the coordinates of p are taken to be $\pm\infty$ (and then $\nu_{+\infty}, \nu_{-\infty}$ have an obvious meaning).

The point p in the above definition can be seen as an asymptotic direction of X, in the sense of the family $\mathcal{F}$, and the curve C_q can be seen as an "$\mathcal{F}$-asymptote of X at p". For example, if $\mathcal{F}$ is the family of affine lines in $\mathbb{R}^2$, given by $C_{a,b} = \{y = ax + b\}$, and X is the curve $(y-1)x = 1$ then the point $p = \langle +\infty, 1\rangle$ is special for (q, X), with $q = \langle 0, 1\rangle$. That is, the curve $y = 1$ is asymptotic to X at $\langle +\infty, 1\rangle$. Indeed, curves of the form $y = ax+b$, for a near 0 and b near 1, will intersect X near $\langle +\infty, 1\rangle$.

Here are some basic observations concerning special points:

Fact 1.37. *Given $\mathcal{F}$ and a curve X as above,*

(1) *If $p \in C_q \cap X$ then p is a special point for (q, X).*
(2) *If $p \in M^k$ is a special point for (q, X) then $p \in \mathrm{cl}(X) \subseteq \mathrm{cl}(P)$. (However, if $\mathcal{F}$ is not given continuously, it is possible that p is not in $\mathrm{cl}(C_q)$).*
(3) *The set of special points for (q, X) is definable.*
(4) *If p is a special point for (q, X) then there exists a definable curve $\gamma : (a, b) \to Q$ tending to q as t tends to a, such that p is in the limit set $\mathcal{F}(\gamma, a)$ (we use here the fact that $\mathcal{M}$ has curve selection).*
(5) *If q is generic in Q (over $\emptyset$) and p is special for (q, X) then $p \in \mathrm{cl}(C_q)$ (see (4) of Section 1.7).*

The following lemma establishes the connection between the notions of a special point and that of a limit set:

Lemma 1.38. *Given $\mathcal{F}$ as above and a definable curve $X \subseteq P$, a point p is special for (q, X) if and only if q belongs to the limit set $\mathcal{L}(X, p)$ of the dual family $\mathcal{L}$ along X at p (see (4) of Section 1.7 for the definition of $\mathcal{L}(X, p)$)*

Proof. By definition, p is special for (q, X) if and only if $X \cap \nu_p \cap C_{q'} \neq \emptyset$ for some $q' \in \nu_q$, if and only if $\ell_{p'} \cap \nu_q \neq \emptyset$ for some $p' \in X \cap \nu_p$. By Claim 1.34(iii), this is equivalent to q belonging to the limit set of $\mathcal{L}$ at p, along the curve X. □

Lemma 1.39. *Given $\mathcal{F}$ as above, let q_0 be generic in Q and let $X \subseteq P$ be a definable curve not $\mathcal{F}$-linear near any of its points. Then:*

(1) *There are at most finitely many points $p_1, \dots, p_r$ (including possibly points with coordinates in $\pm\infty$), which are special for (q_0, X).*
(2) *There are pairwise disjoint definable open neighbourhoods $U_1, \dots, U_r$ of $p_1, \dots, p_r$, respectively, and an open neighbourhood W of q_0 such that for every $q \in W$, every intersection point of C_q with X is contained in one of the U_i.*

Proof. (1) As was observed in 1.37 (5), if p is special for (q_0, X) and q_0 is generic in Q over $\emptyset$ then $p \in \mathrm{cl}(C_{q_0})$. Now, if there were infinitely many special points for (q_0, X) then infinitely many of them were already in $C_{q_0} \cap X$ and in particular, C_{q_0} would equal X in some neighbourhood, contradicting the fact that X is not $\mathcal{F}$-linear.

(2) If the statement failed then, by curve selection, we would get a definable curve $\gamma : (a, b) \to M^2$, $\lim_{t \to a} \gamma(t) = q_0$, with points in $C_{\gamma(t)} \cap X$ tending away from $p_1, \dots, p_r$ as t tends to a. By definition, if p is a limit point for the set $C_{\gamma(t)} \cap X$ then it is a special point for (q_0, X) (some of the coordinates of p could be $\{\pm\infty\}$). This would contradict the assumption that $p_1, \dots, p_r$ were all the special points for (q_0, X). □

The following technical lemma will play a very important role in our subsequent proof of the main theorem. It is here that v.d. Dries' theorem about limit sets is used.

Lemma 1.40. *Given $\mathcal{F}$ as above, let $X \subseteq P$ be an A-definable curve, p_0 generic in X over A, and assume that the $\mathcal{M}$-rank of X at p_0 is $k > 2$. Assume also that for $q_0 \in Q$, the curve C_{q_0} touches X at p_0 (here we identify P locally with an open set in M^2), and in addition, $\dim(p_0, q_0/\emptyset) = 3$. If p is a special point for (q_0, X) then $\dim(p/A) = 1$. In particular, p is a generic point of X.*

Proof. As was observed in Fact 1.37 (2), $p \in \mathrm{cl}(X)$ and hence $\dim(p/A) \leqslant 1$. By 1.7, there are neighborhoods U of p_0 and W of q_0 such that the family $\mathcal{F}_1 = \{C_q \cap U : q \in W\}$ is nice. Consider $\tau_{\mathcal{F}_1}(X \cap U)$ and recall that q_0 is generic in $\tau_{\mathcal{F}_1}(X \cap U)$ over

A (by the discussion following Definition 1.13). The sets U and W can be chosen definable over a set B independent from A and all mentioned elements, such that the $\mathcal{M}$-rank of $X \cap U$ over B is k. By 1.32, the $\mathcal{M}$-rank of $\tau_{\mathcal{F}_1}(X \cap U)$ over some $C \supset B$ at q_0 is also k.

Assume, towards a contradiction, that $\dim(p/A) = 0$. Hence, q_0 is still generic in $\tau_{\mathcal{F}_1}(X \cap U)$ over Ap. By Lemma 1.38, q_0 is in the Ap-definable limit set $\mathcal{L}(X, p)$. But then $\dim(\tau_{\mathcal{F}_1}(X \cap U) \cap \mathcal{L}(X, p)) = 1$.

Consider the family of limit sets $\tilde{\mathcal{L}}$, as in Theorem 1.35, for the dual family $\mathcal{L}$. By that theorem, $\tilde{\mathcal{L}}$ is definable and $\dim(\tilde{\mathcal{L}}) \leqslant \dim(\mathcal{L}) = 2$. The curve $\mathcal{L}(X, p)$ is a finite union of A-definable curves, all in $\tilde{\mathcal{L}}$ (see Section 1.7 (4)).

We can now apply Lemma 1.20 to the curve $\tau_{\mathcal{F}_1}(X \cap U)$ and the family $\tilde{\mathcal{L}}$ and conclude that the $\mathcal{M}$-rank of $\tau_{\mathcal{F}_1}(X \cap U)$ is at most $\dim \tilde{\mathcal{L}} \leqslant 2 < k$. Contradiction. □

Remark The above lemma implies that, under the same assumptions, a point in $Fr(X)$ cannot be special for (q_0, X) (see Figure 7). Similarly, it can be shown that such special points cannot have $\pm\infty$ as one of their coordinates. This can be seen as saying that an $\mathcal{F}$-tangent curve to X cannot be asymptotic to X at $\pm\infty$, for sufficiently generic data.

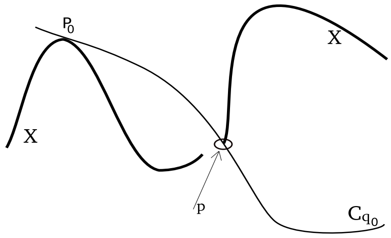


Figure 7:
A problematic special point

We will need the following lemma:

Lemma 1.41. *Let $\mathcal{F}$, X, p_0 and q_0 be as in the last lemma. Assume that p is a special point for (q_0, X) such that $\dim(p/p_0, q_0) = 1$. Then, for all $q \in \nu_{q_0}$, the curve C_q intersects X exactly once in ν_p.*

Proof. By Lemma 1.40, p is generic in X over A and by Lemma 1.31, $\mathcal{C}_{q_0}$ does not touch X at p.

The fact that p is a special point for (q_0, X) implies (see Fact 1.37 (5)) that p is in $\mathrm{cl}(C_{q_0})$. However, since we assume that $\dim(p/q_0) = 1$, p must belong to C_{q_0}. We now apply Lemma 1.16 to $\mathcal{F}_{U,W}$ (see notation of Corollary 1.9) for some neighbourhoods U of p and W of q_0, such that $\mathcal{F}_{U,W}$ is nice. □

2. Stable one dimensional theories in o-minimal structures

We return to the original setting of the main theorem. Namely, $\mathcal{M}$ is an o-minimal structure (expanding a real closed field), $\mathcal{N}$ is definable in $\mathcal{M}$ such that $\dim_{\mathcal{M}}(N) = 1$. In this section we treat the case that $\mathcal{N}$ is stable. We now use $\dim_{\mathcal{M}}$ and $\dim_{\mathcal{N}}$ to denote the dimension in $\mathcal{M}$ and $\mathcal{N}$, respectively.

First note that by cell decomposition, and the fact that we assume that $\mathcal{M}$ expands a field, we may assume that N is a subset of M (later on, in Appendix B, we will show why this can be assumed even without the field assumption).

We will need the following facts about stable theories interpretable in o-minimal structures:

Fact 2.1. *Assume $\mathcal{N}$ is a stable structure interpretable in $\mathcal{M}$. Then:*
(1) *$\mathcal{N}$ is superstable and its* U*-rank is at most the o-minimal dimension of N.*
(2) *If $\mathrm{U}(\mathcal{N}) = 1$ then (i) $Th(\mathcal{N})$ is 1-based iff (ii) every minimal type is locally modular iff (iii) $\mathcal{N}$ is not rich.*

Proof. For (1), see [8]. For (2), the equivalence of (i) and (ii) is general (see 2.5.8 in [24]). The equivalence of (ii) and (iii) goes as follows: If $\mathcal{N}$ is not 1-based then there exists a definable non locally modular strongly minimal $\mathcal{N}' \subseteq \mathcal{N}$ (see 2.3.3 in [24]). This in turn yields a definable pseudoplane (see below) in $(N')^2$, which witnesses the richness of $\mathcal{N}$. For the converse, assume $\mathcal{F}$ is a 2-dimensional nice family and C_q is a generic curve in $\mathcal{F}$. If $\langle a, b\rangle$ is generic in C_q then, because there are infinitely many $\mathcal{F}$-curves through $\langle a, b\rangle$, $Cb(stp(ab/q)) \nsubseteq \mathrm{acl}_{\mathcal{N}}(ab)$ so $Th(\mathcal{N})$ is not 1-based. □

Our goal in this section is to prove;

Theorem 2.2. *If $\dim_{\mathcal{M}}(N) = 1$ and $\mathcal{N}$ is stable then $\mathcal{N}$ is 1-based (and therfore not rich).*

We first observe:

Proposition 2.3. *If $\dim_{\mathcal{M}}(N) = 1$ and $\mathcal{N}$ is stable then no field is interpretable in $\mathcal{N}$.*

Proof. Assume such a field K were interpretable in $\mathcal{N}$. Because $\mathcal{N}$ is superstable K must be algebraically closed ([4]), and because it is interpretable in an o-minimal theory it has to be of characteristic zero. Moreover, as is shown in [22], K has an $\mathcal{N}$-definable real closed subfield R with $[K : R] = 2$, and therefore $\dim_{\mathcal{M}}(K) = 2$. Moreover, because of the stability assumption, K with all its induced $\mathcal{N}$-structure is a pure algebraically closed field ([20]) and therefore strongly minimal. But then, there is an $\mathcal{N}$-definable finite-to-finite correspondence between K and N, implying that $\dim_{\mathcal{M}}(K) = 1$, contradiction. □

We are now ready to start the proof of Theorem 2.2. By Fact 2.1 (1), $\mathrm{U}(\mathcal{N}) = 1$. We assume towards contradiction that $\mathcal{N}$ is not 1-based. By (2), we may assume that $\mathcal{N}$ is strongly minimal and not locally modular. Our strategy will be to contradict strong minimality by one of two ways. Either we find a stationary type with two non-forking extensions, or we will interpret a field in $\mathcal{N}$, contradicting 2.3.

Recall that we use $\dim_{\mathcal{N}}$ to denote the dimension of tuples and sets in the sense of the geometric structure $\mathcal{N}$. Since $\mathcal{N}$ is strongly minimal this is the same as Morley Rank.

Non-local modularity of $\mathcal{N}$ supplies us with a two-dimensional family of $\mathcal{N}$-definable plane curves, to which we intend to apply the results of the previous section. So we remind:

Definition 2.4. A definable (possibly in $\mathcal{N}^{eq}$) *pseudoplane* is given by two sets P, Q, together with a definable set $F \subseteq P \times Q$, such that the following hold:
(i) $\mathrm{MR}(P) = \mathrm{MR}(Q) = 2$.
(ii) For every $q \in Q$, $\mathrm{MR}(F(P, q)) = 1$ (where $F(P, q) = \{p \in P : \langle p, q\rangle \in F\}$).
(iii) For every $p \in P$, $\mathrm{MR}(F(p, Q)) = 1$ (where $F(p, Q) = \{q \in Q : \langle p, q\rangle \in F\}$).
(iv) For every $q_1 \neq q_2$ in Q, $F(P, q_1) \cap F(P, q_2)$ is finite.
(v) For every $p_1 \neq p_2$ in P, $F(p_1, Q) \cap F(p_2, Q)$ is finite.

Given a pseudoplane as above, $p \in P, q \in Q$, we let $C_q = F(P,q)$ and $\ell_p = F(p,Q)$. We also let $\mathcal{F}$ denote the family of curves $\{C_q : q \in Q\}$ in P, and let $\mathcal{L}$ denote the dual family, $\{\ell_p, p \in P\}$.

