Strongly minimal groups in o-minimal structures
(with P. Eleftheriou and A. Hasson)

Kobi Peterzil

Department of Mathematics
University of Haifa

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Zil'ber's Trichotomy Conjecture

The original conjecture, 1984

From Zil'ber's book "Zariski Geometries: Geometry from Logician's Point of View", LMS lecture Notes 360

The geometry of a strongly minimal structure $\mathcal{D}$ is either
(i) trivial
(ii) locally projective , or
(iii) isomorphic to a geometry of an algebraically closed field K definable in $\mathcal{D}$ and the only structure induced on K from $\mathcal{D}$ is definable in the field structure K (i.e. the field is "pure").

The underlying premise

The combinatorial constrains of strong minimality should force the existence of an underlying algebraic structure. Conversely, this algebraic structure should fully explain the combinatorial constraints.

The conjecture's refutation

Hrushovski's Combinatorial constructions, 1993

I. Failure of algebraicity There are strongly minimal non-locally modular structures, without any definable field (or even a group). Namely, the negation of (i) and (ii) does not imply (iii).

II. Failure of purity There are strongly minimal **proper** expansions of algebraically closed fields (in a very strong sense).

Partial conclusions

Combinatorial constraints are neither enough to define a field nor to imply its purity.

Adding algebraic assumptions

Proving the trichotomy in algebraic settings-a partial list

- Zhenia Rabinovich (1991): Reducts of *ACF*.
- Hrushovski-Sokolovich (1993): Any strongly minimal set in *DCF* (full structure)
- Chadzidakis-Hrushovski, C-H-Pe, (1999, 2002): variants in *ACFA*.
- Kowalski-Randriambololona, (2015): variant in $ACVF_0$.
- Hasson-Sustretov (2017): Reducts of any algebraic curve in ACF's of all characteristics.

Applications-partial list

- Diophantine applications (Hrushovski and others).
- "A curve and its abstract Jacobian" (Zil'ber, 2012)

Still open, in algebraically closed fields

Zilber's Trichotomy holds for **any** strongly minimal structure definable in an ACF K. Left to show: if $\dim_K(D) > 1$ then $\mathcal{D}$ is locally modular.

Adding geometric assumptions

Zariski geometries, Hrushovski-Zil'ber (1996)

If $\mathcal{D}$ is a strongly minimal structure, based on closed sets in a Noetherian topology, satisfying a list of axioms (dimension!), then $\mathcal{D}$ satisfies the Trichotomy (up to finite covers).

Strong minimality within o-minimality

Definition

Let $\mathcal{M}$ be an o-minimal structure. We say that a strongly minimal structure $\mathcal{D}$ is **definable (interpretable) in** $\mathcal{M}$ if the universe of $\mathcal{D}$ and all atomic relations are definable (interpretable) in $\mathcal{M}$.

Examples

- $\langle \mathbb{R}; = \rangle$, or $\langle \mathbb{C}; = \rangle$
- $\langle \mathbb{R}; + \rangle$ or $\langle \mathbb{C}; + \rangle$
- $\langle \mathbb{C}; +, \cdot \rangle$ (all the above definable in real closed fields).
- Any compact complex manifold with its analytic structure has finite Morley Rank, definable in $\mathbb{R}_{an}$.

Conjecture: The o-minimal variant

Let $\mathcal{M}$ be o-minimal, and assume that $\mathcal{D}$ is strongly minimal structure, interpretable (definable) in $\mathcal{M}$. Then $\mathcal{D}$ satisfies Zil'ber's Trichotomy.

Results in the o-minimal setting

Fix $\mathcal{M}$ an o-minimal expansion of a RCF field R, and $K = R(\sqrt{-1})$. Let $\mathcal{D} = \langle D; \cdots \rangle$ be a strongly minimal structure definable in $\mathcal{M}$.

Some results

• **Purity.**(Marker, Pe-Starchenko) If $\mathcal{D}$ defines an algebraically closed field F then F is "pure".

• (Hasson-Onshuus-Pe) If D is a one dimensional set in $\mathcal{M}$ then $\mathcal{D}$ is locally modular.

• (Hasson-Kowalski) If $\mathcal{D} = \langle \mathbb{C}; +, f \rangle$, with f a non-affine function then, up to conjugation by some $A \in GL(2, \mathbb{R})$, f is a $\mathbb{C}$-polynomial. It follows that $\mathcal{D}$ is definably isomorphic to the full field structure of $\mathbb{C}$.

Our Theorem

Fix $\mathcal{M}$ is an o-minimal expansion of a real closed field R.

Theorem (Eleftheriou-Hasson-Pe, 2018)

Let $(G; \oplus)$ be a 2-dimensional group definable in $\mathcal{M}$.
Let $\mathcal{D} = \langle G; \oplus, \cdots \rangle$ be a strongly minimal structure non-locally modular, definable in $\mathcal{M}$.
Then there exists a algebraically closed field F definable in $\mathcal{D}$ and an algebraic group H over F, with $RM_{\mathcal{D}}(H) = \dim_F(H) = 1$ such that

$$\langle G; \oplus, \cdots \rangle \cong \langle H; \cdot, \text{ full induced } F\text{-structure} \rangle.$$

Note: The group G is necessarily abelian.

Some discussion of the proof

First, the group G admits the structure of an R-Lie group (Pillay).

