O-MINIMAL FLOWS ON NILMANIFOLDS

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ABSTRACT. Let G be a connected, simply connected nilpotent Lie group, identified with a real algebraic subgroup of $\mathrm{UT}(n,\mathbb{R})$, and let Γ be a lattice in G, with $\pi : G \to G/\Gamma$ the quotient map. For a semi-algebraic $X \subseteq G$, and more generally a definable set in an o-minimal structure on the real field, we consider the topological closure of $\pi(X)$ in the compact nilmanifold G/Γ.

Our theorem describes $\mathrm{cl}(\pi(X))$ in terms of finitely many families of cosets of real algebraic subgroups of G. The underlying families are extracted from X, independently of Γ.

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1. Introduction

Let $\mathrm{UT}(n,\mathbb{R})$ denote the group of real $n\times n$ upper triangular matrices with 1 on the diagonal. Below we say that a group G is *a real unipotent group* if it is a real algebraic subgroup of $\mathrm{UT}(n,\mathbb{R})$, namely a subgroup of matrices which is a solution set to a system of real polynomials in the matrix coordinates. Such subgroups are exactly the connected Lie subgroups of $\mathrm{UT}(n,\mathbb{R})$, and every connected, simply connected nilpotent Lie group is Lie isomorphic to a real unipotent group. For Γ a discrete co-compact subgroup of real unipotent G, the compact manifold G/Γ is called *a compact nilmanifold*. We let $\pi : G \to G/\Gamma$ be the map $\pi(g) = g\Gamma$.

Let $\mathbb{R}_{\mathrm{om}}$ be an o-minimal expansion of the real field and G a real unipotent group. We consider the following problem:

Given $X \subseteq G$ an $\mathbb{R}_{\mathrm{om}}$-definable set (e.g. $X \subseteq G$ a semi-algebraic set), what is the topological closure of $\pi(X)$ in the nilmanifold G/Γ?

A special case of this problem is when the set $X \subseteq G$ is the image of $\mathbb{R}^d$ under a polynomial map (with G viewed in an obvious way as a subset of $\mathbb{R}^{n^2}$). In [13] Shah considers a similar question when G is an arbitrary real algebraic linear group, and in [7] Leibman considers a discrete variant of the problem, when X is the image of $\mathbb{Z}^d$ under certain polynomial maps inside nilpotent Lie groups. Both prove results about equidistribution from which theorems about the closure of $\pi(X)$ can be deduced. Our setting is more general, but the results we obtain answer only the closure problem. In Theorem 1.5 below and in Section 5.1 we show how to deduce closure results similar to theirs from our work.

In order to state our main theorem we set some notation: We fix G a real unipotent group and $\mathbb{R}_{\mathrm{om}}$ an o-minimal expansion of the real field. Given a lattice Γ in G, namely a discrete co-compact subgroup of G, we denote by $M^G_\Gamma = G/\Gamma$ the associated compact nilmanifold and by $\pi^G_\Gamma : G \to M^G_\Gamma$ the quotient map $\pi^G_\Gamma(g) = g\Gamma$. We omit G from the notation when the context is clear. Given an $\mathbb{R}_{\mathrm{om}}$-definable set $X \subseteq G$, we want to describe the topological closure of $\pi_\Gamma(X)$ in M_Γ.

As we shall see, the frontier of $\pi_\Gamma(X)$ is given via families of orbits of real algebraic subgroups of G in M_Γ. For that we make use of the following theorem, which can be viewed as a special case of our problem when X is a real algebraic subgroup of G. For the discrete one-variable case, see Lesigne [8], and for the more general result about closures of orbits of unipotent groups, see Ratner [12].

Theorem 1.1 ([8],[12]). *Let G be a real unipotent group. Assume that Γ is a lattice in G. If $H \subseteq G$ is a real algebraic subgroup then there exists a unique real algebraic group $H_0 \supseteq H$ such that*

$$\mathrm{cl}(\pi_\Gamma(H)) = \pi_\Gamma(H_0).$$

The group H_0 is the smallest real algebraic subgroup of G containing H such that $\Gamma \cap H_0$ is co-compact in H_0.

Let us set aside a specific notation for the above H_0:

Definition 1.2. Given $H \subseteq G$ real unipotent groups and Γ a lattice in G, we let H^Γ denote the smallest real algebraic subgroup of G containing H such that $H^\Gamma \cap \Gamma$ is co-compact in H^Γ.

We can now state our main theorem:

Theorem 1.3. *Let G be a real unipotent group and let $X \subseteq G$ be an $\mathbb{R}_{\mathrm{om}}$-definable set. Then, there are finitely many real algebraic subgroups $L_1, \ldots, L_m \subseteq G$ of positive dimension, and finitely many $\mathbb{R}_{\mathrm{om}}$-definable closed sets $C_1, \ldots, C_m \subseteq G$, such that for every lattice $\Gamma \subseteq G$, we have:*

$$\mathrm{cl}\Big(\pi_\Gamma(X)\Big) = \pi_\Gamma\Big(\mathrm{cl}(X) \cup \bigcup_{i=1}^{m} C_i L_i^\Gamma\Big).$$

In addition, we may choose the sets C_i as so that:

(1) For every $i = 1, \ldots, m$, $\dim(C_i) < \dim X$.
(2) Let L_i be maximal with respect to inclusion among $L_1, \ldots, L_m$. Then C_i is a bounded subset of G, and in particular, $\pi_\Gamma(C_i L_i^\Gamma)$ is closed in M_Γ.

As an immediate corollary we obtain:

Corollary 1.4. *For G real unipotent and $X \subseteq G$ an $\mathbb{R}_{\mathrm{om}}$-definable set, if $\Gamma \subseteq G$ is a lattice then there exists an $\mathbb{R}_{\mathrm{om}}$-definable set $Y \subseteq G$, such that*

$$\mathrm{cl}(\pi_\Gamma(X)) = \pi_\Gamma(Y).$$

As part of our analysis we conclude in Section 5.1 the following variant of theorems of Shah and Leibman:

Theorem 1.5. *Let G be a unipotent group, viewed as a subset of $\mathbb{R}^{n^2}$, and $F\colon \mathbb{R}^d \to \mathbb{R}^{n^2}$ a polynomial map that takes values in G. Let $X \subseteq G$ be the image of $\mathbb{R}^d$ under F. If $cH \subseteq G$ is the smallest coset of a real algebraic subgroup of G with $X \subseteq cH$ then for every lattice $\Gamma \subseteq G$*

$$\mathrm{cl}(\pi_\Gamma(X)) = \pi_\Gamma(cH^\Gamma).$$

We make some comments on Theorem 1.3:

Remark 1.6. (1) If we let X be a definable curve, i.e. $\dim(X) = 1$, then by Theorem 1.3(1) there are finitely many real algebraic subgroups $L_1, \ldots, L_m$, determined by the curve X, and finitely many points $c_1, \ldots, c_m \in G$ such that for every lattice $\Gamma \subseteq G$,

$$\mathrm{cl}(\pi_\Gamma(X)) = \pi_\Gamma(X) \cup \bigcup_{i=1}^{m} \pi_\Gamma(c_i L_i^\Gamma).$$

Thus the closure of $\pi_\Gamma(X)$ is obtained by attaching to it finitely many sub-nilmanifolds of G/Γ (we recall below the definition of a sub-nilmanifold).

(2) In [11] we examined the same problem in the special case when G was abelian, so could be identified with $\langle \mathbb{R}^n, + \rangle$ and the final theorem was very similar to the current one. We also proved there a finer theorem when $G = \langle \mathbb{C}^n, + \rangle$ and $X \subseteq \mathbb{C}^n$ a complex algebraic variety. That work was inspired by questions of Ullmo and Yafaev in [16] and [17].

(3) In the same paper [11] we showed that one cannot in general replace the sets C_i in Theorem 1.3 by finite sets. For a simple example (pointed out to us by Hrushovski) one can just start with the curve $C = \{(t, 1/t) : t > 1\}$ in $\mathbb{R}^2$ and then consider $\pi_{\mathbb{Z}^4}(C \times C)$ inside $\mathbb{R}^4/\mathbb{Z}^4$. If we let $H = \mathbb{R} \times \{0\}$, then the frontier of $\pi_{\mathbb{Z}^4}(C \times C)$ equals

$$\pi_{\mathbb{Z}^4}((C \times H) \cup (H \times C) \cup (H \times H)).$$

(4) Finally, our main theorem only handles the closure problem and not equidistribution questions. In Section 8 we make some remarks on the difference between the two for definable sets in o-minimal structures.

We end this introduction by noting that definable sets in o-minimal structures allow for a richer collection than semialgebraic sets, and thus for example we could take $X \subseteq \mathrm{UT}(3, \mathbb{R})$ to be the following $\mathbb{R}_{\mathrm{an,exp}}$-definable set

$$\left\{ \begin{pmatrix} 1 & e^y & \arctan(y) \\ 0 & 1 & 1/\sqrt{x^2 + y^4} \\ 0 & 0 & 1 \end{pmatrix} : x, y > 0 \right\}.$$

1.1. **On definable subsets of arbitrary nilpotent Lie groups.** Instead of working with real unipotent groups we could have worked in a more general setting:

Let G be a connected, simply connected nilpotent Lie group. It is known (e.g. see [1]) that G is Lie isomorphic to a real algebraic subgroup

G_0 of $\mathrm{UT}(n,\mathbb{R})$. Given an o-minimal structure $\mathbb{R}_{\mathrm{om}}$, we may declare a subset of G to be $\mathbb{R}_{\mathrm{om}}$-definable (or real algebraic) if its image under the above isomorphism is an $\mathbb{R}_{\mathrm{om}}$-definable (or real algebraic) subset of G_0. As noted in Lemma 2.15 below, every Lie isomorphism between real unipotent groups is given by a polynomial map and thus this notion of definability (or algebraicity) does not depend of the choice of G_0 or the isomorphism between G and G_0. It follows from Fact 2.3 below that every closed connected subgroup of G is algebraic in this sense, and thus Theorem 1.3 holds for an arbitrary connected, simply connected nilpotent Lie groups, under the above interpretation of the relevant notions.