The following can be found in [26] (or see Proposition 1.7 on p.155 of [24]):

Fact 2.5. *If $\mathcal{N}$ is any strongly minimal non locally modular structure, then a pseudoplane is interpretable in $\mathcal{N}$. Moreover, (see, e.g.* [11]*), P may be taken to be a subset of N^2 (but without EI, we cannot assume at the same time that Q is a subset of N^2 as well).*

From now on we work with a fixed such pseudoplane that we assume is interpretable in $\mathcal{N}$. From the point of view of $\mathcal{M}$, the sets P and Q are 2-dimensional definable (using $\mathcal{M}$-definable choice for $\mathcal{N}$) sets, with $\mathcal{F}$ and its dual $\mathcal{L}$ definable as well. By (iv) and (v), both $\mathcal{F}$ and $\mathcal{L}$ are normal families. We may assume that P, Q and each generic curve in $\mathcal{F}$ and $\mathcal{L}$ are of Morley degree 1. We may also assume that P, Q and $\mathcal{F}$ are all $\emptyset$-definable.

Our goal is to use the intersection theory developed in Section 1. As we will see, this theory will be most useful when we are in a sufficiently general situation, see Case I below. In the complimentary case (Case II below) we will need to work harder in order to define a field in $\mathcal{N}$ and thus to obtain the desired contradiction. Towards this goal we make the following ad-hoc definition.

Definition 2.6. An $\mathcal{N}$-definable plane curve X will be called *good with respect to* $\mathcal{F}$ if:
(1) The $\mathcal{M}$-rank of X (over $\emptyset$) is at least 3.
(2) There exists a generic $p_0 \in X$ and $C_{q_0} \in \mathcal{F}$, touching X at p_0 such that: For every $s \in N^2$ which is special for (q_0, X), if $s \neq p_0$ then $\dim_{\mathcal{M}}(s/p_0, q_0) = 1$.

Example 2.7. Let us see that in general it is indeed possible, for a fixed family $\mathcal{F}$, to have many definable curves of $\mathcal{M}$-rank 3 which are not good with respect to $\mathcal{F}$. We work in an expansion $\mathcal{M}$ of a real closed field.

Let $\mathcal{L} = \{\ell_p : p \in P\}$ be a definable normal family of plane curves such that for every $q \in Q$ (the domain of $\mathcal{L}$) there are two of the ℓ_p with the same slope at q. (For example, take the union of the family of affine lines with a suitable 2-dimensional family of parabolas). Assume moreover, that the dual family $\mathcal{F} = \{C_{:} q \in Q\}$ is normal as well.

For a definable curve $Y \subseteq Q$, consider the curve $X = \tau(Y) \subseteq M^2$. Because every $q \in Y$ has two different $\mathcal{L}$-touching curves ℓ_p and $\ell_{p'}$, it follows from the Duality Theorem that C_q touches X at both p and p'. We thus have p' special for (q, X), $\dim_{\mathcal{M}}(p'/p, q) = 0$ (using the fact that $\ell_{p'}$ is the only curve whose slope equals to that of ℓ_p at q), implying that X is not good with respect to $\mathcal{F}$.

If Y is a curve of large $\mathcal{M}$-rank (say, a generic polynomial of high degree) then, by Lemma 1.32, the curve X has also large rank, at least locally, and is still not good with respect to $\mathcal{F}$.

We now start the proof of Theorem 2.2. Namely, $\mathcal{N}$ is strongly minimal with a definable 2-dimensional normal family of plane curves (possibly, parameterized over a set in $\mathcal{N}^{eq}$).

Case I Assume that there is an $\mathcal{N}$-definable curve X that is good with respect to $\mathcal{F}$.

We thus assume that X has $\mathcal{M}$-rank $\geqslant 3$, p_0 generic in X and C_{q_0} touches X at p_0. We let $p_0 = s_0, s_1, \ldots, s_r$ be all the special points for (q_0, X). We also assume that every s_i, $i = 1, \ldots, k$, has dimension 1 over p_0, q_0.

By Lemma 1.39 there exist a neighbourhood W of q_0 and pairwise-disjoint neighbourhoods U_i of s_i, $i = 0, \ldots, r$, respectively, such that for every $q \in W$, every intersection point of C_q with X belongs to one of the U_i. Furthermore, by Lemma 1.41 we can choose W and the U_i in such a way that for each $q \in W$ and $i \geqslant 1$, the curve C_q intersects X in U_i exactly once.

Now, by Lemma 1.14, we can find $q_1, q_2 \in W$ both $\mathcal{N}$-generic over all the data, such that, after possibly shrinking U_0 further, we have $C_{q_1} \cap X \cap U_0 = \emptyset$ while $|C_{q_2} \cap X \cap U_0| > 1$.

It follows that

$$|X \cap C_{q_1}| = r,$$

while

$$|X \cap C_{q_2}| \geqslant r + 2.$$

This contradicts the assumption that the Morley degree of Q is 1, ending the proof of Theorem 2.2 in Case I.

For the rest of this section we handle the remaining case.

Case II Assume that there is no $\mathcal{N}$-definable good curve with respect to $\mathcal{F}$.

Our final goal is to reach a contradiction by defining a field in $\mathcal{N}$. This will require the following two configuration results of Hrushovski. (see [1] for the proof of the first statement and, in a slightly different terminology, Theorem 5.4.5 and 1.6.25 of [24], for the proof of the second):

Theorem 2.8. *[Hrushovski] Let D be a strongly minimal set. A* group configuration *in D (over a parameter set A) is a set $Q := \{a_1, a_2, a_3, b_1, b_2, b_3\}$ of tuples in D with the following properties:*

1. $\mathrm{MR}(a_i/A) = \mathrm{MR}(b_i/A) = 1$ *for every* $i \in \{1, 2, 3\}$.
2. $\mathrm{MR}(Q/A) = 3$.
3. *Each of the sets $\{a_1, a_2, a_3\}$, $\{a_1, b_1, b_2\}$, $\{a_2, b_2, b_3\}$,$\{a_3, b_1, b_3\}$ has Morley Rank 2 over A.*
4. *Every other 3-elements subset of Q is independent over A (in particular, any two elements of Q are pairwise independent over A).*

If D admits a group configuration then a strongly minimal group is interpretable in D.

Given a strongly minimal group G, a standard group configuration is:

$$\{a, b, ab, c, ac, abc\},$$

for a, b, c independent in G. We will also need another variant, known as the field configuration:

Theorem 2.9. *[Hrushovski] Let D be a strongly minimal set. A* field configuration *in D (over a parameter set A) is a set $Q := \{a_1, a_2, a_3, b_1, b_2, b_3\}$ of tuples in D with the following properties:*

1. $\mathrm{MR}(a_i/A) = 2$ *and* $\mathrm{MR}(b_i/A) = 1$ *for every* $i \in \{1, 2, 3\}$.

(2) $\mathrm{MR}(Q/A) = 5$.
(3) *The set* $\{a_1, a_2, a_3\}$ *has Morley rank* 4 *over* A
(4) *Each of the sets* $\{b_1, b_2, a_1\}$, $\{b_1, b_3, a_3\}$, $\{b_2, b_3, a_2\}$ *has Morley Rank* 3 *over* A.
(5) *Every other* 3*-elements subset of* Q *is independent over* A.
(6) *The configuration is minimal in the following sense: If* $a'_i \in \mathrm{acl}(Aa_i)$ *and* $\mathrm{MR}(a'_i/A) = 1$ *then the set* $Q := \{a'_1, a'_2, a'_3, b_1, b_2, b_3\}$ *is not a group configuration.*

If D *admits a field configuration then a strongly minimal field is interpretable in* D.

Given a strongly minimal field F, a standard field configuration is given by the natural action of $A = \{cx + d : c, d \in F\}$ on the set F. Namely, we start with $f_1, f_2 \in A$, $b_1 \in F$ independent generics in their respective sets, then

$$\{f_1\,,\, f_2\,,\, f_1 \circ f_2\,,\, b_1\,,\, f_1(b_1)\,,\, f_1 \circ f_2(b_1)\}$$

is a field configuration.

We now fix an $\mathcal{N}$-definable pseudo-plane $\mathcal{F}$ and consider the family

$$\mathcal{G} = \{C_{q_3} \circ C_{q_2}^{-1} \circ C_{q_3} : q_1, q_2, q_3 \in Q\}.$$

(By $C_{q_2}^{-1} \circ C_{q_1}$ we mean the set $\{\langle x_1, x_2\rangle \in P^2 : \exists y(\langle x_1, y\rangle \in C_{q_1} \,\&\, \langle x_2, y\rangle \in C_{q_2})\}$.)

Note that if the projection of the curves C_{q_1} or C_{q_2} on both coordinate axis is finite-to-one then the composition $C_{q_2}^{-1} \circ C_{q_1}$ is a one-dimensional set which again projects finite-to-one on both coordinates. Therefore, since every generic curve in $\mathcal{F}$ has this property (because $\mathcal{F}$ is normal and two-dimensional) all $C_{q_3} \circ C_{q_2}^{-1} \circ C_{q_1}$, for q_1, q_2, q_3 independent generic, are one-dimensional. Hence, without loss of generality, $\mathcal{G}$ is 0-definable and every set in it is 1-dimensional (we throw away all curves from the original $\mathcal{F}$ which don't project finite-to-one on both coordinates).

For $q_1, q_2, q_3 \in Q$, we write $C_{q_1q_2q_3}$ for the curve $C_{q_1}C_{q_2}^{-1}C_{q_3}$, read as the composition of the three curves. While the curve $C_{q_1q_2q_3}$ might not be strongly minimal, hence not have an $\mathcal{M}$-rank, it is not hard to see that for $\langle q_1, q_2, q_3\rangle$ generic in Q^3, the $\mathcal{M}$-rank of $C_{q_1q_2q_3}$ at every generic point exists and is at least 2 (this can be done, using 1.24, by first showing it for the curve $C_{q_1}C_{q_2}^{-1}$).

We start with some basic facts about differentiability properties of curves in $\mathcal{G}$.

Proposition 2.10. *Let* $\langle \hat{q}_3, \hat{q}_2, \hat{q}_1\rangle$ *be* $\mathcal{M}$*-generic in* Q^3, $p_0 = \langle x_0, y_0\rangle$ $\mathcal{M}$*-generic in* $C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ *(over* $\hat{q}_1\hat{q}_2\hat{q}_3$*) and*

$$p_1 = \langle x_0, z_1\rangle \in C_{\hat{q}_1}\ ,\ p_2 = \langle z_2, z_1\rangle \in C_{\hat{q}_2}\ ,\ p_3 = \langle z_2, y_0\rangle \in C_{\hat{q}_3}.$$

Then,

(1) $\dim_{\mathcal{M}}(x_0, z_1, z_2, y_0) = 4$.
(2) *There exist open neighborhoods* $U_0, U_1, U_2, U_3 \subseteq M^2$ *of* p_0, p_1, p_2, p_3*, respectively, and* $W_1, W_2, W_3 \subseteq Q$ *of* $\hat{q}_1, \hat{q}_2, \hat{q}_3$*, respectively, such that for every* $q_i \in U_i$*,* $i = 1, 2, 3$*, the set* $C_{q_i} \cap U_i$ *is the graph of a* C^1 *strictly monotone function* f_{q_i}*, and* $C_{q_3q_2q_1} \cap U_0$ *is the graph of* $f_{q_3} \circ f_{q_2}^{-1} \circ f_{q_1}$.
(3) *Assume that* p_0 *is generic in* C_{q_0} *over* q_0 *and let* d_0 *be the slope of* C_{q_0} *at* p_0*. Assume that* $I \subseteq R$ *is an open neighbourhood of* d_0*.*
Then there are open neighbourhoods W *of* $\langle q_0, q_0, q_0\rangle$ *and* $U \subseteq N^2$ *of* p_0*, such that for every generic triple* $\langle q_1, q_2, q_3\rangle \in W$ *there exists a generic*

$p \in C_{q_3q_2q_1} \cap U$ such that the slope of $C_{q_3q_2q_1}$ at p is in I.

For $i = 1, 2, 3$, let Q_i be the set of all $q \in Q$ such that $p_i \in Q_i$.

(4) *Let d_0 be the slope of $C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ at p_0.*
Then, for every $q_1 \in \nu_{\hat{q}_1}(Q_1)$ and $q_2 \in \nu_{\hat{q}_2}(Q_2)$ there exists a unique $q_3 \in \nu_{\hat{q}_3}(Q_3)$ such that the slope of $C_{q_3q_2q_1}$ at p_0 is d_0. The same is true for any permutation of the q_i.

Proof. (1) This is not hard to compute.

(2) This is immediate from genericty.

(3) Because $p_0 = \langle x_0, y_0\rangle$ is generic in C_{q_0} it is also generic in $C_{q_0q_0q_0}$ (which contains C_{q_0} but is possibly larger). Hence, p_0 is a C^1-point in $C_{q_0q_0q_0}$. By genericity, $d_0 \neq 0$.

We let f_{q_0} denote the function associated to C_{q_0} near p_0. Namely, f_{q_0} is (a strictly monotone) C^1-function and the graph of f_{q_0} agrees with C_{q_0} near p_0. By genericity, there exist neighbourhoods $W_1 \subseteq Q$ of q_0 and $U_1 \subseteq N^2$ of $\langle x_0, y_0\rangle$ such that $C_q \cap U_1$ is the graph of a C^1 function f_q for every $q \in W_1$. Moreover, the map $\langle q, x\rangle \mapsto f'_q(x)$ is C^1 and nowhere vanishing at $\langle q_0, x_0\rangle$.

Clearly, the graph of $f_{q_3} \circ f_{q_2}^{-1} \circ f_{q_1}$ is a 1-dimensional subset of the curve $C_{q_3q_2q_1}$ (however, the curve may contain other branches as well).

By the chain rule for C^1-functions in real closed fields, we have, for $f_{q_1}(x) = z_1$, $f_{q_2}^{-1}(z_1) = z_2$ and $f_{q_3}(z_2) = y$,

$$(f_{q_3} \circ f_{q_2}^{-1} \circ f_{q_1})'(x) = (f'_{q_3}(z_2) \cdot f'_{q_1}(x))/f'_{q_2}(z_2).$$

By the continuity of the map $\langle q, z\rangle \mapsto f'_q(z)$ and the fact that $d_0 \neq 0$ we can find $W \ni \langle q_0, q_0, q_0\rangle$ and $U \ni p_0$ such that, for $\langle q_1, q_2, q_3\rangle \in W$, the slope of $f_{q_3} \circ f_{q_2}^{-1} \circ f_{q_1}$ in U belongs to I. Every generic point on the graph of this function can serve as the desired p.