The Rabinovich-Zil'ber strategy

• Non modularity $\Rightarrow$ there is a $\mathcal{D}$-definable almost normal family $\mathcal{L} = \{\ell_q : q \in Q\}$, with each $\ell_q \subseteq G^2$, and $RM(\ell_q) = 1$, $RM(\mathcal{L}) = 2$ (rich family of "lines").

Consider a smooth $\mathcal{D}$-curve C in $\mathcal{L}$ through $(0,0) \in G^2$, and look at its R-Jacobian $J_0(C) := J_{(0,0)}(C)$.
We have

$$J_0(C_1 \circ C_2) = J_0(C_1) \cdot J_0(C_2) \;\; ; \;\; J_0(C_1 \oplus^* C_2) = J_0(C_1) + J_0(C_2).$$

The heart of the problem

Using intersection theory, identify, **definably in** $\mathcal{D}$, when two $\mathcal{D}$-definable curves are tangent at $(0,0)$, in the sense of R.

Goal: recover "complex-analytic intersection theory"

We need to identify $\mathcal{D}$-definable sets as "complex-like" objects.

Theorem I: On small frontier

Let $S \subseteq G^2$ be an A-definable set in $\mathcal{D}$, with $RM(S) = 1$. Then the set $Cl_{\mathcal{M}}(S) \setminus S$ is finite and contained in $acl_{\mathcal{D}}(A)$.

(See picture).

Poles

Def. Let $S \subseteq G^2$ be a definable set in $\mathcal{D}$. A point $a \in G$ **is a pole** if for every $\mathcal{M}$-definable neighborhood $U \subseteq G$ of a, the set $S \cap (U \times G)$ is unbounded.

Theorem II: On finitely many poles

Every $\mathcal{D}$-definable set $S \subseteq G^2$ with $RM(S) = 1$ has at most finitely many poles.

Example:

Recovering intersection theory-cont

The ring of Jacobians of $\mathcal{D}$-functions

Let $\mathfrak{F}_0$ be the collection of all $f : 0 \in U \subseteq G \to G$ which are C^1, $f(0) = 0$, and $Graph(f) = S \cap W$ for a $\mathcal{D}$-definable $S \subseteq G^2$ and $W \ni 0$ open and $\mathcal{M}$-definable.

- $\mathfrak{F}_0$ is closed under G-addition, under composition, and compositional inverse when defined.
- Let $\mathfrak{R} = \{J_0(f) : f \in \mathfrak{F}\}$. Then $\mathfrak{R}$ is a subring of $M_2(R)$, closed under inverse (when defined).

Theorem III: On the ring $\mathfrak{R}$

The ring $\mathfrak{R}$ is an algebraically closed field. Up to conjugation by a fixed $M \in GL(2, R)$, the matrices are of the form: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, for $a, b \in R$.

If $G = \langle K^2; +\rangle$ then the proof would end here, because all $\mathcal{D}$-definable functions would be K-holomorphic hence algebraic.

Digression: Almost complex structures

Definition and questions

An almost complex structure on a smooth manifold M is a smooth map $J : TM \to TM$ such that for every $a \in M$ and $v \in T_a(M)$, we have $J^2(v) = -v$.

The idea: J turns each $T_a(M)$ into a complex vector space by $J(v) = i \cdot v$.

Example: Any complex manifold is naturally almost complex.

Fact: If M is a 2-dimensional almost complex manifold then it is isomorphic to a complex curve.

Question: Let M be a definable almost complex manifold of dimension 2. Is M definably isomorphic to a complex curve?

Back to our strongly minimal structure

The group G can be endowed with a G-invariant almost K-structure such that every $\mathcal{D}$-function is J-holomorphic.

More intersection theory

In order to imitate complex intersection theory we need:

Theorem IV: On intersection multiplicity

If f is a $\mathcal{D}$-function and $J_0(f) = 0$ then f is not locally injective at 0.

Possible proof: Show that $(G; J)$ is definably isomorphic to a K-complex curve, with f becoming K-holomorphic. Then use complex analysis in K.

Strategy fails: We don't know how to show it. At this stage.

Instead We proved the theorem using differential and algebraic topology.

Finishing the proof: back to Rabinovich-Zil'ber strategy

- Using the established intersection theory, can identify in $\mathcal{D}$ when two "sufficiently generic" curves are tangent.
- Since collection of "derivatives" forms a field, extract in $\mathcal{D}$ a field configuration, and thus interpret a field F in $\mathcal{D}$.
- There is a finite subgroup $G_0 \subseteq G$ such that G/G_0 is $\mathcal{D}$-internal to F.
- There is a $\mathcal{D}$-homomorphism $\alpha : G/G_0 \twoheadrightarrow G$, thus G is internal to F.
- As we noted earlier, F is pure, thus G is algebraic. In fact, F and $\mathcal{D}$ are bi-interpretable. **End of proof**

A by-product result

The almost (K-) complex manifold $(G; J)$ is "definably integrable", namely definably isomorphic to a (K-)complex manifold H.

Further questions

- Prove the o-minimal version of Zil'ber's Trichotomy, for strongly minimal structures on arbitrary 2-dimensional definable sets D in $\mathcal{M}$. **Difficulty: how can we endow D with "a natural topology"?**
- Prove that in higher dimensions, strongly minimal structures are necessarily locally modular.
- Find applications of the o-minimal statement.
- The notion of distality (P. Simon) abstracts "topological settings" among NIP structures.

Zil'ber's conjecture-the distal version

Assume that $\mathcal{D}$ is strongly minimal and definable in a distal structure. Does Zil'ber's Trichotomry hold?