1.2. **On the proof.** Our proof combines model theory with the theory of nilpotent Lie groups. It breaks down into three main parts.

Given an $\mathbb{R}_{\mathrm{om}}$-definable $X \subseteq G$ we examine the contribution of complete types on X (see Preliminaries for more details on the basic notions) to the closure of $\pi_\Gamma(X)$. To each complete type p on X we assign "the nearest coset to p", a coset of a real algebraic subgroup of G, which we denote by c_pH_p (see Section 3). We then prove, see Corollary 5.4, that for every lattice Γ, the closure of $\pi_\Gamma(X)$ is the union of all $\pi_\Gamma(c_pH_p^\Gamma)$, as p varies over all complete types on X. Notice that the coset c_pH_p is independent of the lattice Γ.

Next, in Lemma 6.1, we use model theory to show that the family of nearest cosets

$$\{c_pH_p : p \text{ a complete type on } X\}$$

is itself a definable family in $\mathbb{R}_{\mathrm{om}}$.

Finally, we use Baire Category Theorem to obtain finitely many families of fixed subgroups of G.

2. Preliminaries

2.1. **Lattices and nilmanifolds.** We list some basic notions and properties of lattices in simply connected nilpotent Lie groups. For a reference we use [1] and [6].

We identify the Lie algebra of $\mathrm{UT}(n,\mathbb{R})$ with $\mathrm{ut}(n,\mathbb{R})$, the space of real $n\times n$ upper triangular matrices with 0 on the main diagonal.

The following fact will be used often.

Fact 2.1. *The matrix exponential map restricted to* $\mathrm{ut}(n,\mathbb{R})$ *is polynomial and maps* $\mathrm{ut}(n,\mathbb{R})$ *diffeomorphically onto* $\mathrm{UT}(n,\mathbb{R})$*. Its inverse* $\log\colon \mathrm{UT}(n,\mathbb{R}) \to \mathrm{ut}(n,\mathbb{R})$ *is a polynomial map as well.*

Remark 2.2. If G is a closed subgroup of $\mathrm{UT}(n,\mathbb{R})$ then we identify its Lie algebra $\mathfrak{g}$ with a subalgebra of $\mathrm{ut}(n,\mathbb{R})$. It follows from Fact 2.1 that if G is a connected closed subgroup of $\mathrm{UT}(n,\mathbb{R})$ then the exponential map $\exp_G\colon \mathfrak{g} \to G$ is a polynomial map (in matrix coordinates) that is also a diffeomorphism, and its inverse $\log_G\colon G \to \mathfrak{g}$ is polynomial as well.

We note:

Fact 2.3. *Assume that $G \subseteq \mathrm{UT}(n,\mathbb{R})$ is a subgroup. Then the following are equivalent:*

(1) G is a closed, connected subgroup of $\mathrm{UT}(n,\mathbb{R})$.
(2) G is a real algebraic subgroup of $\mathrm{UT}(n,\mathbb{R})$.
(3) G is definable in $\mathbb{R}_{\mathrm{om}}$.

Proof. The equivalence of (1) and (2) follows from the fact the exponential map and its inverse are polynomial maps.

Clearly, every real algebraic subgroup of $\mathrm{UT}(n,\mathbb{R})$ is $\mathbb{R}_{\mathrm{om}}$-definable, so (2) $\Rightarrow$ (3).

To see that (3) $\Rightarrow$ (1), note that every definable set in an o-minimal structure has finitely many connected components, and every definable subgroup is closed. Thus, every definable subgroup of $\mathrm{UT}(n,\mathbb{R})$ must be connected. □

For the rest of this section we assume that G is a real unipotent group, namely a real algebraic subgroup of $\mathrm{UT}(n,\mathbb{R})$, with $\mathfrak{g}$ its Lie algebra. Since $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism, the group G is a simply connected, and we have ([1, Coroallry 5.4.6]):

Fact 2.4. *A discrete subgroup $\Gamma \subseteq G$ is co-compact (i.e. G/Γ is compact) if and only if the induced Haar measure on G/Γ is finite.*

The above justifies the following definition:

Definition 2.5. A subgroup Γ of G is called *a lattice in G* if Γ is discrete and co-compact. If Γ is a lattice in G then the quotient G/Γ is called *a compact nilmanifold*.

Given a lattice $\Gamma \subseteq G$, a real algebraic subgroup H of G is called a Γ-*rational* if $\Gamma \cap H$ is a lattice in H.

Remark 2.6. In [1] a closed subgroup H of G is defined to be Γ-rational if the Lie algebra $\mathfrak{h}$ of H has a basis in the $\mathbb{Q}$-linear span of $\log_G(\Gamma)$. By [1, Theorem 5.1.11] these two definitions are equivalent.

The following is easy to verify:

Fact 2.7. *If Γ is a lattice in G then there is no real algebraic subgroup of G containing Γ other than G.*

Fact 2.8. *Let $H \subseteq G$ be a real algebraic normal subgroup with $\pi : G \to G/H$ the quotient map. Let $\Gamma \subseteq G$ a discrete subgroup. Then:*

(1) If Γ is a lattice in G and $\Gamma \cap H$ is a lattice in H then $H\Gamma$ is closed in G and $\pi(\Gamma)$ is a lattice in G/H.

(2) If $\Gamma \cap H$ is a lattice in H and $\pi(\Gamma)$ is a lattice in G/H then Γ is a lattice in G.

(3) If Γ is a lattice in G then H is Γ-rational if and only if $\pi_\Gamma(H)$ is closed.

(4) If Γ is a lattice in G then all subgroups in the ascending central series are Γ-rational, in particular $Z(G)$ is Γ-rational. Also, $[G, G]$ and all subgroups in the descending central series are Γ-rational subgroups (in particular closed).

(5) If Γ is a lattice in G and $H_1, H_2 \subseteq G$ are real algebraic Γ-rational subgroups then so is $H_1 \cap H_2$.

Proof. (1) and (2) follow from [1, Lemma 5.1.4].

(3). If H is Γ-rational then $H\Gamma$ is closed in G by (1). Assume $H\Gamma$ is closed in G. Then $\pi_\Gamma(H)$ is closed in G/Γ, hence compact. We can find then a compact subset $K \subseteq H$ such that $\pi_\Gamma(K) = \pi_\Gamma(H)$, i.e. $K\Gamma = H\Gamma$. It is not hard to see that $K\Gamma$ is closed, since it is a product of compact and closed sets.

(4) follows from [1, Proposition 5.2.1].

(5) follows from Remark 2.6. Indeed, since H_1 and H_2 are Γ-rational their Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ both have basis in the $\mathbb{Q}$-vector space $\mathbb{Q}$-span of $(\log_G(\Gamma)$. The Lie algebra of $H_1 \cap H_2$ is $\mathfrak{h}_1 \cap \mathfrak{h}_2$ and it has basis in the same $\mathbb{Q}$-vector space. □

We shall also need the following:

Lemma 2.9. *Let $\Gamma \subseteq G$ be a lattice in G. Let H be a real algebraic normal subgroup of G. Then H^Γ is also normal in G.*

Proof. Since H is invariant by conjugation, and every Γ-conjugate of H^Γ is also Γ-rational, it follows that H^Γ is normalized by Γ. Thus the normalizer of H^Γ is a real algebraic subgroup containing Γ, so by Fact 2.7 equals to G. □

Definition 2.10. Let $M = G/\Gamma$ be a compact nilmanifold. A set $S \subseteq M$ is called a *sub-nilmanifold of N* if there exists $a \in G$ and a Γ-rational group $H \subseteq G$ such that

$$S = \pi_\Gamma(aH).$$

The group G acts on M on the left and the sub-nilmanifold S can also be written as $S = a \cdot \pi_\Gamma(H)$.

Note that a sub-nilmanifold of M is closed in M and can be written as an orbit of the element $\pi_\Gamma(a)$, under the group aHa^{-1}.

We use the following lemma to identify quotients of unipotent group with semialgebraic sets:

Lemma 2.11. *Let G be a real unipotent group and let $H \subseteq G$ be a real algebraic subgroup. Then there exists a closed semialgebraic set $A \subseteq G$ such that the map $f : A \times H \to G$ given by $(a, h) \mapsto \cdot h$ is a diffeomorphism.*

Proof. Let $\mathfrak{h} \subseteq \mathfrak{g} \subseteq \mathrm{ut}(n, \mathbb{R})$ be the Lie algebras of H and G, respectively, and let $n = \dim G$ and $k = \dim H$. By [1, Theorem 1.1.13], there is a weak Malcev basis $\{\xi_1, \dots, \xi_n\}$ for $\mathfrak{g}$ through $\mathfrak{h}$. Namely, $\{\xi_1, \dots, \xi_k\}$ is a basis for $\mathfrak{h}$, and for every $m \le n$, the $\mathbb{R}$-linear span of $\xi_1, \dots, \xi_m$ is a Lie subalgebra of $\mathfrak{g}$.