(4) By (1), we are allowed to use functional notation. By the chain rule, for $q_i \in \nu_{\hat{q}_i}(Q_i)$, the derivative of $C_{q_3q_2q_1}$ at x_0 equals $f'_{q_3}(z_2)f'_{q_1}(x_0)/f'_{q_2}(z_2)$ (using 1.25, we have $f_{\hat{q}_2}(z_2) \neq 0$). By genericity, the map $(q, x) \mapsto f'_q(x)$ is continuous near each of the points $\langle \hat{q}_1, x_0\rangle$, $\langle \hat{q}_2, z_2\rangle$, $\langle \hat{q}_3, z_2\rangle$. Also, by 1.25, the maps $q \mapsto f'_q(x_0)$, $q \mapsto f'_q(z_2)$ and $q \mapsto f'_q(z_2)$ induce continuous monotone bijections between $\nu_{\hat{q}_1}(Q_1)$, $\nu_{\hat{q}_2}(Q_2)$, $\nu_{\hat{q}_3}(Q_3)$ and the infinitesimal neighborhoods of $f'_{\hat{q}_1}(x_0)$, $f'_{\hat{q}_2}(z_2)$ and $f'_{\hat{q}_3}(z_2)$ in the field R, respectively. The result follows. □

The following general proposition and its later additive analogue are the main tools we will be using here.

Proposition 2.11. *Let $q_0 \in Q$ be generic and $p_0 = \langle x_0, y_0\rangle \in C_{q_0}$ generic over q_0. Assume that there exists an $\mathcal{N}$-definable $R_\circ \subseteq Q^3$ such that:*
(i) $\dim_{\mathcal{M}}(R_\circ) = 4$ and
(ii) for some $\mathcal{M}$-generic $\langle \hat{q}_3, \hat{q}_2, \hat{q}_1\rangle \in R_\circ$ over p_0, q_0 and the parameters defining $R_\circ$, we have $p_0 \in C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ and C_{q_0} touches $C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ at p_0.
Let

$$p_3 = \langle z_2, y_0\rangle \in C_{\hat{q}_3}\ ,\ \ p_2 = \langle z_2, z_1\rangle \in C_{\hat{q}_2}\ ,\ p_1 = \langle x_0, z_1\rangle \in C_{\hat{q}_1},$$

and $Q_2 = \{r \in Q : p_2 \in C_r\}$. For $r \in Q_2$, let $d(r) \in R$ denote the slope of C_r at $\langle z_2, z_1\rangle$.

Then there is a set of parameters A such that for all $r_1, r_2, r_3, r_4 \in \nu_{\hat{q}_2}(Q_2)$, if $\dim_{\mathcal{M}}(r_1r_2/A) = \dim_{\mathcal{M}}(r_3r_4/A) = 2$ and if $d(r_1)/d(r_2) = d(r_3)/d(r_4)$ (in the field R) then each r_i is in the $\mathrm{acl}_{\mathcal{N}}$-closure of the set $\{r_j : j \neq i\} \cup A$.

Proof. The set

$$R_t = \{\langle q_1, q_2, q_3\rangle \in Q^3 : C_{q_3q_2q_1} \text{ touches } C_{q_0} \text{ at } p_0\}$$

is definable in $\mathcal{M}$ over p_0q_0 and, as is not hard to see, has dimension 4 in a neighborhood of $\langle \hat{q}_3, \hat{q}_2, \hat{q}_1\rangle$ (the set R_{p_0} of all $q_3q_2q_1$ with $p_0 \in C_{q_3q_2q_1}$ has dimension 5, while the set of $\bar{q} \in R_{p_0}$ such that $C_{\bar{q}}$ has the same slope as C_{q_0} at p_0, has dimension 4). Hence, $\langle \hat{q}_3, \hat{q}_2, \hat{q}_1\rangle$ is, at least locally, $\mathcal{M}$-generic in R_t over p_0q_0 and therefore has a neighborhood $W \subseteq Q^3$ where the two relations $R_\circ$ and R_t coincide.

We let Q_i, $i = 1, 2, 3$, be the set of all q such that $p_i \in C_q$. So, each Q_i is definable in $\mathcal{N}$ (over p_i) and 1-dimensional. Each $\hat{q}_i$ is $\mathcal{M}$-generic in Q_i over $\{x_0, y_0, z_1, z_2\}$, and $\hat{q}_1, \hat{q}_2, \hat{q}_3$ are $\mathcal{M}$-independent over $\{x_0, y_0, z_1, z_2\}$. While each Q_i might not be strongly minimal there is a strongly minimal $Q_i' \subseteq Q_i$, defined over $\mathrm{acl}_{\mathcal{N}}(p_i)$, such hat $\hat{q}_i \in Q_i'$. By replacing Q_i by Q_i' we will assume from now on that Q_i is strongly minimal. We denote elements of Q_2 by r, r_1, r_2, r_3, r_4 and elements of Q_i, $i = 1, 3$, by q_i, q_i'. For $q_1 \in Q_1, r \in Q_2$ and $q_3 \in Q_3$ we denote the composition curves $C_{q_3} \circ C_r^{-1}$ and $C_r^{-1} \circ C_{q_1}$ by C_{q_3r} and C_{rq_1}, respectively. Finally, for $r_1, r_2 \in Q_2$, we let $C_{r_1r_2}$ denote the curve $C_{r_1}C_{r_2}^{-1}$.

Let $\hat{r} := \hat{q}_2$. In abuse of notation, we will use from now on $\nu_{\hat{q}_1}, \nu_{\hat{r}}, \nu_{\hat{q}_3}$ to denote the relative infinitesimal neighbourhoods $\nu_{\hat{q}_1} \cap Q_1$, $\nu_{\hat{r}} \cap Q_2$ and $\nu_{\hat{q}_3} \cap Q_3$, repsectively. As was pointed out above, for $q_3 \in \nu_{\hat{q}_3}, r_2 \in \nu_{\hat{r}}, q_1 \in \nu_{\hat{q}_1}$, we have $R_\circ(q_3, r_2, q_1)$ if and only if C_{q_0} has the same slope as $C_{q_3r_2q_1}$ at p_0.

We fix A a set of parameters containing x_0, y_0, z_1, z_2 and the parameters used to define $R_\circ$.

Claim 2.12. *(1) For every $r_1, r_2 \in Q_2$ with $\dim_{\mathcal{M}}(r_1r_2/A) = 2$, let f_{r_1}, f_{r_2} be the functions associated to C_{r_1} and C_{r_2}, respectively, in a neighbourhood of $\langle z_1, z_2\rangle$. Then the curve $C_{r_1r_2}$, in a neighbourhood of $\langle z_1, z_1\rangle$, is the graph of $f_{r_1} \circ f_{r_2}^{-1}$. In particular, it is a C^1 curve in the same neighbourhood.*
(2) For every $r_1, r_2, r_3 \in \nu_{\hat{r}}$ $\mathcal{M}$-independent over A there exists $r_4 \in \nu_{\hat{r}}$, such that every triple in $\{r_1, r_2, r_3, r_4\}$ is $\mathcal{M}$-independent over A and $C_{r_1r_2}$ and $C_{r_3r_4}$ have the same slope at $\langle z_1, z_1\rangle$.

Proof. (1) Note that $\langle z_1, z_1\rangle$ is not generic in $C_{r_1r_2}$ so indeed some argument is needed here. Since $f_{r_1} \circ f_{r_2}$ is the graph of a C^1-function near $\langle z_1, z_1\rangle$ and its graph is contained in the 1-dimensional set $C_{r_1r_2}$, the only way that the claim can fail is if $C_{r_1r_2}$ has another local component in a neighbourhood of $\langle z_1, z_1\rangle$. This in turn implies that there exists a neighbourhood $U \ni z_2$ and there are functions $z_1(t), \hat{z}_1(t)$, $z_2(t)$ for $t \in (0, 1)$ such that $z_1(t), \hat{z}_1(t)$ tend to z_1 as $t \to 0$, $z_2(t) \notin U$ for every t and $\langle z_2(t), z_1(t)\rangle, \langle z_2(t), \hat{z}_1(t)\rangle \in C_{r_1}$.

We let $\hat{z}_2 \in M \cup \{\pm\infty\}$ be the limit of $z_2(t)$ as t tends to 0. Then $\hat{z}_2 \neq z_2$ and $\langle \hat{z}_2, z_1\rangle \in \mathrm{cl}(C_{q_1}) \cap \mathrm{cl}(C_{q_2})$.

If $\hat{z}_2 = \pm\infty$, or if $\langle \hat{z}_2, z_1\rangle$ belongs to the frontier of either C_{r_1} or C_{r_2} then z_1 would not be generic over the parameters defining this curve, contradiction. So we may assume that $\langle \hat{z}_2, z_1\rangle \in C_{r_1} \cap C_{r_2}$. Since the set $\{z : \langle z, z_1\rangle \in C_{r_1}\}$ is finite we have $z_2, \hat{z}_2 \in \mathrm{dcl}_{\mathcal{M}}(z_1r_1)$. Because r_2 was chosen generic in Q_2 over z_1, r_1, p_2, we

get infinitely many $r \in Q_2$ such that $\langle z_2, z_1\rangle, \langle \hat{z}_2, z_1\rangle \in C_r$. This contradicts the normality of the dual family $\mathcal{L}$.
(2) This follows, using the chain rule, from 1.25 and 1.26, just like the proof of 2.10(4). End of Claim.

We define a relation $T_\circ$ as follows: For $r_1, r_2, r_3, r_4 \in Q_2$: $T_\circ(r_1, r_2, r_3, r_4)$ if there are $q_1, q_1' \in Q_1$ and $q_3, q_3' \in Q_3$ such that

$$\langle q_3, r_1, q_1\rangle, \langle q_3, r_2, q_1'\rangle, \langle q_3', r_3, q_1\rangle, \langle q_3', r_4, q_1'\rangle \in R_\circ$$

Clearly, $T_\circ$ is $\mathcal{N}$-definable over the parameter set A. The proposition will be proved once we show:

Claim 2.13. *Take $r_1, r_2, r_3, r_4 \in \nu_{\hat{r}}$, with $\dim_{\mathcal{M}}(r_1r_2A) = \dim_{\mathcal{M}}(r_3r_4A) = 2$. If $d(r_1)/d(r_2) = d(r_3)/d(r_4)$ then: (i) $\langle r_1, r_2, r_3, r_4\rangle \in T_\circ$ and (ii) for each $i = 1, 2, 3, 4$, r_i is in the $\mathrm{acl}_{\mathcal{N}}$-closure of the set $\{r_j : j \neq i\} \cup A$.*

Proof. By the chain rule (and 2.12(1)), the curves $C_{r_1r_2}$ and $C_{r_3r_4}$ have the same slope at $\langle z_2, z_1\rangle$. We start with an arbitrary $q_3 \in \nu_{\hat{q}_3}$ $\mathcal{M}$-generic over $Ar_1r_2r_3r_4$.

We still use d_0 to denote the slope of $C_{\hat{q}_3\hat{r}\hat{q}_1}$ at p_0. By 2.10(4), there are $q_1, q_1' \in \nu_{\hat{q}_1}$ such that $C_{q_3r_1q_1}$ and $C_{q_3r_2q_1'}$ have slope d_0 at p_0. By Theorem 1.28, the curve C_{q_0} touches $C_{q_3r_1q_1}$ and $C_{q_3r_2q_1'}$ at p_0. It follows that $\langle q_3, r_1, q_1\rangle$ and $\langle q_3, r_2, q_1'\rangle$ belong to $R_\circ$. Applying 2.10 again, we can find $q_3' \in \nu_{q_3}$ such that the slope of $C_{q_3'r_3q_1}$ at p_0 is d_0. Again, by 1.28, C_{q_0} touches $C_{q_3'r_3q_1}$ at p_0, implying that $\langle q_3', r_3, q_1\rangle$ is in $R_\circ$.

We proceed using functional notation. By our assumption on the r_i,

$$(f_{q_1}^{-1}f_{r_1}f_{r_2}^{-1}f_{q_1'})'(x_0) \;=\; (f_{q_1}^{-1}f_{r_3}f_{r_4}^{-1}f_{q_1'})'(x_0) = y_0.$$

Therefore,

$$[(f_{q_1}^{-1}f_{r_1}f_{q_3}^{-1})(f_{q_3}f_{r_2}^{-1}f_{q_1'})]'(x_0) \;= [(f_{q_1}^{-1}f_{r_3}f_{q_3'}^{-1})(f_{q_3'}f_{r_4}^{-1}f_{q_1'})]'(x_0).$$

Because $C_{q_3r_1q_1}$, $C_{q_3r_2q_1'}$ and $C_{q_3'r_3q_1}$ all have slope d_0 at p_0 it follows from the above (using compositional inverse and, again, the chain rule), that $C_{q_3'r_4q_1'}$ also has the same slope at p_0 and, as before, that $\langle q_3', r_4, q_1'\rangle \in R_\circ$. We therefore showed that $r_1r_2r_3r_4$ satisfies the definition of $T_\circ$.

Let's see why $r_4 \in \mathrm{acl}_{\mathcal{N}}(r_1, r_2, r_3A)$: First note that, in the above argument, the fact that $\langle r_1, r_2, r_3, r_4\rangle \in T_\circ$ was witnessed by an arbitrary $q_3 \in \nu_{\hat{q}_3}$ and by $q_1, q_1' \in \nu_{\hat{q}_1}$ and $q_3' \in \nu_{\hat{q}_3}$. By 2.10(4), $q_1 \in \mathrm{acl}_{\mathcal{N}}(q_3r_1A)$ (for otherwise there will be infinitely many $q \in \nu_{\hat{q}1}$ with $\langle q_3, r_1, q\rangle \in R_\circ$). Similarly, $q_1' \in \mathrm{acl}_{\mathcal{N}}(q_3r_2A)$, and because $q_1, q_1' \in \nu_{\hat{q}_1}$, we also have $q_3' \in \mathrm{acl}_{\mathcal{N}}(r_3q_1'A)$ and $r_4 \in \mathrm{acl}_{\mathcal{N}}(q_3'q_1'A)$. It follows that $r_4 \in \mathrm{acl}_{\mathcal{N}}(q_3r_1r_2r_3A)$. Because q_3 was chosen generic in Q_3 (over $r_1r_2r_3r_4A$) it follows that $r_4 \in \mathrm{acl}_{\mathcal{N}}(r_1r_2r_3A)$. We similarly prove all the other $\mathrm{acl}_{\mathcal{N}}$-requirements.

This ends the proof of the claim and therefore also of Proposition 2.11. □

We are now ready to prove:

Lemma 2.14. *A strongly minimal group is interpretable in $\mathcal{N}$.*

Proof. We first want to produce a generic triple $\hat{q}_3, \hat{q}_2, \hat{q}_1 \in Q$ and a generic $p_0 \in C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ with a touching curve C_{q_0} at p_0.

Let q be generic in Q and p generic in C_q, and let d_0 be the slope of C_q at p. By Lemma 1.25 there exists an interval I with $d_0 \in I$ such that every $d \in I$ is realized as the slope at p of some curve $C_{q'}$ in $\mathcal{F}$. Using the genericity of p in P, we can find an open interval $I_1 \ni d_0$, $I_1 \subseteq I$ and a neighbourhood U of p such that for every $p' \in U$, every $d \in I_1$ is realized as the slope of some curve $C_{q'} \in \mathcal{F}$ at p'.

By 2.10(3), there exist a neighbourhood $W \subseteq Q$ of q and a neighbourhood $U_1 \subseteq U$ of p, such that for every generic triple $q_1, q_2, q_3 \in W$ there exists a generic $p_0 \in C_{q_3q_2q_1} \cap U$ for which the slope of $C_{q_3q_2q_1}$ at p_0 is in I_1.

We now take $\hat{q}_1, \hat{q}_2, \hat{q}_3 \in W$ to be $\mathcal{M}$-generic and independent and p_0 generic in $U \cap C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ (over $\hat{q}_1\hat{q}_2\hat{q}_3$) as above and let $d \in I_1$ be the slope of $C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ at p'. By the above, there exists some $C_{q_0} \in \mathcal{F}$ whose slope at p_0 is d.

As pointed out earlier, the $\mathcal{M}$-rank of $C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ at p_0 is $\geqslant 2$ so we can apply Theorem 1.28 and conclude that C_{q_0} touches $C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ at p_0.

We still use $p_0 = \langle x_0, y_0\rangle$, $p_3 = \langle z_2, y_0\rangle \in \mathcal{C}_{\hat{q}_3}$, $p_2 = \langle z_2, z_1\rangle \in C_{\hat{q}_2}$, $p_3 = \langle x_0, z_1\rangle \in \mathcal{C}_{\hat{q}_1}$ and let $Q_i = \{q \in Q : p_i \in C_q\}$. We fix the parameter set $A = \{x_0, y_0, z_1, z_2\}$ over which the Q_i's are defined.

Let $X \subseteq C_{\hat{q}_3\hat{q}_2\hat{q}_1}$ be a strongly minimal set, defined in $\mathcal{N}$ over $\mathrm{acl}_{\mathcal{N}}(\hat{q}_3\hat{q}_2\hat{q}_1)$ and containing p_0. In particular, p_0 is still generic in X over $\hat{q}_3\hat{q}_2\hat{q}_1$. By 1.24, the curve X has a $\mathcal{M}$-rank, which, as we said already, is at least 2 (because $\mathcal{F}$ was 2-dimensional). Hence, there exists an $\mathcal{M}$-definable normal family $\mathcal{Y} = \{X_t : t \in T\}$ over $\emptyset$, with $\dim(T) \geqslant 2$, $X \sim X_{t_0}$ and t_0 $\mathcal{M}$-generic in T. We consider two cases, both yielding in an $\mathcal{N}$-definable relation $R_\circ$ as required in Proposition 2.11.

We first prove a simple dimension fact.

Fact 2.15. *For any $p' \neq p'' \in N^2$, the set $Z = \{\langle q_3, q_2, q_1\rangle \in Q^3 : p', p'' \in C_{q_3q_2q_1}\}$ has dimension $\leqslant 4$.*

Proof. It is sufficient to see that for any fixed $q_2, q_1 \in Q^2$, the set $Z_0 = \{q \in Q : p', p'' \in C_{qq_2q_1}\}$ is finite. Let $Y = C_{q_2}^{-1}C_{q_1}$ and recall that Y projects finite-to-one on each of the coordinates (because the same is true for each of the curves in $\mathcal{F}$). If $p' = \langle x', y'\rangle$ and $\langle x', y'_1\rangle, \ldots, \langle x', y'_\ell\rangle$ are all the points of Y which project onto x', then for every C_q with $p' \in C_q \circ Y$, we have some $\langle y'_i, y'\rangle \in C_q$. Similarly, if $p'' = \langle x'', y''\rangle$ then there are $y''_1, \ldots, y''_r$ such that for each C_q with $p'' \in C_q \circ Y$, we have some $\langle y''_j, y''\rangle \in C_q$ for some j. Hence, if Z_0 is infinite then there are two points $\langle y'_i, y'\rangle$ and $\langle y''_j, y''\rangle$ belonging to infinitely many of the C_q. This contradicts the normality of $\mathcal{L}$ unless $y'_i = y''_j$ and $y' = y''$.

But now, we repeat this process with $X = C_{q_3}C_{q_2}^{-1}$ and the family $Z_1 = \{q \in Q : p', p'' \in X \circ C_q\}$. If Z_1 were infinite then, as above, we must have $x' = x''$.

Put together we obtain: If $p' \neq p''$ then $\dim(Z) = 4$. □

Assume first that the $\mathcal{M}$-rank of X is 2, so $\dim(T) = 2$. Consider the $\mathcal{N}$-definable set

$$R_\circ = \{\langle q_3, q_2, q_1\rangle \in Q^3 : X \sqsubseteq C_{q_3q_2q_1}\},$$

where '$X \sqsubseteq Y$' means 'all but finitely many points of X are in Y'. Notice that by replacing X with X_{t_0}, the set $R_\circ$ set can also be defined, in $\mathcal{M}$, over t_0.

We now calculate dimension in $\mathcal{M}$. It is easy to see that t_0 is $\mathcal{M}$-algebraic over $\hat{q}_3\hat{q}_2\hat{q}_1$ and therefore a simple computation shows that $R_\circ$ has dimension 4. We have $\dim_{\mathcal{M}}(\hat{q}_3\hat{q}_2\hat{q}_1p_0q_0) = 7$, $\dim_{\mathcal{M}}(q_0p_0) = 3$ and therefore $\dim_{\mathcal{M}}(\hat{q}_3\hat{q}_2\hat{q}_1/p_0q_0) = 4$. Also, by applying 1.26 to the family $\mathcal{Y}$, t_0 and the slope of C_{q_0} at p_0, we see that $t_0 \in \mathrm{acl}_{\mathcal{M}}(p_0q_0)$. It follows that $\langle\hat{q}_3, \hat{q}_2, \hat{q}_1\rangle$ is $\mathcal{M}$-generic in $R_\circ$ over $t_0p_0q_0$, so we have all the assumptions of Proposition 2.11.

Assume now that the $\mathcal{M}$-rank of X is $\geqslant 3$. By what we have done so far, we have p_0 generic in X and C_{q_0} touching X at p_0. By our standing assumption of Case II, X is not a good curve with respect to $\mathcal{F}$ and therefore there exists s_1 which is special for (q_0, X) and such that $s_1 \in \mathrm{dcl}_{\mathcal{M}}(p_0 q_0)$. By Lemma 1.40, the point s_1 is generic in X over $\hat{q}_3\hat{q}_2\hat{q}_1$ and in particular $s_1 \in X$, and is not in Fr(X) (this is a crucial application of v.d. Dries' theory of limit sets). Notice that $s_1 \notin \mathrm{dcl}_{\mathcal{M}}(p_0)$ for otherwise $\dim_{\mathcal{M}}(q_0/p_0, s_1) = 1$ and hence there will be infinitely many C_q from $\mathcal{F}$ going through p_0 and s_1, contradicting the normality of the dual family $\mathcal{L}$.

We can now define $R_\circ$ as follows:

$$R_\circ = \{\langle q_3, q_2, q_1\rangle :\in Q^3 : p_0, s_1 \in C_{q_3q_2q_1}\}.$$

Clearly, $R_\circ$ is $\mathcal{N}$-definable over $p_0 s_1$, and contains $\langle \hat{q}_3, \hat{q}_2, \hat{q}_1\rangle$. As we saw above, $\dim_{\mathcal{M}}(\hat{q}_3\hat{q}_2\hat{q}_1/p_0q_0) = 4$ and since $s_1 \in \mathrm{acl}_{\mathcal{M}}(p_0q_0)$, we also have $\dim_{\mathcal{M}}(\hat{q}_3\hat{q}_2\hat{q}_1/p_0q_0s_1) = 4$. By 2.15, $\dim R_\circ = 4$ so we have the assumptions of Proposition 2.11

We can now return to the task of finding a group configuration in $\mathcal{N}$, applying Proposition 2.11

For r in Q_2 near $\hat{r} := \hat{r}_2$, let $d(r)$ denote the slope of C_r at $\langle z_2, z_1\rangle$. Notice that the slope of $C_{r_1r_2}$ at $\langle z_1, z_1\rangle$ is $d(r_1)/d(r_2)$ and, by continuity of the map $r \mapsto f'_r(z_2)$, when r_1, r_2 vary in $\nu_{\hat{r}}$, the family of slopes of $C_{r_1r_2}$ is exactly the infinitesimal group ν_1 (the infinitesimal neighbourhood of 1 in the real closed field R). Moreover, it is not hard to see that the family of slopes of $C_{r_1\hat{r}}$, as r_1 varies in $\nu_{\hat{r}}$, equals ν_1 as well.

We start with a group configuration in the sense of $\mathcal{M}$: Namely we take a_1, a_2, b_1 infinitesimally close to 1 of the field R (namely, in ν_1) and $\mathcal{M}$-independent over $\hat{r}A$. Let $a_3 = a_1 \cdot a_2$, $b_2 = a_1 \cdot b_1$ and $b_3 = a_1 \cdot a_2 \cdot b_1$. Note that

$$S_1 = \{a_1, a_2, a_3, b_1, b_2, b_3\} \subseteq \nu_1$$

and forms a group configuration over $\hat{r}A$ with respect to $\mathrm{acl}_{\mathcal{M}}$ (by that we mean that the conditions in 2.8 hold when MR is replaced with $\dim_{\mathcal{M}}$ and independence is taken in $\mathcal{M}$). We now choose $r_i \in \nu_{\hat{r}}$, $i = 1, 2, 3$, with $d(r_i)/d(\hat{r}) = a_i$ and $s_i \in \nu_{\hat{r}}$, $i = 1, 2, 3$, with $d(t_i)/d(\hat{r}) = b_i$.

Because the map $x \mapsto d(x)/d(\hat{r})$ is a bijection of S_1 and $S = \{r_1, r_2, r_3, s_1, s_2, s_3\}$ definable in $\mathcal{M}$ over $\hat{r}A$, it follows that S is a group configuration over $\hat{r}A$ with respect to $\mathrm{acl}_{\mathcal{M}}$. We now claim that S is also a group configuration in the sense of $\mathcal{N}$.

Indeed, Clause (1), (2), of 2.8 follow from the fact that $\dim_{\mathcal{M}}(-/\hat{r}A) \leqslant \dim_{\mathcal{N}}(-/\hat{r}A)$. Similarly, $\dim_{\mathcal{N}}(S/\hat{r}A) \geqslant 3$.

By 2.11, we have

($\diamond$)

$$r_3 \in \mathrm{acl}_{\mathcal{N}}(r_1r_2\hat{r}A)\,,\ t_2 \in \mathrm{acl}_{\mathcal{N}}(r_1s_1\hat{r}A)\,,\ \text{ and } s_3 \in \mathrm{acl}_{\mathcal{N}}(s_1r_3\hat{r}A) \cap \mathrm{acl}_{\mathcal{N}}(r_2s_2\hat{r}A),$$

which implies that S is a group configuration in $\mathcal{N}$, over $\hat{r}A$. Since the latter is strongly minimal, by Theorem 2.8, a strongly minimal abelian group $\langle G, \oplus\rangle$ is interretable in $\mathcal{N}$. With that we end the proof of Lemma 2.14. $\square$

Lemma 2.16. *A strongly minimal field is interpretable in $\mathcal{N}$.*

Proof. In the proof of Lemma 2.14 we pulled back the group configuration of the multiplicative group of the underlying real closed field, using the definability of tangency applied to composition of curves. We are about to repeat the process and in addition pull back the additive group of the field, using, instead of composition, the operation $\oplus$ we have just defined.

We start with an $\mathcal{N}$-definable 1-dimensional abelian group $\langle G, \oplus\rangle$. Because G is non-orthogonal to $\mathcal{N}$, the $\mathcal{N}$-induced structure on G is non-locally modular as well. Hence, we may assume that $\mathcal{N}$ itself is a strongly minimal non-locally modular expansion of an abelian group G. As before, we may assume that the universe of $\mathcal{N}$ is a subset of $\mathcal{M}$. By [23], we may assume, after possibly translating the group in R, that the zero of G agrees with that of the field R, denoted both by 0, and that $\oplus$ is a C^1-function in a neighbourhood $I \times I \subseteq G^2$ of $\langle 0, 0\rangle$.

We will use x_0, y_0, z_1, z_2 exactly as in the construction of the group operation, only this time we choose them in I (indeed, it is easy to verify that z_1, z_2 could have been chosen as close as we wish to the zero of the field).

Given two curves $X, Y \in \mathcal{F}$, we write (with functional notation in mind):

$$X \oplus Y = \{\langle x, y_1 \oplus y_2\rangle : \langle x, y_1\rangle \in X, \langle x, y_2\rangle \in Y\}$$

and

$$X \ominus Y = \{\langle x, y_1 \ominus y_2\rangle : \langle x, y_1\rangle \in X, \langle x, y_2\rangle \in Y\}$$

(where $\ominus$ is subtraction in G).

Our goal is to use curves of the form $C_{q_1} \ominus C_{q_2} \oplus C_{q_3}$ in the same way we used $C_{q_1} \circ C_{q_2}^{-1} \circ C_{q_3}$. We first establish some basic differential properties of $\oplus$.