By [1, Proposition 1.2.8], the map $\psi : \mathbb{R}^n \to G$ defined by

$$\psi(s_1, \dots, s_n) = \exp_G(s_1\xi_1) \cdot \ \dots \ \cdot \exp_G(s_n\xi_n)$$

is a polynomial diffeomorphism. It sends $\mathbb{R}^k \times \{0_{n-k}\}$ onto the group H and the subspace $\{0_k\} \times \mathbb{R}^{n-k}$ onto a closed semialgebraic subset of G, which we call A'. We have $G = H \cdot A'$, and if we now let $A = \{a^{-1} : a \in A'\}$ and replace $\psi(\bar{s})$ by $\psi(\bar{s})^{-1}$, then we see that $G = A \cdot H$ and the result follows. □

Recall that in any nilpotent group G, if $H \subseteq G$ is a proper subgroup then H is contained in a proper normal subgroup of G. Let us see that this remains true when restricting to real unipotent groups:

Claim 2.12. *If G is a real unipotent group and $H \subseteq G$ is a proper real algebraic subgroup then H is contained in a proper normal real algebraic subgroup of G.*

Proof. By [1, Theorem 1.1.13], there is a chain of real algebraic subgroups,

$$\{e\} = H_0 \subseteq \ \cdots \ \subseteq H = H_m \subseteq H_{m+1} \subseteq \cdots \subseteq H_n = G,$$

with $n = \dim G$, and $\dim H_{i+1} = \dim H_i + 1$. It follows from [1, Corollary 1.15], that H_{n-1} is normal in G, so we are done. □

Finally, we want to show that the collection of all cosets of real algebraic subgroup of G is itself a semi-algebraic family. By Fact 2.1, $\exp : \mathrm{ut}(n, \mathbb{R}) \to \mathrm{UT}(n, \mathbb{R})$ is a polynomial diffeomorphism. It induces a bijection between the Lie subalgebras of $\mathrm{ut}(n, \mathbb{R})$ and the connected closed subgroups of $\mathrm{UT}(n, \mathbb{R})$. Because the family of all Lie subalgebras of $\mathrm{ut}(n, \mathbb{R})$ is semi-algebraic we obtain:

Fact 2.13. *The family $\mathcal{F}_n$ of all cosets of real algebraic subgroups of* $\mathrm{UT}(n,\mathbb{R})$ *is semi-algebraic. Namely, there exists a semi-algebraic set* $S \subseteq M_n(\mathbb{R}) \times \mathbb{R}^k$*, for some* k*, such that*

$$\mathcal{F}_n = \{P \in M_n(\mathbb{R}) : \exists \bar{b} \in \mathbb{R}^k \, (P, \bar{b}) \in S\}.$$

In fact, by Definable Choice, we may choose the above family so that every coset is represented exactly once.

2.2. **Maps between real unipotent groups.**

Definition 2.14. Let G be a real unipotent group. A map $f : \mathbb{R}^d \to G$ is called *polynomial* if, when we view G as a subset of $\mathbb{R}^{n^2}$, the coordinate functions of f are real polynomials in $x_1, \ldots, x_d$. A map $f : G \to \mathbb{R}^d$ is *polynomial* if f is the restriction to G of a polynomial map from $\mathbb{R}^{n^2}$ into $\mathbb{R}^d$.

We note:

Lemma 2.15. *(1) If G_1 and G_2 are real unipotent groups and $f : G_1 \to G_2$ is a Lie homomorphism then f is a polynomial map.*
(2) Let G be a real unipotent group and v an arbitrary element in its Lie algebra $\mathfrak{g} \subseteq \mathrm{ut}(n,\mathbb{R})$. If $p : \mathbb{R}^d \to \mathbb{R}$ is a polynomial function then $f(\bar{x}) = \exp_G(p(\bar{x})v)$ is a polynomial map from $\mathbb{R}^d$ into G.

Proof. (1) By standard Lie theory we have $f = \exp_{G_2}, df, \log_{G_1}$, where $df : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map. Since $\log_{G_1}$ and $\exp_{G_2}$ are polynomials, f is polynomial as well.

(2) By Fact 2.1, the map $\exp : \mathrm{ut}(n,\mathbb{R}) \to \mathrm{UT}(n,\mathbb{R})$ is polynomial, and $\exp_G$ is its restriction to $\mathfrak{g}$ is clearly polynomial as well. The map $f : \mathbb{R}^d \to G$ is thus a composition of polynomial maps. □

2.3. **Model theoretic preliminaries.** We use the same set-up as in [11, Section 2]. We refer to [2] and [4] for introductory material on o-minimal structures, as well as examples. We let

$$\mathcal{L}_{\mathrm{sa}} = \langle +, -, \cdot, <, 0, 1 \rangle$$

be the language of ordered rings (as the subscript suggests, the definable sets in the ordered field $\mathbb{R}$ are the semialgebraic sets). We let $\mathcal{L}_{\mathrm{om}} \supseteq \mathcal{L}_{\mathrm{sa}}$ be the language of our o-minimal structure $\mathbb{R}_{\mathrm{om}}$. We let $\mathcal{L}_{\mathrm{full}}$ be the language in which every subset of $\mathbb{R}^n$ has a predicate symbol, and let $\mathbb{R}_{\mathrm{full}}$ be the corresponding structure on $\mathbb{R}$. Clearly, every $\mathbb{R}_{\mathrm{om}}$-definable set is also $\mathbb{R}_{\mathrm{full}}$-definable.

All definable sets are definable *with parameters*. The dimension of a definable set in an o-minimal structure is defined using the cell decomposition theorem. In our setting it is enough to know that an

$\mathbb{R}_{\text{om}}$-definable $X \subseteq \mathbb{R}^n$ has dimension k if and only if it can be decomposed into finitely many C^1-submanifolds of $\mathbb{R}^n$, whose maximal dimension is k.

2.3.1. *Elementary extensions and some valuation theory.* We let $\mathfrak{R}_{\text{full}} = \langle \mathfrak{R}, \ldots \rangle$ be an elementary extension of $\mathbb{R}_{\text{full}}$ which is $|\mathbb{R}|^+$-saturated, or alternatively a sufficiently large ultra-power of $\mathbb{R}_{\text{full}}$. We let $\mathfrak{R}_{\text{om}}$ and $\mathfrak{R}$ be reducts of $\mathfrak{R}_{\text{full}}$ to the languages $\mathcal{L}_{\text{om}}$ and $\mathcal{L}_{\text{sa}}$, respectively. Given any set $X \subseteq \mathbb{R}^n$, we denote by $X^\sharp = X(\mathfrak{R})$ its realization in $\mathfrak{R}_{\text{full}}$. We use roman letters X, Y, Z etc. to denote subsets of $\mathbb{R}^n$ and script letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ to denote subsets of $\mathfrak{R}^n$, when not of the form $X^\sharp$ for some $X \subseteq \mathbb{R}^n$.

The underlying field $\langle \mathfrak{R}; +, \cdot \rangle$ of $\mathfrak{R}_{\text{full}}$ is real closed and we let

$$\mathcal{O}(\mathfrak{R}) = \{\alpha \in \mathfrak{R} : \exists n \in \mathbb{N}\ |\alpha| < n\}.$$

It is a valuation ring of $\mathfrak{R}$ and its the maximal ideal $\mu(\mathfrak{R})$ is the set of infinitesimal elements, namely

$$\mu(\mathfrak{R}) = \{\alpha \in \mathfrak{R} : \forall n \in \mathbb{N}\, |\alpha| < 1/n\}.$$

Mostly, for a real unipotent group G, we shall use a group variant $\mathcal{O}(G)$ and $\mu(G)$ of the above, defined as follows. Because G is closed subset of $\mathrm{UT}(n, \mathbb{R})$, it can be viewed as a closed subset of $\mathbb{R}^{n^2}$, and then $G^\sharp$ is a subset of $\mathfrak{R}^{n^2}$. In the definitions below we let I denote the identity matrix and use $+$ for the usual addition in $\mathbb{R}^{n^2}$.

We let

$$\mathcal{O}(G) = \mathcal{O}(\mathfrak{R})^{n^2} \cap G^\sharp \ \text{ and } \mu(G) = (I + \mu(\mathfrak{R})) \cap G^\sharp.$$

Both $\mathcal{O}(G)$ and $\mu(G)$ are subgroups of $G^\sharp$, and $\mu(G)$ is normal in $\mathcal{O}(G)$. In fact $\mathcal{O}(G)$ is a semi-direct product of $\mu(G)$ and G, so given $\beta \in \mathcal{O}(G)$ there exists a unique $b \in G$ such that

$$\beta \in \mu(G)b = b\mu(G).$$

We call b *the standard part of* β, denoted as $b = \mathrm{st}(\beta)$. The map $\mathrm{st} : \mathcal{O}(G) \to G$ is a surjective group homomorphism whose kernel is $\mu(G)$. It coincides with the the standard part map on $\mathcal{O}(\mathfrak{R})^{n^2}$, when restricted to $\mathcal{O}(G)$. We thus have, for $g = (g_{i,j})_{1 \le i,j \le n} \in G^\sharp$,

$$g \in \mathcal{O}(G) \Leftrightarrow \forall i, j,\ g_{i,j} \in \mathcal{O}(\mathfrak{R}) \Leftrightarrow |g| \in \mathcal{O}(\mathfrak{R}),$$

where $|g|$ is the Euclidean norm computed in $\mathfrak{R}^{n^2}$.

For $\mathcal{X} \subseteq G^\sharp$, we let

$$\mathrm{st}(\mathcal{X}) := \mathrm{st}(\mathcal{X} \cap \mathcal{O}(G)).$$

When our setting is clear we shall omit G from the notation and use $\mathcal{O}$ and μ instead.

We shall be using extensively the following simple observation:

Fact 2.16. *If $X \subseteq G$ is an arbitrary set then* $\mathrm{cl}(X) = \mathrm{st}(X^\sharp)$. *In particular, if $\Gamma \subseteq G$ is a subgroup then*

$$\mathrm{cl}(X\Gamma) = \mathrm{st}(X^\sharp \Gamma^\sharp).$$

2.3.2. *Types.* If $\mathcal{L}_\bullet$ is any of our languages then *an $\mathcal{L}_\bullet$-type $p(x)$* over $\mathbb{R}$ is a consistent collection of $\mathcal{L}_\bullet$-formulas with free variables x and parameters in $\mathbb{R}$, or equivalently, a collection of sets defined by $\mathcal{L}_\bullet$-formulas, such that the intersection of any finitely many of them is non-empty. When $p(x)$ contains a formula saying $x \in X$ then we write $p \vdash X$ and say that *p is a type on X*.

An $\mathcal{L}_\bullet$-type $p(x)$ is *complete* if for every $\mathcal{L}_\bullet$-definable $X \subseteq \mathbb{R}^n$, where $n = \mathit{length}(x)$, either X or its complement belongs to p. For $p(x)$ an $\mathcal{L}_\bullet$-type over $\mathbb{R}$, we denote by $p(\mathfrak{R})$ its realization in $\mathfrak{R}$, namely the intersection of all $X^\sharp$, for $X \in p$.

Given $\alpha \in \mathfrak{R}^n$, we let $\mathrm{tp}_\bullet(\alpha/\mathbb{R})$ be the collection of all $\mathcal{L}_\bullet$-definable subsets $X \subseteq \mathbb{R}^n$ with $\alpha \in X^\sharp$. It is easily seen to be a complete type.

For G a real unipotent group, we denote by $S_G(\mathbb{R})$ the collection of all complete $\mathcal{L}_{\mathrm{om}}$-types p over $\mathbb{R}$ such that $p \vdash G$.

Finally, if $p \in S_G(\mathbb{R})$, then we let $\mu \cdot p$ be the (partial) type whose realization is $\mu(G)\, p(\mathfrak{R})$. The type $\mu \cdot p$ is not a complete type, and we call it *a μ-type*. We identify two μ-types $\mu \cdot p, \mu \cdot q$ if $\mu(G)\, p(\mathfrak{R}) = \mu(G)\, q(\mathfrak{R})$. The group G acts on the set of all μ-types on the left, since $g \cdot (\mu \cdot p) = \mu \cdot (g \cdot p)$. See [10] for all the above.

The following definition and subsequent theorem, from [10], will play a significant role in our proof. Given $p \in S_G(\mathbb{R})$, we let

$$\mathrm{Stab}^\mu(p) = \{g \in G(\mathbb{R}) : g \cdot (\mu \cdot p) = \mu \cdot p\}.$$

It is easy to see that $g \in \mathrm{Stab}^\mu(p)$ if and only if g leaves the set $(\mu{\cdot}p)(\mathfrak{R})$ invariant, when acting on the left.

The main theorem of [10] is:

Fact 2.17. [10] *For every $p \in S_G(\mathbb{R})$, the group* $\mathrm{Stab}^\mu(p)$ *is and $\mathcal{L}_{\mathrm{om}}$-definable over $\mathbb{R}$. Moreover, if p is unbounded (namely, $p(\mathfrak{R})$ is not contained in $\mathcal{O}(G)$) then* $\dim \mathrm{Stab}^\mu(p) > 0$.

The above theorem holds for arbitrary definable groups in o-minimal structures, and then $\mathrm{Stab}^\mu(p)$ is also torsion-free. However, when G is a real algebraic subgroup of $\mathrm{UT}(n, \mathbb{R})$ then necessarily $\mathrm{Stab}^\mu(p)$ is real algebraic, even if the type p is in a richer language.

3. The nearest coset of a type

The goal of this section is to prove that to each complete $\mathcal{L}_{\text{om}}$-type p on a real unipotent group G one can associate a coset gH of a real algebraic subgroup $H \subseteq G$, which is "nearest" to p in a precise sense.

Recall that below we are using H, G etc. to denote the $\mathbb{R}$-points of real groups, and use $H^\sharp, G^\sharp$ etc to denote $\mathfrak{R}$-points of the same groups.

Definition 3.1. For G a real unipotent group, $\alpha \in G^\sharp$, $g \in G$, and $H \subseteq G$ a real algebraic subgroup, we say that gH *is near* α if $\alpha \in \mu(G)\, gH^\sharp$.

Note that $\alpha \in \mathcal{O}(G)\, H^\sharp$ if and only if there exists $g \in G$ such that gH is near α. Also, if $\text{tp}_{\text{sa}}(\alpha/\mathbb{R}) = \text{tp}_{\text{sa}}(\beta/\mathbb{R})$ then gH is near α if and only if gH is near β. Our ultimate goal is to show that there exists a minimal coset near α.

Lemma 3.2. *Let G be a real unipotent group and let $H, N \subseteq G$ be real algebraic subgroups with N normal in G. Assume that $\alpha \in H^\sharp$ and there is $b \in G$ such that the coset bN is near α. Then $bN \cap H \neq \emptyset$ and the coset $bN \cap H$ is near α as well.*

Proof. We have
$$\alpha = \epsilon b n$$
for some $\epsilon \in \mu(G)$ and $n \in N^\sharp$.

We first claim that both b and ϵ belongs to the group $(NH)^\sharp$. Indeed, $b = \epsilon^{-1}\alpha n^{-1}$, so $b = \text{st}(\alpha n^{-1})$. The element αn^{-1} belongs to $(NH)^\sharp$, and since NH is a closed subset of G, it follows from Fact 2.16, that $b \in NH$. Hence, $\epsilon = \alpha n^{-1} b^{-1}$ is in $(NH)^\sharp$ as well.

Thus, we may work entirely in the group NH, so we may assume that $G = NH = HN$.

Claim 3.3. *If $G = NH$ then*
$$\mu(G) = \mu(N)\,\mu(H) = \mu(H)\,\mu(N).$$

Proof. By continuity of multiplication, $\mu(N)\,\mu(H) \subseteq \mu(G)$. For the opposite inclusion, it is enough to show that for every $\mathcal{L}_{\text{om}}$-definable $U \subseteq N$, $V \subseteq H$, neighborhoods of e, we have $\mu(G) \subseteq (U\,V)^\sharp$. For that, it suffices to show that the set $U\,V$ contains an open neighborhood of e in G. This follows from the fact that the map $(x, y) \mapsto xy$ from $N \times H$ into G, is a submersion at (e, e). □

We are now ready to prove the lemma. We start with $\alpha = \epsilon b n$, with $\epsilon \in \mu(G)$ and $n \in N^\sharp$. Using the above Claim, $\epsilon = \epsilon_h \epsilon_n$ with $\epsilon_h \in \mu(H)$ and $\epsilon_n \in \mu(N)$. We also write $b = b_h b_n$, with $b_h \in H$ and $b_n \in N$.

So, $\alpha = \epsilon_h \epsilon_n b_h n'$, with $n' \in N^\sharp$. Since N is normal, $\epsilon_n b_h = b_h n^*$, for $n^* \in N^\sharp$, so

$$\alpha = \epsilon_h b_h n^* n'.$$

Clearly, $b_h n^* n'$ is in $b_h N^\sharp$ and since α and ϵ_h are in $H^\sharp$, we also have $b_h n^* n' \in H^\sharp$. So, $\alpha \in \mu(G)\,(b_h N^\sharp \cap H^\sharp)$, and in particular $b_h N \cap H$ is nonempty, and hence a left coset of $N \cap H$. This ends the proof of Lemma 3.2. □

Corollary 3.4. *Let G and $H, N \subseteq G$ be as above. Assume that there are $b, c \in G$ such that the cosets bN and cH are near α. Then $bN \cap cH \neq \emptyset$ and the coset $bN \cap cH$ is near α.*

Proof. Note first that for any $\epsilon \in \mu(G)$, $\alpha \in \mu(G)\,(bN^\sharp \cap cH^\sharp)$ if and only if $\epsilon\alpha \in \mu(G)\,(bN^\sharp \cap cH^\sharp)$. Thus, we may replace the assumption that $\alpha \in \mu(G)cH^\sharp$ by $\alpha \in (cH)^\sharp$, so $c^{-1}\alpha \in H^\sharp \cap \mu(G)\,(c^{-1}bN^\sharp)$. We apply Lemma 3.2 and conclude that $c^{-1}\alpha \in \mu(G)(c^{-1}bN \cap H)^\sharp$. It follows that $\alpha \in \mu(G)(bN \cap cH)^\sharp$. In particular, $bN \cap cH \neq \emptyset$, so it is a left coset of $N \cap H$. □

We also need:

Lemma 3.5. *Let G be a real unipotent group and $H \subseteq G$ a real algebraic subgroup. For $g_1, g_2 \in G$, assume that $\mu(G)\,g_1H^\sharp \cap \mu(G)\,g_2H^\sharp \neq \emptyset$. Then $g_1H = g_2H$.*

Proof. We let

$$\alpha = \epsilon_1 g_1 h_1 = \epsilon_2 g_2 h_2,$$

where $h_1, h_2 \in H^\sharp$ and $\epsilon_1, \epsilon_2 \in \mu(G)$. It follows that $g_2^{-1}g_1 = \epsilon h_2 h_1^{-1}$ for some $\epsilon \in \mu(G)$. But then $g_2^{-1}g_1 = \mathrm{st}(h_2h_1^{-1}) \in H$, so $g_1H = g_2H$. □

Remark 3.6. Although we proved lemmas 3.2–3.5 for real unipotent groups, the results hold for an arbitrary definable group in an o-minimal structures, with exactly the same proofs. See [9] for more on definable groups in o-minimal structures.

We are ready to prove the main result of this section.

Theorem 3.7. *Let G be a real unipotent group and let $\alpha \in G^\sharp$.*

(1) If H_1, H_2 are real algebraic subgroups of G and $g_1, g_2 \in G$ such that the cosets g_1H_1 and g_2H_2 are near α then $g_1H_1 \cap g_2H_2 \neq \emptyset$ and the coset $g_1H_1 \cap g_2H_2$ is near α as well.

(2) There exists a smallest left coset of real algebraic subgroup of G, among all such cosets that are near α.

Proof. (1) We use induction on $\dim G$ and note that the result is obviously true when $\dim G = 1$.

We may clearly assume that H_1, H_2 are both proper subgroups of G. So by Claim 2.12 there exists a proper normal real algebraic $N_1 \subseteq G$ containing H_1. Obviously g_1N_1 is near α. By Corollary 3.4, $\mathfrak{g}_1 N_1 \cap g_2H_2 \neq \emptyset$ and for $d \in g_1N_1 \cap g_2H_2$ the coset $d(N_1 \cap H_2)$ is near α.