Fact 2.17. *Let $M_\oplus(x, y) = x \oplus y$ and $M_\ominus(x, y) = x \ominus y$ and assume that $z_1 \in I$. All functions below are assumed to be definable and differentiable with respect to R.*

(1)
$$\frac{\partial M_\ominus}{\partial y}(z_1, z_1) = -\frac{\partial M_\ominus}{\partial x}(z_1, z_1).$$

(2) *Assume $f_1(z_1) = g_1(z_1)$, $f_1'(z_1) = g_1'(z_1)$, $f_2(z_1) = g_2(z_1)$ and $\frac{\partial M_\oplus}{\partial y}(f_1(z_1), f_2(z_1)) \neq 0$. Then*

$$f_2'(z_1) = g_2'(z_1) \Leftrightarrow (f_1 \oplus f_2)'(z_1) = (g_1 \oplus g_2)'(z_1) \Leftrightarrow$$
$$(f_1 \ominus f_2)'(z_1) = (g_1 \ominus g_2)'(z_1).$$

(3) *If $f_1(z_2) = f_2(z_2) = z_1$ and $h(x) = f_1(x) \ominus f_2(x)$ then*

$$h'(z_2) = \frac{\partial M_\ominus}{\partial x}(z_1, z_1) \cdot (f_1'(z_2) - f_2'(z_2)).$$

Proof. Consider the function $h(x) = x \ominus x = M_\ominus(x, x)$. For every z_1 we have $h'(z_1) = 0$, hence, by the chain rule we get:

$$0 = \frac{\partial M_\ominus}{\partial x}(z_1, z_1) + \frac{\partial M_\ominus}{\partial y}(z_1, z_1).$$

(2) By the chain rule,

$$(f_1 \oplus f_2)'(z_1) = \frac{\partial M_\oplus}{\partial x}(f_1(z_1), f_2(z_1))f_1'(z_1) + \frac{\partial M_\oplus}{\partial y}(f_1(z_1), f_2(z_1))f_2'(z_1).$$

By similarly expressing the derivative of $g_1(x) \oplus g_2(x)$, the result follows.

(3) is immediate from (1). □

Notation Given $q_1, q_2, q_3 \in Q$, we let $C^{\oplus}_{q_3q_2q_1}$ denote the curve $\mathcal{C}_{q_3} \ominus C_{q_2} \oplus C_{q_1}$.

We still let $A = \{z_1, z_2, x_0, y_0\}$, fix $y_1, y_2 \in I$, $\mathcal{M}$-generic and independent over A, such that $z_1 \ominus y_1$, $y_1 \oplus y_2$ and $z_1 \ominus y_1 \oplus y_2$ are all in I. We let

$$T_1 = \{t_1 \in Q : \langle z_2, y_1\rangle \in C_{t_1}\},\ Q_2 = \{r \in Q : \langle z_2, z_1\rangle \in C_q\},$$
$$T_3 = \{t_3 \in Q : \langle z_2, y_2\rangle \in C_{t_3}\}.$$

We let $w = z_1 \ominus y_1 \oplus y_2$, and then $\langle z_2, w\rangle \in C_{t_3} \ominus C_{r_2} \oplus C_{t_1}$, for every $t_3 \in T_3, r_2 \in Q_2$ and $t_1 \in T_1$. We let $p'_0 = \langle z_2, w\rangle$. We work with our fixed $\hat{r} \in Q_2$ from the earlier construction of the group and still use for short $\nu_{\hat{r}} = \nu_{\hat{r}}(Q_2)$.

We can now state the analogue of 2.10:

Proposition 2.18. *Let $\langle \hat{t}_3, \hat{t}_1\rangle$ be $\mathcal{M}$-generic in Q^2 over $\hat{r}A$ such that $p'_0 = \langle z_2, w\rangle$ is $\mathcal{M}$-generic in $C^{\oplus}_{\hat{t}_3\hat{r}\hat{t}_1}$ (over $\hat{t}_3, \hat{r}, \hat{t}_1, x_0, y_0, z_2, z_1$). Take*

$$p'_1 = \langle z_2, y_1\rangle \in C_{\hat{t}_1}\ ,\ p'_2 = \langle z_2, z_1\rangle\ ,\ p'_3 = \langle z_2, y_2\rangle \in C_{\hat{t}_3}.$$

Then,

(1) *Each quadruple of the set $\{z_2, y_1, z_1, y_2, w\}$ has $\mathcal{M}$-dimension 4 over x_0, y_0.*
(2) *There exist open neighborhoods $U_0, U_1, U_2, U_3 \subseteq M^2$ of p'_0, p'_1, p'_2, p'_3, respectively, and $W_1, W_2, W_3 \subseteq Q$ of $\hat{t}_1, \hat{r}, \hat{t}_3$, respectively, such that for every $t_i \in W_i$, $i = 1, 2, 3$, the set $C_{t_i} \cap U_i$ is the graph of a C^1 strictly monotone function f_{t_i}, and $C^{\oplus}_{t_3t_2t_1} \cap U_0$ is the graph of $f_{t_3} \ominus f_{t_2} \oplus f_{t_1}$.*
(3) *Assume that p'_0 is generic in C_{t_0} over t_0 and let d'_0 be the slope of C_{t_0} at p'_0. Assume that $I \subseteq R$ is an open neighbourhood of d'_0.*
 Then there are open neighbourhoods W of $\langle t_0, t_0, t_0\rangle$ and $U \subseteq N^2$ of p'_0, such that for every generic triple $\langle t_1, t_2, t_3\rangle \in W$ there exists a generic $p \in C^{\oplus}_{t_3t_2t_1} \cap U$ such that the slope of $C^{\oplus}_{t_3t_2t_1}$ at p is in I.
(4) *Let d'_0 be the slope of $C^{\oplus}_{\hat{t}_3\hat{r}\hat{t}_1}$ at p_0.*
 Then, for every $t_1 \in \nu_{\hat{t}_1}(T_1)$ and $r \in \nu_{\hat{r}}(Q_2)$ there exists a unique $t_3 \in \nu_{\hat{t}_3}(T_3)$ such that the slope of $C^{\oplus}_{t_3rt_1}$ at p'_0 is d'_0. The same is true for any permutation of t_3, r, t_1.

Proof. Using the notation of 2.17, we can express the function associated with $C^{\oplus}_{q_3q_2q_1}$ as $M_{\oplus}(M_{\ominus}(f_{q_1}, f_{q_2}), f_{q_3})$. Applying the chain rule to this expression allows us to imitate the proof of 2.10 with the help of 2.17.

□

Next, we state the analogue of Proposition 2.11:

Proposition 2.19. *Let $t_0 \in Q$ be generic and $p'_0 = \langle z_2, w\rangle \in C_{t_0}$ generic over t_0. Assume that there exists an $\mathcal{N}$-definable $R_{\oplus} \subseteq Q^3$ such that:*
(i) $\dim_{\mathcal{M}}(R_{\oplus}) = 4$ and
(ii) $\langle \hat{t}_3, \hat{r}, \hat{t}_1\rangle$ is generic in $R_{\oplus}$ over p'_0, t_0 and the parameters defining $R_{\oplus}$ and C_{t_0} touches $C_{\hat{t}_3\hat{r}\hat{t}_1}$ at p'_0.
For $r \in \nu(\hat{r})$, let $d(r)$ denote the slope of C_r at $\langle z_2, z_1\rangle$.
Then there is a set of parameters B such that for all $r_1, r_2, r_3, r_4 \in \nu_{\hat{r}}$, if $\dim_{\mathcal{M}}(r_1r_2/A) = \dim_{\mathcal{M}}(r_3r_4/A) = 2$ and if $d(r_1) - d(r_2) = d(r_3) - d(r_4)$ in the field R then each r_i is in the $\mathrm{acl}_{\mathcal{N}}$-closure of the set $\{r_j : j \neq i\} \cup B$.

The proof of the above is very similar to that of 2.11. Along the way, one needs to formulate the analogue of 2.12, with $\langle z_2, 0\rangle \in C_{r_1} \ominus C_{r_2}$ instead of $\langle z_1, z_1\rangle \in C_{r_1} \circ C_{r_2}^{-1}$, and we leave the details to the reader.

The next step is to produce a generic triple $\hat{t}_3, \hat{r}, \hat{t}_1 \in Q$ and a generic $p'_0 \in C^{\oplus}_{\hat{t}_3\hat{r}\hat{t}_1}$ with a curve C_{t_0} touching it at p'_0. This is a repetition of the argument in the beginning of the proof of 2.14, with 2.18 replacing 2.10, so we omit its proof.

We now aim to produce the assumptions of 2.19. We start with a strongly minimal set $X \subseteq C_{\hat{t}_3\hat{r}\hat{t}_1}$ containing p'_0 (and defined over the same parameters). Then, we consider two cases, where X is either of $\mathcal{M}$-rank $\leqslant 2$ or greater than 3. In analogy to 2.15 we have:

Fact 2.20. *For any $p' \neq p'' \in N^2$ if $\dim(p', p'') \geqslant 3$ then the set $Z = \{\langle q_3, q_2, q_1\rangle \in Q^3 : p', p'' \in C^{\oplus}_{q_3q_2q_1}\}$ has dimension $\leqslant 4$.*

Proof. It would sufficient to see that for any fixed $q_2, q_1 \in Q^2$, the set $Z_0 = \{q \in Q : p', p'' \in C^{\oplus}_{qq_2q_1}\}$ is finite. Let $p' = \langle x', y'\rangle$, $p'' = \langle x'', y''\rangle$.

Let $Y = C_{q_1} \ominus C_{q_2}$ and note that Y projects finite-to-one on its first coordinates (because the same is true for each of the curves in $\mathcal{F}$). If $p' = \langle x', y'\rangle$ and $\langle x', y'_1\rangle, \ldots, \langle x', y'_\ell\rangle$ are all the points of Y which project onto x', then for every C_q with $p' \in C_q \ominus Y$, we have some $\langle x', y'_i \oplus y'\rangle \in C_q$. Similarly, if $p'' = \langle x'', y''\rangle$ then there are $y''_1, \ldots, y''_r$ such that for each C_q with $p'' \in C_q \ominus Y$, we have some $\langle x'', y''_j \oplus y''\rangle \in C_q$ for some j. Hence, if Z_0 were infinite then there would be two points $\langle x', y'_i \oplus y'\rangle$ and $\langle x'', y''_j \oplus y''\rangle$ in N^2 which belong (both) to infinitely many C_q's. This would contradict the normality of the dual family $\mathcal{L}$, unless $x' = x''$ (and $y'_i \oplus y' = y''_j \oplus y''$).

It follows that either $\dim Z = 4$ or $x' = x''$. If we assume the latter then necessarily $\dim_{\mathcal{M}}(y'y''/x') = 2$. The set

$$\{\langle q_3, q_2, q_1, y_1, y_2\rangle \in Q^3 \times M^2 : \langle x', y_1\rangle, \langle x', y_2\rangle \in C^{\oplus}_{q_3q_2q_1}\}$$

is of dimension 6 (every $q_3q_2q_1$ have finitely many corresponding y_1, y_2) therefore, for the independent pair y', y'', the dimension of the $\langle q_3, q_2, q_1\rangle$'s in Q^3 such that $\langle x', y'\rangle, \langle x'', y''\rangle \in C_{q_3q_2q_1}$ is four, as required. ☐

In the case where the $\mathcal{M}$-rank of X is 2, we define

$$R^{\oplus}_{\circ} = \{\langle q_3, q_2, q_1\rangle \in Q^3 : X \sqsubseteq C^{\oplus}_{q_3q_2q_1}\}.$$

By 2.20 (applied to p'_0 and any other independent point of X) the set has dimension 4 and clearly contains $\langle \hat{t}_3, \hat{r}, \hat{t}_1\rangle$, which is generic over p_0, t_0, yielding the assumptions of 2.11.

If, on the other hand, X has rank $\geqslant 3$ then by our standing assumption of Case II, X is not a good curve with respect to $\mathcal{F}$ and therefore there exists s_2 which is special for (t_0, X) and such that $s_2 \in \operatorname{dcl}_{\mathcal{M}}(p'_0 t_0)$. As in the compositional case, we have $\dim_{\mathcal{M}}(p'_0, s_2) \geqslant 3$. Now, the relation

$$R^{\oplus}_{\circ} = \{\langle q_3, q_2, q_1\rangle :\in Q^3 : p'_0, s_2 \in C^{\oplus}_{q_3q_2q_1}\}$$

provides us with the assumptions of 2.19 (using Fact 2.20).

We are now ready to construct a field configuration in $\mathcal{N}$, based on a field configuration in R, similarly to the way we produced the first configuration. We will use 2.11, 2.19.

We work over the parameter set $B = \hat{r}y_1y_1A$.

Let $a_1 = \langle u_1, v_1\rangle$, $a_2 = \langle u_2, v_2\rangle$ with $u_1, u_2 \in \nu_1$ and $v_1, v_2 \in \nu_0$ all $\mathcal{M}$-independent over B, and let $a_3 = \langle u_1 \cdot u_2, u_1 \cdot v_2 + v_1\rangle$. Take $b_1 \in \nu_0$ generic over $a_1a_2a_3B$ and let $b_2 = u_1b_1 + v_1$ (also in ν_0). Finally, let $b_3 = u_2 \cdot u_1 \cdot b_1 + u_2 \cdot v_1 + v_2 \in \nu_0$. This is the standard field configuration (with $\dim_{\mathcal{M}}$ instead of Morley rank). We now want to realize it with $\mathrm{acl}_{\mathcal{N}}$ instead of $\mathrm{acl}_{\mathcal{M}}$.

We choose $r_i, r_i' \in \nu_{\hat{r}}$, $i = 1, 2, 3$ as follows: $d(r_i)/d(\hat{r}) = u_i$ for $i = 1, 2$, $d(r_i') - d(\hat{r}) = v_i$, for $i = 1, 2$, $d(r_3)/d(\hat{r}) = u_1 \cdot u_2$, $d(r_3') - d(\hat{r}) = u_1 \cdot v_2 + v_1$. We then choose $w_1, w_2, w_3 \in \nu_{\hat{r}}$ by $d(w_i) - d(\hat{r}) = b_i$.

We now claim that

$$C = \{\langle r_1, r_1'\rangle, \langle r_2, r_2'\rangle, \langle r_3, r_3'\rangle, w_1, w_2, w_3\}$$

is a field configuration in $\mathcal{N}$ over B. Clauses (1)-(5) of 2.9 follow from 2.11 and 2.19 together with the fact the $\mathcal{M}$-definable injection $r \mapsto d(r)/d(\hat{r})$ sends the elements of C to the corresponding elements in $\mathcal{M}$ (everything over B).

It is thus left to prove minimality. Assume towards contradiction that there were $t_1 \in \mathrm{acl}_{\mathcal{N}}(r_1r_1'B)$, $t_2 \in \mathrm{acl}_{\mathcal{N}}(r_2r_2'B)$ and $t_3 \in \mathrm{acl}_{\mathcal{N}}(r_3r_3'B)$ such that $C_1 = \{t_1, t_2, t_3, w_1, w_2, w_3\}$ is a group configuration in $\mathcal{N}$. We claim that

$$C_2 := \{t_1, t_2, t_3, w_1, w_2, w_3\}$$

is a group configuration in the sense of $\mathcal{M}$ (i.e. that Clause (1)-(4) of 2.8 hold with $\dim_{\mathcal{M}}$ instead of Morley rank).

Indeed, because each w_i, $i = 1, 2, 3$, is inter-definable in $\mathcal{M}$ with b_i over B, every $\mathcal{N}$-dependence in C_1 translates to an $\mathcal{M}$-dependence in C_2. This suffices also to ensure that no additional dependencies (except those explicitly coming from the pre-image in C_1) occur in C_2. For example, because $C_2 \subseteq \mathrm{acl}_{\mathcal{M}}(t_1t_2w_1)$ over B, and because $\dim_{\mathcal{M}}(w_1w_2w_3/B) = 3$, we must have $\dim_{\mathcal{M}}(t_1t_2b_1/B) \geqslant 3$ and so on. Hence, C_2 is indeed a group configuration in the sense of $\mathcal{M}$. This contradicts the minimality of the standard field configuration in $\mathcal{M}$.

Therefore, C is a field configuration in $\mathcal{N}$. By Theorem 2.9, a field is definable in $\mathcal{N}$. This finishes the proof of Lemma 2.16. □

As we have already seen in 2.3, the definability of a field in $\mathcal{N}$ leads to a contradiction. We may therefore conclude that the structure $\mathcal{N}$ is necessarily locally modular, ending the proof of Theorem 2.2. □

3. Appendix A: A proof of v. d. Dries' Theorem on limit sets

In this section we give a proof of the first part of Theorem 1.35. The proof is due to v.d. Dries, and relies in parts on [5], with minor changes intended to make it more self-contained.

We restate the theorem:

Theorem 3.1. *Let $\mathcal{M}$ be an o-minimal expansion of a real closed field, $\mathcal{S}$ a $\emptyset$-definable family of definable closed subsets of M^k, parameterised by Z. Let $\tilde{\mathcal{S}} := \{\mathcal{S}(\gamma) \mid \gamma : (0,1) \to Z$, definable$\}$. Then $\tilde{\mathcal{S}}$ is $\mathcal{M}$-definable and* $\dim(\tilde{\mathcal{S}}) \leqslant \dim \mathcal{S}$.

Because of the presence of field, we may assume that all curves in Z are given by $\gamma : (0, +\infty) \to Z$. It is not hard to check, using definable choice, that the collection $\{\mathcal{S}(\gamma) \mid \gamma : (0, +\infty) \to Z\}$ depends only on the class $\{\mathcal{S}_a : a \in Z\}$ and not on the parametrisation. Namely, if $S' := \{S_b : b \in Z'\}$ is such that for all $b \in Z'$ there is

$a \in Z$ such that $S_a = S_b$ and vice versa, then $\tilde{S}' = \tilde{S}$. Hence, we may assume that every set in $\mathcal{S}$ appears exactly once in the family and in particular $\dim \mathcal{S} = \dim Z$ (otherwise re-parameterise, see §3 of [5]).

The proof goes through the theory of tame elementary pairs (see §8 of [5] for the details). Recall that for an o-minimal expansion $\mathcal{N}$ of a real closed field, a tame extension $\mathcal{N} \precneqq \mathcal{R}$ is one in which $\mathcal{N}$ is Dedekind complete (namely, for every $x \in N$ if x is in the convex hull of M then either $\{m \in M : m \geqslant x\}$ has a minimum or $\{m \in M : m \leqslant x\}$ has a maximum.). The theory of tame elementary pairs, T_{tame} is the theory of those structures $(\mathcal{R}, \mathcal{N}, \mathrm{st})$ where $\mathcal{R}$ is a tame extension of $\mathcal{N}$ and $\mathrm{st} : R \to N$ is the standard part map. Note that st is defined on V, the convex hull of N in R. To simplify the notation, whenever $Y \subseteq R$ we will write $\mathrm{st}(Y)$ for $\mathrm{st}(Y \cap V)$.

The crucial property of the theory of tame pairs is (see Proposition [5] for the proof):

Proposition 3.2. *Suppose that* $(\mathcal{R}, \mathcal{N}, \mathrm{st}) \models T_{tame}$ *and* $Y \subseteq N^k$ *is definable in* $(\mathcal{R}, \mathcal{N}, \mathrm{st})$*, then* Y *is definable in* $\mathcal{N}$*. Moreover, the theory* T_{tame} *is complete modulo* $Th(\mathcal{N})$*.*

We can now prove Theorem 3.1. By replacing $\mathcal{S}$ with the family $\{\mathrm{cl}(A) : A \in \mathcal{S}\}$ we may assume that all the sets in $\mathcal{S}$ are closed (it is easy to check that the family of limit sets does not change). By compactness it suffices to prove the theorem under the assumption that $\mathcal{M}$ is saturated enough: it is ω-saturated and every M-definable set has (in M) a generic point. Let $\tau > M$ (namely, τ realizes the type $\{x > a : a \in M\}$) and $N = M\langle\tau\rangle$, the prime model over $M \cup \{\tau\}$. Then M is Dedekind complete in N and $(\mathcal{N}, \mathcal{M}, \mathrm{st}) \models T_{tame}$. Let $\gamma : (0, +\infty) \to Z$ be any curve definable over M. It is not hard to verify that $\mathcal{S}(\gamma) = \mathrm{st}\, \mathcal{S}_{\gamma(\tau)}$. By Proposition 3.2, and by compactness, the definable family $\{\mathrm{st}\, \mathcal{S}_a : a \in Z(N)\}$ is $\mathcal{M}$-definable. As $N = \mathrm{dcl}(M, \tau)$ every $a \in Z(N)$ is of the form $\gamma(\tau)$ for some $\mathcal{M}$-definable curve γ, and the definability of $\tilde{\mathcal{S}}$ in $\mathcal{M}$ follows. Because every automorphism of $\mathcal{M}$ leaves $\mathcal{S}$ invariant it also fixes $\tilde{\mathcal{S}}$. By ω-saturation of $\mathcal{M}$ it follows that $\tilde{\mathcal{S}}$ is actually $\emptyset$-definable.

Now let Z_1 be a parameter set for $\tilde{\mathcal{S}}$ (so Z_1 is $\emptyset$-definable in $\mathcal{M}$). By definable choice, we may assume that every set in $\tilde{\mathcal{S}}$ is represented exactly once in the family. Let $b \in Z_1(M)$ be generic over $\emptyset$. There is some $\gamma : (0, +\infty) \to Z$ such that $\tilde{\mathcal{S}}_b = \mathcal{S}(\gamma) = \mathrm{st}\, \mathcal{S}_{\gamma(\tau)}$. If $\gamma(\tau) \in M^k$ then γ is eventually constant and hence $\mathcal{S}_\gamma \in \mathcal{S}$. Assume then that $\gamma(\tau) \notin M$ and let $\mathcal{P} = \langle\gamma(\tau)\rangle$, the model generated by $\gamma(\tau)$ over $\emptyset$. Consider the tame pair $(\mathcal{P}, \mathrm{st}\, \mathcal{P}, \mathrm{st})$. It follows from the completeness part of Proposition 3.2 that $\mathrm{st}\, \mathcal{S}_{\gamma(\tau)} = \tilde{\mathcal{S}}_{b'}$ for some $b' \in Z_1(\mathrm{st}\, \mathcal{P})$. By our assumption, we must have $b' = b$.

Since $\gamma(\tau) \in Z$, we have $\dim(\gamma(\tau)/\emptyset) \leqslant \dim(\mathcal{S})$, and therefore

$$\dim(\mathcal{S}) \geqslant \dim(\mathcal{P}/\emptyset) \geqslant \dim(\mathrm{st}\, \mathcal{P}/\emptyset).$$

We conclude that $\dim(b/\emptyset) \leqslant \dim(\mathcal{S})$ therefore $\dim(\tilde{S}) = \dim(Z_1) \leqslant \dim(\mathcal{S})$. □

4. Appendix B: The general o-minimal case

In this appendix we adapt the main results of this paper to the general o-minimal context (i.e. removing the assumption that $\mathcal{M}$ expands a real closed field).

In the first part of this appendix we will prove two important technical facts for $\mathcal{N}$ strongly minimal, 1-dimensional and definable in a sufficiently saturated o-minimal structure $\mathcal{M}$:

- $\mathcal{N}$ has finitely many non-orthogonality classes (in the o-minimal sense).
- $\mathcal{M}$ has definable choice for $\mathcal{M}$-definable subsets of N^k.

The second part of the appendix uses these two facts in order to prove Lemma 1.40 without the field assumption. This lemma relies heavily on v.d. Dries' Theorem on limit sets (1.35), which is only known to be true in expansions of real closed fields.

Finally, we will go systematically through all places where the existence of a field was used, and explain briefly how to avoid it.

Throughout this section $\mathcal{N}$ will be a 1-dimensional structure *definable* in a densely ordered o-minimal $\mathcal{M}$.

Claim 4.1. *Without loss of generality, N is a dense subset of M.*

Proof. Since $\dim_{\mathcal{M}} N = 1$ it is a finite union of 1-cells and 0-cells (in M^k for some k). Note that removing finitely many points cannot alter the truth of Theorem 2.2. Hence, we may assume that N is given by a union of 1-cells only. Each 1-cell is definably homeomorphic to an interval in M and therefore inherits an ordering. Thus each 1-cell, with its induced structure from $\mathcal{M}$ can be viewed as an o-minimal structure on its own. We now take the disjoint union of the 1-cells of N, ordering them together arbitrarily, so that each cell is an open interval in this ordering. We add endpoints between these intervals and obtain an o-minimal structure $\mathcal{M}'$, such that N, without its 0-cells, is a definable dense subset of $\mathcal{M}'$. We can now replace $\mathcal{M}$ by $\mathcal{M}'$. □

4.1. **Non-orthogonality and definable choice.** Since we can no longer assume that $\mathcal{M}$ has elimination of imaginaries, and as the work throughout the paper was carried out in $\mathcal{M}$ (and not in $\mathcal{M}^{eq}$) we should be careful when using $\mathcal{N}$-interpretable subsets - as these will not, a priori, be definable in $\mathcal{M}$. Recall:

Definition 4.2. A theory T has *weak elimination of imaginaries* if for every $M \models T$ and $e \in M^{eq}$ there are $a_1, \dots, a_n \in M$ such that $e \in \mathrm{dcl}(a_1, \dots, a_n)$ and $a_i \in \mathrm{acl}(e)$ for all $1 \leqslant i \leqslant n$.

It is fairly easy to check (See, e.g., [12]) that:

Lemma 4.3. *Assume T is strongly minimal and* $\mathrm{acl}(\emptyset)$ *is infinite then T has weak elimination of imaginaries.*

Hence, by adding constants to $\mathcal{N}$ we may assume that it has weak EI.

Definition 4.4. Let $\mathcal{M}$ be an o-minimal structure. $a_1, a_2 \in M$ *are non-orthogonal* if there exists a continuous monotone definable bijection $f : I_{a_1} \to I_{a_2}$ for some open intervals $I_{a_i} \ni a_i$ sending a_1 to a_2 (see more in [19]).

If $S \subseteq M$ is a definable set and $a \in M$ any element, we say that $b \not\perp a$ *for every $b \in S$, uniformly in b*, if there exists a definable family of definable functions $F : M^2 \to M$ such that for all $x \in S$ the function $f_x(y) := F(x, y)$ witnesses that $x \not\perp a$.

The notion of (non) orthogonality plays an important role in arguments that follow. The key observation, which makes everything else work is:

Lemma 4.5. *If $\mathcal{N}$ is non-locally modular then there exists $a \in \mathcal{N}$ such that $a \not\perp b$ uniformly in b (in the o-minimal sense, obviously) for all $b \in M$ outside a finite set.*

Proof. Choose $a \in \mathcal{N}$ generic (in the o-minimal sense). By weak EI we can find $\mathcal{F}$, an almost normal 2-dimensional family of $\mathcal{N}$-definable plane curves, as provided by non local modularity (see Fact 2.5). Hence for $b \in \mathcal{N}$, o-minimally generic over a, the point $\langle a, b\rangle$ is generic on a generic $\mathcal{F}$-curve through $\langle a, b\rangle$, implying that such a curve is, locally near $\langle a, b\rangle$, the graph of a continuous monotone function with $f(a) = b$, so $a \not\perp b$.

Consider now the set S of all $b \in \mathcal{N}$ such that there is no $\mathcal{F}$-line through $\langle a, b\rangle$ which is locally near $\langle a, b\rangle$ the graph of a continuous monotone function. This is an o-minimally definable set, and by what we have just shown, it must be finite. □

The following is easy:

Lemma 4.6. *If $\mathcal{M}$ has finitely many non-orthogonality classes then $\mathcal{M}$ does not have a generic trivial type.*

Proof. Assume that $e \in M$ is a generic trivial point. Let $I \ni e$ be a closed interval such that every point in I is trivial (see, e.g., the introduction to [19]). We claim that every $a, b \in I$ that are independent must be orthogonal to each other. Indeed, if not then there is a definable, continuous, strictly monotone function f sending a to b. It follows that for every b' near b there is, uniformly in b', a definable $f_{ab'}$ witnessing the non-orthogonality of a and b'. It follows that b is nonorthogonal, uniformly, to every element in some neighbourhood of it. It is not difficult to see that this contradicts the triviality of b. □

We can now prove that $\mathcal{M}$ has definable choice:

Lemma 4.7. *If $\mathcal{N}$ is strongly minimal and non-trivial (and we still assume that N is a dense subset of M) then $\mathcal{M}$ has definable Skolem functions. I.e. for every $\mathcal{M}$-definable formula $\varphi(\bar{x}, \bar{y})$ there is an $\mathcal{M}$-definable function f_φ such th-at $\exists \bar{y}\varphi(\bar{x}, \bar{y}) \to \varphi(\bar{x}, f_\varphi(\bar{x}))$.*

Proof. Since $M \setminus N$ is finite, it is enough to consider definable subsets of N^k. By cell decomposition, we only need to consider cells, and therefore, it suffices to prove that given a z-definable interval I_z we can find a z-definable point in I_z.