Obviously, for $d \in g_1N_1 \cap g_2H_2$ we have $g_1H_1 \cap g_2H_2 = g_1H_1 \cap d(N_1 \cap H_2)$. Replacing H_2 by $N_1 \cap H_2$ and g_2 by $d \in N_1 \cap H_2$, if needed, we may assume that $H_2 \subseteq N_1$.

By assumption, $\alpha \in \mu(G)\, g_1H_1^\sharp \cap \mu(G)\, g_2H_2^\sharp$, so $\alpha \in \mu(G)g_1N_1^\sharp \cap \mu(G)g_2N_1^\sharp$. By Claim 3.5, $g_1N_1 = g_2N_1$, hence $g_1^{-1}g_2 \in N$.

We now consider $\alpha' = g_1^{-1}\alpha \in N_1^\sharp$, and note that

$$\alpha' \in \mu(G)\, H_1^\sharp \cap \mu(G)\, g_1^{-1}g_2H_2^\sharp.$$

(2) The existence of a smallest coset immediately follows from (1). □

The above theorem allows us to define:

Definition 3.8. Given real unipotent G, and $\alpha \in G^\sharp$, we denote by A_α the smallest coset near α. We call it *the nearest coset to* α. We denote by H_α the associated group, so $A_\alpha = gH_\alpha$ for any $g \in A_\alpha$. For p the complete type $\mathrm{tp}_{\mathrm{om}}(\alpha/\mathbb{R})$, we also use $A_p := A_\alpha$ and write $A_p = gH_p$.

Note that if $\alpha \in \mathcal{O}(G)$ then the nearest coset to α is just $\{\mathrm{st}(\alpha)\}$, which can be viewed as a coset of the identity of G. On the other hand, if $\alpha \notin \mathcal{O}(G)$ then no element in G is near α and therefore $\dim A_\alpha > 0$. We thus have:

Lemma 3.9. *For $\alpha \in G^\sharp$, $\alpha \in \mathcal{O}(G)$ if and only if $A_\alpha = \{\mathrm{st}(\alpha)\}$.*

We also need:

Lemma 3.10. *Assume that G and G_1 are real unipotent groups and $f : G \to G_1$ is a surjective Lie homomorphism. Then*
(1) $f(\mu(G)) = \mu(G_1)$ and $f(\mathcal{O}(G)) = \mathcal{O}(G_1)$.
(2) If $\alpha \in G^\sharp$ and $\beta = f(\alpha)$ then $f(A_\alpha) = A_\beta$.

Proof. By Lemma 2.15, f is a polynomial map and hence has a natural extension to $\mathfrak{R}_{\mathrm{om}}$, which is still denoted by $f : G^\sharp \to G_1^\sharp$, so $\beta = f(\alpha)$ is defined and belongs to $G_1^\sharp$.

(1) The map f is continuous and open (by its surjectivity), and hence we have $f(\mu(G)) = \mu(G_1)$ and $f(\mathcal{O}(G)) = \mathcal{O}(G_1)$.

(2) It follows from (1) that if gH is near α then $f(gH)$ is near β, and therefore $A_\beta \subseteq f(A_\alpha)$. For the opposite inclusion, assume that $A_\beta =$

$g_1H_1 \subseteq G_1$. We have $\beta \in \mu(G_1)A_\beta$ and therefore $\alpha \in \mu(G)f^{-1}(A_\beta)$ (here we use that $f(\mu(G)) = \mu(G_1)$). By the minimality of A_α, we have $A_\alpha \subseteq f^{-1}(A_\beta)$ and therefore $f(A_\alpha) \subseteq A_\beta$. □

We end this section with an example which shows that Theorem 3.7 fails for arbitrary real algebraic groups.

Example 3.11. We work with $G = SL(2,\mathbb{R})$. For ε an infinitesimally small element of $\mathfrak{R}$, we let

$$\alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$$

be an element of $SL(2,\mathfrak{R})$. We show that there is no minimal coset near α.

We denote by D the diagonal subgroup of $SL(2,\mathbb{R})$. Since $\alpha \in D^\sharp$, we have that D is a coset near α.

Let

$$b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and H be the conjugate of D by b, namely $H = b^{-1}Db$.

We consider the coset $bH = Db$, and claim that it is near α. Obviously, the element $\beta = \alpha b$ is in $D^\sharp b = bH^\sharp$, so it is enough to see that $\alpha\beta^{-1}$ is in $\mu(G)$. We have

$$\alpha\beta^{-1} = \begin{pmatrix} 1 & \varepsilon^2 \\ 0 & 1. \end{pmatrix},$$

clearly in $\mu(G)$.

Thus, both D and bH are near α, but $D \cap bH = D \cap Db = \emptyset$, so there is no minimal coset near α.

The above example takes place entirely in the solvable group of upper triangular matrices, thus we see that Theorem 3.7 fails even for solvable linear Lie groups.

4. The algebraic normal closure of a set

We still assume here that G is a real unipotent group. All definability is in $\mathbb{R}_{\mathrm{om}}$.

Definition 4.1. Given a definable set $X \subseteq G$, we let $\langle X\rangle_{alg}$ be the minimal real algebraic subgroup of G containing X.

We call the smallest algebraic normal subgroup of G containing X *the algebraic normal closure of X*.

Lemma 4.2. *Let $P \subseteq G$ a real algebraic subgroup and assume that $U \subseteq G$ is a nonempty open subset of G. Then the group $\langle \bigcup_{g\in U} P^g \rangle_{alg}$ is normal in G, and in particular, equals the algebraic normal closure of P.*

Proof. For subsets $A, S \subseteq G$ we write A^S for $\bigcup_{g\in S} A^g = \bigcup_{g\in S} g^{-1}Ag$.

Let $u \in U$. Since $\langle P^U \rangle_{alg} = \langle\ (P^u)^{u^{-1}U} \rangle_{alg}$, replacing P by P^u and U by $u^{-1}U$, if needed, we may assume that U is an open neighborhood of e.

Clearly, for $V \subseteq V_1 \subseteq G$ we have $\langle P^V \rangle_{alg} \subseteq \langle P^{V_1} \rangle_{alg}$, and $\langle P^V \rangle_{alg}$ is normal in G if and only if $\langle P^V \rangle_{alg} = \langle P^G \rangle_{alg}$. Thus, to show that $\langle P^U \rangle_{alg}$ is normal in G, it is sufficient to find a non-empty $B \subseteq U$ such that $\langle P^B \rangle_{alg}$ is normal in G.

By DCC on real algebraic subgroups, we can find an open neighborhood U_0 of e with $U_0 \subseteq U$ such that $\langle P^V \rangle_{alg} = \langle P^{U_0} \rangle_{alg}$ for any open neighborhood V of e with $V \subseteq U_0$. Let $N = \langle P^{U_0} \rangle_{alg}$. We claim that N is normal in G.

Indeed, choose open $B \ni e$ with. $B^{-1} = B$ and $BB \subseteq U_0$. Since for any $b \in B$ we have $e \in Bb \subseteq U_0$, it follows that

$$N^b = (\langle P^B \rangle_{alg})^b = \langle P^{Bb} \rangle_{alg} = N.$$

Thus the normalizer of N contains an open neighborhood of e and therefore equals the whole of G, hence N is normal in G. □

As a corollary we obtain the following proposition. Recall that for a subgroup $N \subseteq G$ and a lattice $\Gamma \subseteq G$, the group N^Γ is the smallest Γ-rational subgroup of G containing N.

Proposition 4.3. *Let G be a real unipotent group, P a real algebraic subgroup of G, and N be the algebraic normal closure of P. Let Γ be a lattice in G. Then the set $X = \{g \in G : (P^g)^\Gamma = N^\Gamma\}$ is dense in G.*

Proof. It is sufficient to prove that the complement of X is nowhere dense in G. Since every conjugate of P is contained in N, this complement can be written as the union over all proper Γ-rational subgroup L of N^Γ, of the semialgebraic sets

$$X_L = \{g \in G : (P^g)^\Gamma \subseteq L\} = \{g \in G : P^g \subseteq L\}.$$

By Remark 2.6, there are at most countably many Γ-rational subgroups of G, so by Baire Categoricity Theorem, it is enough to prove that each of the sets X_L is nowhere dense. Since X_L is semialgebraic we just need to see that it does not contain any nonempty open set.

Assume towards contradiction that for some proper Γ-rational subgroup $L \subseteq N^\Gamma$, X_L contained an open set U. Then $\langle \bigcup_{g\in U} P^g \rangle_{alg}$ is contained in L. But, by Lemma 4.2, $\langle\bigcup_{g\in U} P^g\rangle_{alg} = N$, so $N \subseteq L$ and hence $N^\Gamma \subseteq L$, contradicting our choice of L. □

5. The main result for complete types

We assume in this section that G is a real unipotent group.

Lemma 5.1. *Let H be a real unipotent group, $f : G \to H$ a surjective homomorphism of Lie groups, and $\mathcal{X}$ a subset of $G^\sharp$.*

Then, for every lattice $\Gamma \subseteq G$, if $f(\Gamma)$ is closed in H then

$$f(\mathrm{st}(\mathcal{X}\Gamma^\sharp)) = \mathrm{st}(f(\mathcal{X})f(\Gamma^\sharp)).$$

Proof. By Lemma 2.15, f is polynomial so in particular definable in $\mathbb{R}_{\mathrm{om}}$. By Lemma 3.10, f sends $\mathcal{O}(G)$ to $\mathcal{O}(H)$ and $\mu(G)$ to $\mu(H)$. It follows that for $\alpha \in \mathcal{O}(G)$ we have $f(\mathrm{st}(\alpha)) = \mathrm{st}(f(\alpha))$.

Let $D_{\mathcal{X},\Gamma} = \mathrm{st}(\mathcal{X}\,\Gamma^\sharp)$. We need to show that $f(D_{\mathcal{X},\Gamma}) = \mathrm{st}(f(\mathcal{X})f(\Gamma^\sharp))$.