If $\mathcal{N}$ is locally modular, then since it is non-trivial there are an $\mathcal{N}$-definable vector space V, a finite equivalence relation E on N and a definable injection $f : N/E \to V$ (see, [26]) whose image is co-finite in V. By [7], $\mathcal{M}$ has definable choice for definable subsets of V, this gives an element $z_0 \in f(I)$ and we take $z = \min\{f^{-1}(z_0)\} \in I$.

So we may assume that $\mathcal{N}$ is not locally modular. Let $(c, d) \subseteq N$ be any interval. We show that we can choose a point $e \in (c, d)$ uniformly in c, d. Combining the previous lemma with Lemma 4.6 for all but finitely $x \in N$ there exists, uniformly in x, an interval I_x on which a group-interval (not a group!) is defined. If c, d are not both of the finitely many exceptional points, there is (without loss) an interval $I_c \ni c$ as above. If $d \notin \mathrm{cl}(I_c)$ then setting $e := \sup I_c \in (c, d)$ we are done. Otherwise $(c, d) \subseteq I_c$ and we can set $e = (c +_c d)/_c 2_c$ (in the sense of the local group defined on I_c) if $d < \sup I_c$ or $c +_c 1_c$ otherwise. Since the number of exceptional points, and hence the number of intervals with exceptional end-points, is finite, this gives us definable choice for sub intervals in N. □

4.2. **Limit sets in structures without fields.** Our next task is to slightly refine the notion of non-orthogonality in o-minimal structures.

Definition 4.8. Let $\mathcal{M}$ be an o-minimal (saturated enough) structure, $a, b \in M \cup \{\pm\infty\}$. We say that $a^+ \not\perp b^+$ if there exist intervals $I_a^+ := (a, a_\epsilon)$, $I_b^+ := (b, b_\epsilon)$ and a definable (with parameters) continuous monotone bijection $f : I_a^+ \twoheadrightarrow I_b^+$ (so $\lim_{t\to a} f(t) = b$)). For $\varepsilon_i \in \{+,-\}$ we define analogously, $a^{\varepsilon_1} \not\perp b^{\varepsilon_2}$ and $a^\varepsilon \not\perp \pm\infty$.

Consider the structure $\mathcal{M}^\pm$ consisting of $\mathcal{M}$ with all its structure, as well as a new sort $M^\pm = (M \cup \{\pm\infty\}) \times \{+,-\}$ and the projection $\pi : M^\pm \to M \cup \{\pm\infty\}$. The relation $a^{\varepsilon_1} \not\perp b^{\varepsilon_2}$ is an equivalence relation on the elements of $M \times \{+,-\}$, so we can define:

Definition 4.9. A complete set of representatives of the non-orthogonality classes of $\mathcal{M}^\pm$ is a subset of $\mathcal{M}^\pm$ in which every nonorthogonality class is represented.

Definition 4.10. Assume that $\mathcal{M}$ is o-minimal and every a, b outside a finite set F are non-orthogonal to each other, uniformly in a, b. For the purposes of this note, we call such $\mathcal{M}$ *uniformly unidimensional.*

Lemma 4.11. *Let $\mathcal{M}$ be uniformly unidimensional, then $\mathcal{M}$ has at most finitely many trivial points.*

Proof. This is immediate from 4.6. □

Note that by the previous lemma and (the proof of) Lemma 4.7 a uniformly unidimensional structure $\mathcal{M}$ has definable choice. Moreover, uniformly unidimensional structures have curve selection for non-degenerate points. More precisely, if $S \subseteq M^r$ is a definable set and $q := (q_1, \ldots, q_r) \in \partial S$ is such that $q_i \notin F$, then there are $a < b \in M$ and a definable curve $\gamma : (a, b) \to S$ such that $\lim_{t\to a} \gamma(t) = q$.

Definition 4.12. Let $\mathcal{M}$ be uniformly unidimensional. $a \in M$ is *degenerate* if one of a^+, a^- is not in the generic non-orthogonality class of $\mathcal{M}$.

A few clarifications may be in place:

Remark
(1) If $\mathcal{M}$ has finitely many non-orthogonality classes so does $\mathcal{M}^\pm$. The converse is true as well: Let $\{a_1^{\varepsilon_1}, \ldots, a_k^{\varepsilon_k}\}$ be a set of representatives of the non-orthogonality classes of $\mathcal{M}^\pm$. For $b \in \mathcal{M}$ define the non-orthogonality type of b to be the pair (i, j) $(1 \leqslant i, j \leqslant k)$ such that $b^+ \not\perp a_i^{\varepsilon_i}$ and $b^- \not\perp a_j^{\varepsilon_j}$. It will be enough to show that if b, d have the same non-orthogonality type, they are non-orthogonal in the usual sense, which is a simple exercise in concatenation and composition of monotone functions.

(2) The number of non-orthogonality classes, when finite, depends only on $Th(M)$. Indeed, assume that $\{a_1^{\varepsilon_1}, \ldots, a_k^{\varepsilon_k}\}$ is a complete set of representatives of the non-orthogonality classes of $\mathcal{M}$. Because of the saturation of $\mathcal{M}$ the type stating that $x^+ \in M$ is orthogonal to each of the $a_i^{\varepsilon_i}$ is inconsistent. It follows that, uniformly in a, every a^+ is nonorthogonal to one of the $a_i^{\varepsilon_i}$, and similarly for a^-. By quantifying over the a_i we get in every model $\mathcal{M}_0$ of $Th(\mathcal{M})$ k-many non-orthogonality classes in $\mathcal{M}_0^\pm$.
(3) It is possible that a^{ε_1} and b^{ε_2} are non-orthogonal to each other, while a is a degenerate point and b is a non-degenerate point. For example, consider $\langle \mathbb{R}; <, +_{|(\mathbb{R}_{>0})^2} \rangle$. We have $0^+ \not\perp r$ for every positive r, but 0 is a degenerate point.

Fix $\mathcal{M}$, a uniformly unidimensioinal structure.

We now start our treatment of limit sets in $\mathcal{M}$. Let $F^\epsilon = \{a_1^{\epsilon_1}, \dots, a_k^{\epsilon_k}\}$ be the set of representatives of non-orthogonality classes in $\mathcal{M}^{\pm}$ which are orthogonal to c^+ for a fixed generic c. In particular, $F^\epsilon \cup \{c^+\}$ is a complete set of such representatives. We let $F = \{a_1, \dots, a_k\}$. Clearly, $F \subseteq \mathrm{dcl}_{\mathcal{M}}(\emptyset)$.

Recall that in the case of o-minimal expansions of fields, given a definable family of definable sets $\mathcal{S}$, with parameter set Z_1, the limit family $\bar{\mathcal{S}}$ was defined as $\{\mathcal{S}(\gamma) : \gamma : (0,1) \to Z_1 \text{ definable}\}$. In the presence of more than one non-orthogonality class not all definable curves γ with image in Z_1 can have the same domain. In uniformly unidimensional structures, we handle separately limits along curves $\gamma : (a,b) \to Z_1$ with a^+ degenerate and those curves for which a^+ is non-degenerate. Clearly, we can handle similarly curves $\gamma : (b,a) \to Z_1$ and a^-.

Finally, it may happen that for some curve γ the limit set $\mathcal{S}(\gamma)$ is finite. To avoid unpleasant technicalities, and since in the application we will always be interested solely in limit sets that are infinite, we will restrict ourselves to those. We start with the former case:

Lemma 4.13. *Let $\mathcal{M}$ be be as above, $\mathcal{S}$ an* almost normal *family of definable curves with parameter set Z_1, and $\gamma : (a,b) \to Z_1$ a definable curve, $a^+ \in F^\epsilon$. Assume that the limit set $\mathcal{S}(\gamma)$ is infinite and let $x = \langle x_1, \dots, x_n\rangle$ be a generic element in $\mathcal{S}(\gamma)$. Then there exists some i such that $x_i \in \mathrm{dcl}(\emptyset)$. In particular, by genericity $x_i' = x_i$ for all $x' \in \mathcal{S}(\gamma) \cap U$ for some open $U \ni x$.*

Proof. As before we may assume that all the curves in $\mathcal{S}$ are closed (after moving to the family of closures we still have an almost normal family). Assume that the lemma fails. Then for some generic $x \in \mathcal{S}(\gamma)$ all coordinates are non-orthogonal to some generic $c \in M$. Since c is not trivial, there is, by the Trichotomy Theorem, a group interval defined around c. Pulling back this group structure to x (on each coordinate separately) we may assume that x lives in an expansion of a group interval. In particular, there is a definable family of open sets $U_r \ni x$, with $r \in M$ such that $\lim_{r\to c} U_r = \{x\}$ (the existence of this family is our only use of the group structure near x). Since $x \in \mathcal{S}(\gamma)$, for every r there exists $t(r) \in (a,b)$ such that $S_{\gamma(t(r))} \cap U_r \neq \emptyset$. Since $a^+ \in F$, it cannot be that $t(r) \to a$ as $r \to c$. Therefore, $\lim_{r\to c} t(r) = a_0 > a$. It follows that $x \in S_{\gamma(a_0)}$. But this is impossible: we can repeat the process, finding $a < \dots < a_n < \dots < a_1 < a_0$ with $x \in S_{\gamma(t(a_i))}$ for all i, i.e. x is in infinitely many of the $S_{\gamma(t)}$. So we can find a 1-dimensional $\mathcal{S}'$, subset of $\mathcal{S}$, defined over parameters independent from x, such that $x \in C$ for all $C \in \mathcal{S}'$. As x was generic in $S(\gamma)$ and the latter was assume to be infinite, the same is true of any $x' \in U$ for some relatively open $U \ni x$. This contradicts the (almost) normality of $\mathcal{S}$ $\square$

$>$From now one we also assume: **Every generic type in $\mathcal{M}$ is rich, namely it is contained in an $\mathcal{M}$-definable real closed field.**

For $\gamma : (a,b) \to M^2$ a definable curve, we say that γ *is of $\mathcal{M}$-rank k over A* if the image of the interval (a,b) under γ is a curve of $\mathcal{M}$-rank k over A.

Lemma 4.14. *Let $\mathcal{M}$ be as above. Let $\gamma = \langle \gamma_1, \gamma_2\rangle : (a,b) \to M^2$ be a definable curve, with $a^+ \notin F^\epsilon$. If γ is of $\mathcal{M}$-rank $k \geqslant 3$ over A then there are open intervals $I_1, I_2 \subseteq M$ such that:*
(i) For some $a < b_1 < b$, the curve $\gamma(a, b_1)$ is contained in $I_1 \times I_2$;

(ii) I_1, I_2 are definable over $B \supseteq A$ and the $\mathcal{M}$-rank of $\gamma|(a, b_1)$ over B is still k.
(iii) I_1, I_2 are definably isomorphic to some open intervals in a definable real closed field R.

Proof. Since γ has $\mathcal{M}$-rank k over A, there exists an A-definable almost normal family of curves $\mathcal{X} = \{X_d : d \in D\}$, such that $\gamma(a, b) = X_d$ for some generic $d \in D$, and $\dim(D) = k$. We first note that for every generic $x = \langle x_1, x_2 \rangle \in X_d$, we have $x_1, x_2 \notin \mathrm{dcl}(\emptyset)$. Indeed, otherwise X_d would be locally the graph of a constant function and hence, in some M^2-neighbourhood of x, the family $\mathcal{X}$ will be 1-dimensional, contradicting the fact that $\mathcal{X}$ is almost normal and of $\mathcal{M}$-rank at least 3.

Let $\langle p_1, p_2 \rangle \in (M \cup \{\pm\infty\})^2$ be the limit of $\gamma(t)$ as t tends to a. Because of our last remark, the functions γ_1, γ_2 are nowhere constant hence for some $\epsilon_1, \epsilon_2 \in \{\pm\}$ we have $p_1^{\epsilon_1} = \lim_{t \to a} \gamma_1(t)$ and $p_2^{\epsilon_2} = \lim_{t \to a} \gamma_2(t)$ (by $p^+(p^-) = \lim_{t \to a} \gamma_1(t)$ we mean that $\gamma(t)$ tends to p from above (below)). To simplify notation, let us assume that $\epsilon_1 = \epsilon_2 = +$. Since $a^+ \notin F^\epsilon$, it is nonorthogonal to c^+, for our fixed generic c. Let R denote a fixed definable real closed field such that $c \in R$.

There are now several cases to consider:

Case 1 $p_1, p_2 \in \mathrm{dcl}(\emptyset)$.

We consider intervals $I_1 = (p_1, p_1')$, $I_2 = (p_2, p_2')$, for p_1', p_2' generic and independent from d over A. By choosing b_1 sufficiently small, we have $\gamma(a, b_1) \subseteq I_1 \times I_2$. Because γ_1, γ_2 witness the fact that $p_1^+, p_2^+ \not\perp c^+$, I_1, I_2 are definably isomorphic to open intervals in the field R.

Case 2 $p_1, p_2 \notin \mathrm{dcl}(\emptyset)$.

In this case, p_1 and p_2 are nonorthogonal to c and therefore each is contained in a definable real closed field. We can then find open intervals $I_1 \ni p_1, I_2 \ni p_2$, definable over generic parameters which are independent from d over A, such that each I_i is contained in a definable real closed field. Because all generic points are uniformly non-orthogonal to each other, the two intervals are definably isomorphic to intervals in R.

Case 3 $p_1 \in \mathrm{dcl}(\emptyset)$ while $p_2 \notin \mathrm{dcl}(\emptyset)$.