$\subseteq$: If $a = \mathrm{st}(\alpha\gamma^*) \in D_{\mathcal{X},\Gamma}$, with $\alpha \in \mathcal{X}$ and $\gamma^* \in \Gamma^\sharp$ then $f(a) = \mathrm{st}(f(\alpha\gamma^*)) = \mathrm{st}(f(\alpha)f(\gamma^*)) \in \mathrm{st}(f(\mathcal{X})\,f(\Gamma^\sharp))$.

$\supseteq$: Assume that $a_1 = \mathrm{st}(f(\alpha)\,f(\gamma^*))$, for some $\alpha \in \mathcal{X}$ and $\gamma^* \in \Gamma^\sharp$. We want to show that $a_1 \in f(D_{\mathcal{X},\Gamma})$.

Since G/Γ is compact, there exists a compact semi-algebraic set $K \subseteq G$ such that for every $g \in G$, there exists $\gamma \in \Gamma$ with $g\gamma \in K$. This remains true for $G^\sharp$, $\Gamma^\sharp$ and $K^\sharp$. Thus, we can find $\gamma_1^* \in \Gamma^\sharp$ such that

$$(\alpha\gamma^*)\gamma_1^* \in K^\sharp \subseteq \mathcal{O}(G).$$

We may therefore take the standard part and get $a := \mathrm{st}(\alpha\gamma^*\gamma_1^*) \in D_{\mathcal{X},\Gamma}$. It follows that

$$f(a) = f(\mathrm{st}(\alpha\gamma^*\gamma_1^*)) = \mathrm{st}(f(\alpha\gamma^*\gamma_1^*)) = \mathrm{st}(f(\alpha)f(\gamma^*\gamma_1^*)) \in \mathrm{st}(f(\mathcal{X})\,f(\Gamma^\sharp)).$$

Writing $f(a)$ differently we have

$$f(a) = \mathrm{st}(f(\alpha\gamma^*)f(\gamma_1^*)) = \mathrm{st}(f(\alpha\gamma^*))\,\mathrm{st}(f(\gamma_1^*)).$$

Note that we are allowed to write this since indeed $f(\gamma_1^*) \in \mathcal{O}(H)$, because both $f(a)$ and $f(\alpha\gamma^*)$ are in $\mathcal{O}(H)$). So, the term on the right equals $a_1\,\mathrm{st}(f(\gamma_1^*))$.

Finally, since $f(\Gamma)$ is closed in H, we have

$$f(\Gamma) = \mathrm{st}(f(\Gamma)^\sharp) = \mathrm{st}(f(\Gamma^\sharp)),$$

hence $\mathrm{st}(f(\gamma_1^*)) = f(\gamma)$, for some $\gamma \in \Gamma$.

Because $D_{\mathcal{X},\Gamma}$ is right-invariant under Γ, its image is right-invariant under $f(\Gamma)$ and hence $f(a)f(\gamma)^{-1} = a_1$ is in $f(D_{\mathcal{X},\Gamma})$, as we wanted. □

Recall that for a complete type $p \in S_G(\mathbb{R})$ we let A_p be the nearest coset to p. We can now prove:

Theorem 5.2. *Assume that p is a type in $S_G(\mathbb{R})$. Then for every lattice $\Gamma \subseteq G$ we have*

$$\mathrm{st}(p(\mathfrak{R})\Gamma^\sharp) = \mathrm{cl}(A_p\Gamma).$$

Proof. We write $A_p = gH_p$. To simplify notation we let

$$D_{p,\Gamma} = \mathrm{st}(p(\mathfrak{R})\Gamma^\sharp).$$

We first handle a special case.

Proposition 5.3. *Assume that $A_p = H_p$ is a subgroup of G and that $H_p^\Gamma = G$. Then $D_{p,\Gamma} = G$.*

Proof of Proposition. We prove the proposition by induction on $\dim G$, starting from $\dim G = 0$, for which the result is trivially true. We assume then that $\dim G > 0$.

Since $H_p^\Gamma = G$, the group H_p must have positive dimension, hence p is not a bounded type, so by Fact 2.17, the group $P := \mathrm{Stab}^\mu(p)$ is a definable subgroup of positive dimension.

We consider the algebraic normal closure of P, call it N and then N^Γ. By Lemma 2.9, N^Γ is normal, hence it is the minimal normal Γ-rational subgroup of G containing P. Since G is nilpotent, the intersection any nontrivial normal subgroup with the center $Z(G)$ is nontrivial (see for example [15, Proposition 7.13]), so $N_0 = N^\Gamma \cap Z(G)$ is nontrivial. Since G is torsion-free, N_0 is a real algebraic subgroup of positive dimension, so $\dim G/N_0 < \dim G$.

We consider the quotient map

$$f : G \to G/N_0.$$

The group G/N_0 is again a connected, simply connected nilpotent Lie group and hence Lie isomorphic to a real unipotent group. By Lemma 2.15, the composition of this isomorphism with f is a polynomial map. Thus, we identify G/N_0 with a real unipotent group, and still denote the homomorphism from G onto this unipotent group by f.

We let q be the image of the type p under f. By that we mean that for some (equivalently any) $\alpha \in p(\mathfrak{R})$ we let $q = \mathrm{tp}_{om}(f(\alpha)/\mathbb{R}) \vdash G/N_0$. We let $\Gamma_1 = f(\Gamma)$. Since both $Z(G)$ and N^Γ are Γ-rational then so is N_0. It follows that Γ_1 is a lattice in G/N_0 (for both, see Fact 2.8).

Let $A_q = g_qH_q$ be the nearest coset of q. We claim that $A_q^{\Gamma_1} = G/N_0$, namely $H_q^{\Gamma_1} = G/N_0$. Indeed, first note that by Lemma 3.10, we have $f(A_p) = A_q$, so $f(H_p) = A_q$ and hence $A_q = H_q$ is a group. Next, since N_0 is Γ-rational the pre-image under f of the Γ_1-rational group $H_q^{\Gamma_1}$ is a Γ-rational subgroup of G containing H_p, so by our assumptions on p it equals to G. It follows that $H_q^{\Gamma_1} = G/N_0$.

Since $\dim G/N_0 < \dim G$, we may apply induction to $q \vdash G/N_0$ and Γ_1 and conclude that $\mathrm{st}(q(\mathfrak{R})\Gamma_1^\sharp) = G/N_0$. Therefore, by Lemma 5.1,

$$f(D_{p,\Gamma}) = G/N_0.$$

Next, we claim that $D_{p,\Gamma}$ is left-invariant under $P = \mathrm{Stab}^\mu(p)$. Indeed, if $a \in D_{p,\Gamma} = \mathrm{st}(p(\mathfrak{R})\Gamma^\sharp)$ then $a = \epsilon\alpha\gamma^*$ for $\epsilon \in \mu(G)$, $\alpha \in p(\mathfrak{R})$ and $\gamma^* \in \Gamma^\sharp$. By definition, for every $h \in P$, there exists $\epsilon' \in \mu(G)$ and $\alpha' \in p(\mathfrak{R})$ such that $h\alpha = \epsilon'\alpha'$. But then, for some $\epsilon'' \in \mu(G)$,

$$ha = h\epsilon\alpha\gamma^* = \epsilon'' h\alpha\gamma^* = \epsilon''\epsilon'\alpha'\gamma^*.$$

Since $ha \in G$, we have $ha = \mathrm{st}(ha) = \mathrm{st}(\alpha'\gamma^*) \in D_{p,\Gamma}$, so $D_{p,\Gamma}$ is left-invariant under P.

By definition, $D_{p,\Gamma}$ is also right-invariant under Γ.

We now consider the set

$$Y = \{g \in G : (P^g)^\Gamma = N^\Gamma\}.$$

By Proposition 4.3, the set Y is dense in G.

Claim *The set Y is contained in $D_{p,\Gamma}$.*

Proof of Claim. We will show that $Y \cap D_{p,\Gamma}$ is left-invariant under $N_0 = \ker(f)$ and that $f(Y \cap D_{p,\Gamma}) = f(Y)$. The result follows (since we conclude that $Y = Y \cap D_{p,\Gamma}$).

First, let us note that $N_0Y = Y$: Because N_0 is central, for every $n \in N_0$ and $g \in G$, $P^g = P^{ng}$, so by the definition of Y, if $g \in Y$ then so is ng.

In order to show that $Y \cap D_{p,\Gamma}$ is left-invariant under N_0 it is enough to show that for every $g \in Y \cap D_{p,\Gamma}$, we have $N_0 g \subseteq D_{p,\Gamma}$. So fix $g \in Y \cap D_{p,\Gamma}$.

Since $D_{p,\Gamma}$ is left-invariant under P and right-invariant under Γ, we have $Pg\Gamma = gP^g\Gamma \subseteq D_{p,\Gamma}$. Because it is also closed, we have $\mathrm{cl}(gP^g\Gamma) \subseteq D_{p,\Gamma}$. Since $g \in Y$,

$$\mathrm{cl}(P^g\Gamma) = (P^g)^\Gamma\Gamma = N^\Gamma\Gamma,$$

and hence

$$gN^\Gamma\Gamma = \mathrm{cl}(gP^g\Gamma) \subseteq D_{p,\Gamma}.$$

Because $N_0 \subseteq N^\Gamma$ and is normal in G, we have

$$N_0 g = gN_0 \subseteq gN_0^\Gamma \subseteq D_{p,\Gamma},$$

thus completing the proof that $Y \cap D_{p,\Gamma}$ is left-invariant under N_0.