In this case, we take $I_1 = (p_1, p_1')$ and I_2 an interval containing p_2 and proceed as before. We handle similarly the symmetric fourth case. □

Lemma 4.15. *Let $\mathcal{M}$ be as above. Let $\mathcal{S}$ be a definable two-dimensional* almost normal *family of curves, all contained in a two dimensional set Q, and having a parameter set $P \subseteq M^2$. Let $\gamma : (a, b) \to P$ be a definable curve of $\mathcal{M}$-rank $k \geqslant 3$.*

Assume that S_γ is infinite. If q is a generic element in $\mathcal{S}(\gamma)$, with $\dim(q) = 2$ *then there exists an open neighbourhood U of q such that the $\mathcal{M}$-rank of $\mathcal{S}(\gamma) \cap U$ is at most* 2.

Proof. By Lemma 4.13, $a^+ \notin F^\epsilon$, for otherwise we will have $\dim(q) \leqslant 1$.

If $p = \lim_{t \to a} \gamma(t)$ then, by Lemma 4.14, there exists $b_1 \in (a, b)$ and an open rectangular box $V = I_1 \times I_2$ such that $\gamma(a, b_1) \subseteq V$, and V is definably isomorphic to an open rectangular box in a definable real closed field R. Moreover, the $\mathcal{M}$-rank of $\gamma|(a, b_1)$ over all parameters is still $k \geqslant 3$. Assume that $\gamma(a, b_1) = X_d$ for X in a definable family of curves $\mathcal{X} = \{X_{d'} : d' \in D\}$. Then, after possibly shrinking D (but still with $\dim(D) = k$, we may assume that for every $d' \in D$ the curve $X_{d'}$ is contained in V.

Because $\dim(q) = 2$, we may assume that $q = \langle q_1, q_2 \rangle \in M^2$, where q_1, q_2 are nonorthogonal to c. Hence, there exists an open rectangular box $U = J_1 \times J_2 \ni q$ such that J_1, J_2 are definably isomorphic to open intervals in the field R.

We can now restrict ourselves to $\mathcal{S}(U, V) = \{\mathcal{S}_{p'} \cap U : p' \in V\}$. Because U, V are definably isomorphic to open rectangular boxes in the field R, we may assume that $\mathcal{S}(U, V)$ and every limit set of this family are definable in R. We can now apply Theorem 1.35 and conclude that the family of all possible limit sets for the family $\mathcal{S}(U, V)$ is definable of dimension at most 2. The limit curve $\mathcal{S}(\gamma) \cap U$ belongs to this family and therefore its $\mathcal{M}$-rank is at most 2. □

Finally, we can prove Lemma 1.40 without the field assumption. We formulate it here again:

Lemma 4.16. *For $\mathcal{M}$ uniformly unidimensional, let $\mathcal{F}$ be an almost normal family of plane curves parameterised by Q, $X \subseteq P$ an A-definable curve of $\mathcal{M}$-rank $k > 2$, $p_0 \in X$ generic and assume that for $q_0 \in Q$, the curve C_{q_0} touches X at p_0 (here we identify P locally with an open set in M^2), and in addition,* $\dim(p_0, q_0/\emptyset) = 3$. *If p is a special point for (q_0, X) then* $\dim(p/A) = 1$. *In particular, p is a generic point of X.*

Proof. As was pointed out earlier on, $p \in \mathrm{cl}(X)$ and hence $\dim(p/A) \leqslant 1$. By 1.7, we may assume that in a neighbourhood U of p_0 and V of q_0, the family $\mathcal{F}$ is nice. Consider $\tau(X \cap U)$ and recall that q_0 is generic in $\tau(X \cap U)$ over A (by the Duality Theorem).

Assume, towards a contradiction, that $\dim(p/A) = 0$. Hence, q_0 is still generic in $\tau(X \cap U)$ over Ap. As was pointed out in clause (3) of the discussion following Definition 1.36, it follows that q_0 is in the Ap-definable limit set $\mathcal{L}(X, p)$. But then, by the genericity of q_0 over pA, there exists a neighbourhood $V_1 \subseteq V$ of q_0, such that $\tau(X \cap U) \cap V_1 = \mathcal{L}(X, p) \cap V_1$. In particular, $\mathcal{L}(X, p)$ is infinite.

The limit set $\mathcal{L}(X, p)$ can also be written as $\mathcal{L}(\gamma)$ with $\gamma : (a, b) \to P$ a curve of $\mathcal{M}$-rank $k > 2$ over A (γ determines a branch of X which contains p in its frontier). By Lemma 4.15, there exists an open neighbourhood V' of q_0, such that $\mathcal{L}(\gamma) \cap V'$ has $\mathcal{M}$-rank at most 2 over A. Without loss of generality, $V' = V_1$ and therefore the $\mathcal{M}$-rank of $\tau(X \cap U) \cap V_1$ over A is at most 2. However, since the $\mathcal{M}$-rank of X over A is $k > 2$, then the $\mathcal{M}$-rank of $X \cap U$ is k as well and, by Lemma 1.32, the $\mathcal{M}$-rank of $\tau(X \cap U) \cap V_1$ is k as well (see also 1.21), contradiction. □

4.3. **Some extra details.** In this final subsection, we explain - claim by claim - how to avoid any use we made of the ambient field structure on $\mathcal{M}$ in the proof of Theorem 2.2. Throughout this section we will assume that $\mathcal{N}$ is a strongly minimal non-locally modular structure *definable* in an arbitrary o-minimal structure $\mathcal{M}$.

The following lemma shows that under these (a posteriori inconsistent) assumptions fields are (locally) definable in $\mathcal{M}$:

Lemma 4.17. *Let $\mathcal{M}$ be an o-minimal structure, $\mathcal{N}$ a 1-dimensional non-locally modular strongly minimal set definable in $\mathcal{M}$ such that N is dense in M. Then $\mathcal{M}$ is uniformly unidimensional and every generic type in $\mathcal{M}$ is contained in an $\mathcal{M}$-definable real closed field.*

Proof. We already established that $\mathcal{M}$ is uniformly unidimensional (Lemma 4.5 and Lemma 4.7).

Adding constants, we may assume that $\mathcal{N}$ has weak EI. The non-local modularity of $\mathcal{N}$ gives rise to a 2-dimensional family of plane curves in $\mathcal{N}$ (and hence in $\mathcal{M}$). Such a family cannot be definable in trivial structures or in ordered vector spaces, hence by the Trichotomy Theorem for o-minimal structures a real closed field R is definable in $\mathcal{M}$. By the unidimensionality of $\mathcal{M}$ every generic type in $\mathcal{M}$ is nonorthogonal to a generic type in R. □

$>$From the last lemma it follows that all the results of Sections 1 and 2 which uses the field structure only in a local way transfer to the present context. Here are some examples:

(1) In Lemma 1.25: The fact that X is a graph of a C^1-function is meaningless without the existence of an ambient field. But since we are interested in the result only locally near a generic point $p_0 = \langle x_0, y_0 \rangle \in X$ this can be solved as follows. Since $x_0, y_0 \in N$ are generic in $\mathcal{M}$ they are non-orthogonal to each other and each has an $\mathcal{M}$-definable real closed field containing it. By non-orthogonality there exist closed (and non-trivial) $\mathcal{M}$-intervals $I_1 \ni x_0, I_2 \ni y_0$, which are definably homeomorphic to each other. We may assume that I_1, I_2 are definably homeomorphic to open intervals in the same field R. Hence, differentiation can be take with respect to that field structure. Note, also, that by stable embeddedness, $X \cap I_1 \times I_2$ is definable, locally near p_0 using only parameters from $I_1 \cup I_2$. Thus, the lemma clearly remains true in this new formulation.
(2) Lemma 1.26: Again, we use the fact that x_0, y_0 are nonorthogonal to generic elements in the same real closed field.
(3) The two lemmas discussed above are crucial to prove Theorem 1.28, where the assumptions are only meaningful in the presence of an ambient field, but whose conclusion is independent of that field. Thus, we can reformulate the theorem as follows: Let $\mathcal{F}$ be a $\emptyset$-definable (almost) normal family of plane curves of dimension greater than 1. Let X be an A-definable plane curve of $\mathcal{M}$-rank greater than 1, $p_0 = \langle x_0, y_0 \rangle \in X$ generic in over A. Endow $\emptyset$-definable neighbourhoods of x_0 and y_0 with a similar field structure, as above, and let d be the derivative - with respect to that field structure - of the function associated to X at x_0. If $p_0 \in C_{q_0}$ for $q_0 \in Q$ and if $f'_{q_0}(x_0) = d$, then C_{q_0} and X touch each other at p_0 and $\dim_{\mathcal{M}}(p_0, q_0/\emptyset) = 3$.

This covers all occurrences of the field structure in Section 1. We now turn to Section 2.

(1) The construction of the curve X, depends only on the $\mathcal{N}$-structure and so causes no problem. We have to make sure, however, that there exists a curve from the family $\mathcal{F}$ which touches X at a generic point. This follows, essentially, from the version of Theorem 1.28 we just formulated above. However, an alternative approach is also possible. Note that in the construction, we only need the points z_1, z_2, x_0, y_0 to be independent generics. By choosing them all very close to each other, and restricting our attention to a small interval containing them all, we may in fact assume (using stable embeddedness, as usual) that in that part of the proof we are working in an expansion of a field.
(2) Since we checked that the results of Section 1 go through, the proof of Theorem 2.2 in Case I needs no alteration. As for Case II, we note that the

argument is only concerned with what is going on near (in the o-minimal sense) the point p_0. Thus choosing z_1, z_2, x_0, y_0 as suggested above, will assure that the argument will need no alteration.

References

[1] Elisabeth Bouscaren. The group configuration—after E. Hrushovski. In *The model theory of groups (Notre Dame, IN, 1985–1987)*, volume 11 of *Notre Dame Math. Lectures*, pages 199–209. Univ. Notre Dame Press, Notre Dame, IN, 1989.

[2] Egbert Brieskorn and Horst Knörrer. *Plane algebraic curves.* Birkhäuser Verlag, Basel, 1986. Translated from the German by John Stillwell.

[3] Steven Buechler. *Essential Stability Theory.* Springer, 1996.

[4] G. Cherlin and S. Shelah. Superstable fields and groups. *Ann. Math. Logic*, 18(3):227–270, 1980.

[5] Lou Van Den Dries. Limit sets in o-minimal structures. In *Proceedings of RAAG summer school, Lisbon 2003*, pages 159–215, 2005.

[6] Lou Van Den Dries. A remark on limit sets. 2006.

[7] Mário J. Edmundo. Solvable groups definable in o-minimal structures. *J. Pure Appl. Algebra*, 185(1-3):103–145, 2003.

[8] Jerry Gagelman. Stability in geometric theories. *Ann. Pure Appl. Logic*, 132(2-3):313–326, 2005.

[9] Assaf Hasson and Piotr Kowalski. Strongly minimal expansions of $(\mathbb{C}, +)$ definable in o-minimal fields. *Proc. Lond. Math. Soc. (3)*, 97(1):117–154, 2008.

[10] Assaf Hasson and Alf Onshuus. Minimal types in super dependent theories. arXiv:0711.0122, 2008.

[11] Ehud Hrushovski. Unidimensional theories. an introduction to geometric stability theory. In H.D. Ebbinghaus et Al., editor, *Logic Colloquium '87 (Granda 1987)*, pages 73–103. North-Holland, Amsterdam, 1989.

[12] Ehud Hrushovski. Strongly minimal expansions of algebraically closed fields. *Israel Journal of Mathematics*, 79:129–151, 1992.

[13] Ehud Hrushovski. A new strongly minimal set. *Annals of Pure and Applied Logic*, 62:147–166, 1993.

[14] Ehud Hrushovski and Anand Pillay. Groups definable in local fields and pseudo-finite fields. *Israel J. Math.*, 85(1-3):203–262, 1994.

[15] Ehud Hrushovski and Boris Zilber. Zariski geometries. *Journal of the American Mathematical Society*, 9(1):1–55, 1996.

[16] Alf Onshuus. Properties and consequences of thorn-independence. *J. Symbolic Logic*, 71(1):1–21, 2006.

[17] Margarita Otero, Ya'acov Peterzil, and Anand Pillay. On groups and rings definable in o-minimal expansions of real closed fields. *Bull. London Math. Soc.*, 28(1):7–14, 1996.

[18] Ya'acov Peterzil, Anand Pillay, and Sergei Starchenko. Simple algebraic and semialgebraic groups over real closed fields. *Trans. Amer. Math. Soc.*, 352(10):4421–4450 (electronic), 2000.

[19] Ya'acov Peterzil and Sergei Starchenko. A trichotomy theorem for o-minimal structures. *Proc. London Math. Soc. (3)*, 77(3):481–523, 1998.

[20] Ya'acov Peterzil and Sergei Starchenko. Expansions of algebraically closed fields in o-minimal structures. *Selecta Math. (N.S.)*, 7(3):409–445, 2001.

[21] Yaacov Peterzil and Sergei Starchenko. Complex analytic geometry and analytic-geometric categories. To appear in Journal fur die reine und angewandte Mathematik (Crelle's Journal), 2007.

[22] Ya'acov Peterzil and Charles Steinhorn. Definable compactness and definable subgroups of o-minimal groups. *J. London Math. Soc. (2)*, 59(3):769–786, 1999.

[23] Anand Pillay. On groups and fields definable in *o*-minimal structures. *J. Pure Appl. Algebra*, 53(3):239–255, 1988.

[24] Anand Pillay. *Geometric stability theory*, volume 32 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1996. Oxford Science Publications.

[25] E. D. Rabinovich. *Definability of a field in sufficiently rich incidence systems*, volume 14 of *QMW Maths Notes*. Queen Mary and Westfield College School of Mathematical Sciences, London, 1993. With an introduction by Wilfrid Hodges.
[26] Boris Zilber. *Uncountably Categorical Theories*, volume 117. Translations of Mathematical Monographs, American Mathematical Society, Providence, RI., 1993.

University of Oxford, Mathematical Institute, 24-29 St Giles', Oxford, OX1 3LB, UK

Current address: Department of Mathematics, Ben Gurion University of the Negev, Be'er Sheva, ISRAEL

E-mail address: hassonas@math.bgu.ac.il

Universidad de los Andes, Departemento de Matemáticas, Cra. 1 No 18A-10, Bogotá, Colombia

E-mail address: onshuus@gmail.com

Department of Mathematics, University of Haifa, Haifa, ISRAEL

E-mail address: kobi@math.haifa.ac.il