Now, since $N_0Y = Y$, we have $f(Y \cap D_{p,\Gamma}) = f(Y) \cap f(D_{p,\Gamma})$. We already saw that $f(D_{p,\Gamma}) = G/N_0$, and therefore $f(Y \cap D_{p,\Gamma}) = f(Y)$. Because $Y \cap D_{p,\Gamma}$ is left-invariant under N_0 it follows that $Y \subseteq D_{p,\Gamma}$, completing the proof of the claim. □

Because Y is dense in G and $D_{p,\Gamma}$ is closed we have $D_{p,\Gamma} = G$. This ends the proof of Proposition 5.3. □

In order to complete the proof of Theorem 5.2, consider now an arbitrary type $p \in S_G(\mathbb{R})$, with $A_p = gH_p$. By replacing p with $g^{-1}p$ and $D_{p,\Gamma}$ with $D_{g^{-1}p,\Gamma} = g^{-1}D_{p,\Gamma}$, we may assume that $A_p = H_p$. For every $\alpha \in p(\mathfrak{R})$ there is $\epsilon \in \mu(G)$ such that $\epsilon\alpha \in H_p^\sharp$. Since $\mathrm{st}(\epsilon\alpha) = \mathrm{st}(\alpha)$, replacing α with $\epsilon\alpha$ we may assume that $p \vdash H_p$, and thus $\mathrm{st}(p(\mathfrak{R})\Gamma) \subseteq \mathrm{cl}(H_p\Gamma) = H_p^\Gamma\Gamma$.

Let $G_0 = H_p^\Gamma$ and $\Gamma_0 = G_0 \cap \Gamma$, a lattice in G_0. Notice that $\mathrm{cl}(H_p\Gamma_0) = H_p^{\Gamma_0}\Gamma_0 = H_p^\Gamma = G_0$. Thus, in order to prove the theorem it is sufficient to show that $\mathrm{st}(p(\mathfrak{R})\Gamma_0^\sharp) = G_0$. This is exactly Proposition 5.3 (for G_0 and Γ_0 instead of G and Γ), so we are done. □

Returning to the setting of Theorem 1.3, we start with a given definable set $X \subseteq G$, and define the associated family of nearest cosets:

$$\mathcal{A}(X) = \{A_\alpha : \alpha \in X^\sharp\}.$$

By Lemma 3.9, the 0-dimensional elements of $\mathcal{A}(X)$ are exactly the singletons $\{g\}$ for $g \in G$.

For $\alpha \in X^\sharp$, let $A_\alpha = g_\alpha H_\alpha$, where g_α is any element in A_α. For every lattice $\Gamma \subseteq G$, we have

$$\mathrm{cl}(A_\alpha\Gamma) = \mathrm{cl}(g_\alpha H_\alpha\Gamma) = g_\alpha(H_\alpha)^\Gamma\Gamma.$$

We let A_α^Γ denote the coset $g_\alpha H_\alpha^\Gamma$. We can now describe the closure of $\pi_\Gamma(X)$ as follows:

Corollary 5.4. *For every lattice $\Gamma \subseteq G$,*

$$\mathrm{cl}(X\Gamma) = \bigcup_{\alpha \in X^\sharp} g_\alpha(H_\alpha)^\Gamma\Gamma = \bigcup_{\alpha \in X^\sharp} A_\alpha^\Gamma\Gamma,$$

and

$$\mathrm{cl}(\pi_\Gamma(X)) = \bigcup_{\alpha \in X^\sharp} \pi_\Gamma(g_\alpha H_\alpha^\Gamma) = \bigcup_{\alpha \in X^\sharp} \pi_\Gamma(A_\alpha^\Gamma).$$

Proof. As we saw,

$$\mathrm{cl}(X\Gamma) = \mathrm{st}(X^\sharp\Gamma^\sharp) = \bigcup_{p \vdash X} \mathrm{st}(p(\mathfrak{R})\Gamma^\sharp).$$

By Theorem 5.2, we have

$$\mathrm{cl}(X\Gamma) = \bigcup_{p \vdash X} (A_p^\Gamma)\Gamma.$$

Since $A_\alpha = A_\beta$ whenever α and β realize the same complete type, we can write the same union as $\bigcup_{\alpha \in X^\sharp} A_\alpha^\Gamma \Gamma$. The result follows. □

5.1. **Digression, the connection to the work of Leibman and Shah.** Our goal here is to deduce Theorem 1.5 from Corollary 5.4. Before doing that, we briefly discuss the connection between our notion of "a polynomial map" and that of [7].

Given G a connected, simply connected nilpotent Lie group, let $a_1, \ldots, a_n$ be some elements of G, and let $p : \mathbb{R}^d \to \mathbb{R}^n$ be a polynomial map, such that $p(\mathbb{Z}^d) \subseteq \mathbb{Z}^n$. Then the map $f : \mathbb{Z}^d \to G$, defined by

$$f(\bar{k}) = a_1^{p_1(\bar{k})} \cdots a_n^{p_n(\bar{k})}$$

is said to be a polynomial map in [7]. Note that this definition is invariant under an isomorphism of G thus we may assume that G is a real unipotent group. By Lemma 2.15 (2), there is a map $F\colon \mathbb{R}^d \to G$, polynomial in matrix coordinates, such that $f(\bar{k}) = F(\bar{k})$ for $\bar{k} \in \mathbb{Z}^d$.

We prove:

Theorem 5.5. *Let G be a real unipotent group. Assume that $f : \mathbb{R}^d \to G$ is a polynomial map in matrix coordinates and let $X = f(\mathbb{R}^d) \subseteq G$. Let gH be the minimal coset among all left cosets of real algebraic subgroups of G with $X \subseteq gH$. Then for every lattice $\Gamma \subseteq G$,*

$$\mathrm{cl}(\pi_\Gamma(X)) = \pi_\Gamma(gH^\Gamma).$$

Proof. Note first that for every $\alpha \in X^\sharp$, its nearest coset A_α is contained in gH. Thus, by Corollary 5.4, for every lattice Γ,

$$\mathrm{cl}(\pi_\Gamma(X)) = \bigcup_{\alpha \in X^\sharp} \pi_\Gamma(A_\alpha^\Gamma) \subseteq \pi_\Gamma(gH^\Gamma).$$

It is therefore sufficient to prove:

Lemma 5.6. *Under the above assumptions, there exists $\alpha \in X^\sharp$ such that $A_\alpha = gH$.*

Proof of Lemma. We use induction on $\dim G$, with $\dim G = 0$ being a trivial case. Since left translation by g^{-1} is a polynomial map from G to G, we may replace X by $g^{-1}X$ and assume that the minimal coset containing X is H.

If H is a proper subgroup of G then by induction there exists $\alpha \in X^\sharp$ such that $A_\alpha = H$. Thus, we may assume that $H = G$, and we wish to find $\alpha \in X^\sharp$ such that the nearest coset to α is G. We define α as follows:

We choose $\beta = (\beta_1, \dots, \beta_d) \in \mathfrak{R}^d$ with $0 << \beta_1 << \beta_2 << \cdots << \beta_d$. By that we mean $\beta_1 > \mathbb{R}$, and for every $i = 1, \dots, d-1$, and every polynomial $q(x_1, \dots, x_i) \in \mathbb{R}[x_1, \dots, x_i]$ we have $\beta_{i+1} > q(\beta_1, \dots, \beta_i)$. We can find such a tuple β because $\mathfrak{R}$ is $|\mathbb{R}|^+$-saturated. The following is easy to verify:

Claim 5.7. *If $q(x_1, \dots, x_d) \in \mathbb{R}[x_1, \dots, x_d]$ is a non-constant polynomial then $q(\beta) \notin \mathcal{O}(\mathfrak{R})$.*

We now claim that $\alpha = f(\beta)$ is the desired element. Towards that we prove the following general claim:

Claim 5.8. *For $\beta \in \mathfrak{R}^d$ and G as above, if $q : \mathbb{R}^d \to G$ is a polynomial map, and gH_0 is near $q(\beta)$, for some real algebraic $H_0 \subseteq G$ and $g \in G$, then $q(\beta) \in gH_0$.*

Before proving the claim let us see that it implies Lemma 5.6. Indeed, the above claim implies that when gH_0 is any coset near α then $\alpha \in gH_0$. We now consider the set $S = \{x \in \mathbb{R}^d : q(x) \in gH_0\}$. Since H_0 is a real algebraic group, the set S is also real algebraic, defined over $\mathbb{R}$. The transcendence degree of β over $\mathbb{R}$ is d, and since $\alpha \in H_0$ and $\beta \in S^\sharp$, we must have $S = \mathbb{R}^d$. It follows that $X \subseteq gH_0$, and therefore the nearest coset to α must contain X. By our assumptions, it follows that $A_\alpha = G$, thus ending the proof of Lemma 5.6, and with it the proof of Theorem 5.5.

Thus, we are left to prove Claim 5.8, and we do so by induction on the $\dim G$. We may assume that gH_0 equals A_α, and by replacing the map q with the polynomial map $g^{-1}q$, we may assume that the group $A_\alpha = H_0$. We want to show that $\alpha \in H_0$. Without loss of generality, H_0 is a proper subgroup of G, for otherwise we are done.

We may further assume that there is no proper algebraic subgroup $H_1 \subseteq G$ such that $q(\mathbb{R}^d) \subseteq H_1$ (for otherwise H_0 is also contained in H_1 and we may replace G with H_1 and finish by induction). Let N be a proper real algebraic normal subgroup of G containing H_0 and consider the map $\pi \circ q$, where $\pi : G \to G/N$ is the quotient map. By Lemma 2.15 (1), the map $\pi \circ q$ is still polynomial, and by our assumptions the trivial group $\{e\}$ is near $\pi \circ q(\beta)$, and in particular $q(\beta) \in \mathcal{O}(\mathfrak{R})$. By Claim 5.7, the map $\pi \circ q$ must be a constant map, which is necessarily e. It follows that $q(\mathbb{R}^d) \subseteq N$, contradicting our assumption. This ends the of Claim 5.8 and with it the proofs of Lemma 5.6 and Theorem 5.5. □

6. Neat families of cosets

The work here is similar to the work in [11, Sectin 7.1-7.2]. We assume that G is a real unipotent group.

Our first goal is to show that the family $\mathcal{A}(X)$ of all nearest cosets to elements in $X^\sharp$, is an $\mathbb{R}_{\text{om}}$-definable subfamily of the family of all cosets of real algebraic subgroups of G (see Fact 2.13). This is very similar to the work in [10]. We expand the structure $\mathfrak{R}_{\text{om}}$ by adding a predicate symbol for the set of reals $\mathbb{R}$. We are thus working in the structure $\mathfrak{R}_{pair} = \langle \mathfrak{R}_{\text{om}}, \mathbb{R}_{\text{om}} \rangle$, in which $\mathbb{R}_{\text{om}}$ is an elementary substructure of $\mathfrak{R}_{\text{om}}$. Such structures are called tame pairs of o-minimal structures and were studied in [3].

Note first that since the standard part map is definable in $\mathfrak{R}_{pair}$, the family $\mathcal{A}(X)$ is definable in $\mathfrak{R}_{pair}$. By [3, Proposition 8.1] we may conclude:

Lemma 6.1. *The family of cosets $\mathcal{A}(X)$ is definable in $\mathbb{R}_{\text{om}}$. Namely, there exists in $\mathbb{R}_{\text{om}}$ a definable set T and a formula $\phi(x, t)$, with x and t tuples of variables, such that*

$$\mathcal{A}(X) = \{\phi(G, t) : t \in T\}.$$

Our next goal is to replace $\mathcal{A}(X)$ by a family of cosets of finitely many subgroups.

Definition 6.2. Let $\mathcal{F} = \{g_t H_t : t \in T\}$ be an $\mathbb{R}_{\text{om}}$-definable family of cosets of real algebraic subgroups of G. We say that $\mathcal{F}$ is *neat* if the following hold:

(1) For $t_1 \neq t_2$, $g_{t_1} H_{t_1} \neq g_{t_2} H_{t_2}$.
(2) There exists k, such that T is a connected submanifold of $\mathbb{R}^k$.
(3) There exists a definable continuous function from T to G, $t \mapsto h_t \in G$, such that for every $t \in T$, $h_t H_t = g_t H_t$.
(4) For every nonempty open $U \subseteq T$,

$$\langle \bigcup_{t \in U} H_t \rangle_{alg} = \langle \bigcup_{t \in T} H_t \rangle_{alg}.$$

For $\mathcal{F}$ a neat family of of cosets as above, we denote by $H_{\mathcal{F}}$ the group $\langle \bigcup_{t \in T} H_t \rangle_{alg}$.

Lemma 6.3. *Let $\mathcal{F}$ be a neat family of algebraic subgroups of G. Then for every lattice $\Gamma \subseteq G$, the set $T_\Gamma = \{t \in T : H_t^\Gamma = (H_{\mathcal{F}})^\Gamma\}$ is dense in T.*

Proof. For a Γ-rational subgroup L of G, let

$$T(L) = \{t \in T : H_t \subseteq L\}.$$

Clearly, if $t \in T \setminus T_\Gamma$ then H_t^Γ is a proper subgroup of $(H_\mathcal{F})^\Gamma$, hence $T \setminus T_\Gamma$ can be written as a union of all sets $T(L)$, as L varies over all Γ-rational proper subgroups of $(H_\mathcal{F})^\Gamma$.

By Remark 2.6, there are countably many Γ-rational subgroups of G, thus the union is countable. So, in order to show that T_Γ is dense in T it is sufficient, by Baire Categoricity Theorem, to show that every $T(L)$ is nowhere dense. Since this is a definable set it is sufficient to prove that $T(L)$ does not contain any nonempty open subset of T. But, by definition of $H_\mathcal{F}$, for every $U \subseteq T$ nonempty open set, the group $\langle\bigcup_{t\in U} H_t\rangle_{alg}$ is the whole of $H_\mathcal{F}$, so $\langle\bigcup_{t\in U} H_t\rangle_{alg}^\Gamma = (H_\mathcal{F})^\Gamma$. On the other hand, for every $V \subseteq T(L)$, we have $\langle\bigcup_{t\in V} H_t\rangle_{alg}^\Gamma \subseteq L \neq (H_\mathcal{F})^\Gamma$, so no open nonempty subset of T is contained in $T(L)$. Therefore, T_Γ is indeed dense in T. ☐

Lemma 6.4. *Let $\{g_t H_t : t \in T\}$ be a definable family of pairwise distinct cosets of algebraic subgroups of $G \subseteq \mathrm{UT}(n,\mathbb{R})$. Then*

(1) there is a definable partition of $T = T_1 \cup \cdots \cup T_r$, such that for each $i = 1, \ldots, r$ the family $\{g_t H_t : t \in T_i\}$ is neat.

(2) For each $i = 1, \ldots, r$, let

$$L_i = \langle \bigcup_{t\in T_i} H_t\rangle_{alg}.$$

Then for every lattice $\Gamma \subseteq G$,

$$\mathrm{cl}(\bigcup_{t\in T_i} g_t H_t^\Gamma) = \mathrm{cl}(\bigcup_{t\in T_i} g_t L_i^\Gamma).$$

Proof. (1) We use induction on $\dim T$. By o-minimality, we may assume that T is a connected submanifold of some $\mathbb{R}^k$ and that the function $t \mapsto g_t$ is continuous on T. Given $t \in T$, it follows from DCC for real algebraic subgroups that there exists a subgroup $G_t \subseteq G$ such that for all sufficiently small open $t \in U \subseteq T$, $\langle\bigcup_{t\in U} H_t\rangle_{alg} = G_t$.

Because the family of all real algebraic subgroups of G is definable the family $\{G_t : t \in T\}$ is also definable, thus we may divide T into finitely many definable submanifolds, $T_1, \ldots, T_m$, on each of which $\dim G_t$ is constant. By induction, it is sufficient to handle those T_i whose dimension equals that of T. Notice that for such a T_i, and $t \in T_i$, it is still the case that for all sufficiently small open $U \subseteq T_i$, a neighborhood of t, we have

$$G_t = \langle \bigcup_{t\in U} H_t\rangle_{alg}$$

(this might not be the case for those T_i's with $\dim T_i < \dim T$).

Thus, without loss of generality, $\dim G_t$ is constant as t varies in T. We claim that now the group G_t is the same for all $t \in T$ (and hence $\{g_tH_t : t \in T\}$ is a neat family). Indeed, fix $t_0 \in T$ and let

$$T_0 = \{t \in T; G_t = G_{t_0}\}.$$

The set T_0 is closed in T: Let $t_1 \in \text{cl}(T_0)$ and fix $U \ni t_1$ such that $G_{t_1} = \langle \bigcup_{t\in U} H_t\rangle_{alg}$. For every $t \in U \cap T_0$, we have $G_t = G_{t_0} \subseteq G_{t_1}$, but since $\dim G_t$ is constant in T we must have $G_{t_1} = G_{t_0}$, so $t_1 \in T_0$.

Let us see that T_0 is also open in T. For $t_2 \in T_0$ let $t_2 \in U \subseteq T$ be an open set such that $G_{t_2} = G_{t_0} = \langle \bigcup_{t\in U} H_t\rangle_{alg}$. By dimension considerations, for all $t \in U$, $G_t = G_{t_0}$, so $U \subseteq T_0$, and thus T_0 is open.

Because T is connected, $T_0 = T$. It follows that for every open nonempty sets $U \subseteq T$

$$\langle \bigcup_{t\in U} H_t\rangle_{alg} = \langle \bigcup_{t\in T} H_t\rangle_{alg}.$$

(2) Fix $i = 1, \ldots, r$ so the family $\{g_tH_t : t \in T_i\}$ is neat. First note that for $t \in T_i$, each $g_tH_t^\Gamma$ is contained in $g_tL_i^\Gamma$, so it is sufficient to show that $\bigcup_{t\in T} g_tH_t^\Gamma$ is dense in $\bigcup_{t\in T} g_tL_i^\Gamma$.

By Lemma 6.3, the set $T_0 = \{t \in T : H_t^\Gamma = L_i^\Gamma\}$ is dense in T_i. Let $g_{t_0}h_0$ be an arbitrary element of $g_{t_0}L_i^\Gamma$, for some $t_0 \in T_i$, and choose $t_n \in T_0$ a sequence converging to t_0. For each t_n we have $g_{t_n}h_0 \in g_{t_n}L_i^\Gamma = g_{t_n}H_{t_n}^\Gamma$. Because the map $t \mapsto g_t$ is continuous, $g_{t_n}h_0$ tends to $g_{t_0}h_0$, so indeed the union of $g_tH_t^\Gamma$ is dense in the union of $g_tL_i^\Gamma$. □

7. The main theorem

We are now ready to prove Theorem 1.3. We find it convenient to reformulate the result within G and not in G/Γ. The equivalence of the theorem below to Theorem 1.3 follows from the definition of the quotient topology on G/Γ. Namely, for every $X \subseteq G$, $\pi_\Gamma(X)$ is closed in G/Γ if and only if $X\Gamma$ is closed in G.

All definability below is taken in the o-minimal structure $\mathbb{R}_{\text{om}}$.

Theorem 7.1. *Let G be a real unipotent group and let $X \subseteq G$ be a definable set. Then there are finitely many definable real algebraic subgroups $L_1, \ldots, L_m \subseteq G$ of positive dimension, and finitely many definable closed sets $C_1, \ldots, C_m \subseteq G$, such that for every lattice $\Gamma \subseteq G$,*

$$\text{cl}(X\Gamma) = (\text{cl}(X) \cup \bigcup_{i=1}^{m} C_iL_i^\Gamma)\Gamma.$$

In addition, the C_i's can be chosen to satisfy:

(1) For every $i = 1, \ldots, m$, $\dim(C_i) < \dim X$.

(2) Let L_i be a maximal subgroup with respect to inclusion, among $L_1, \dots, L_m$. Then C_i is a bounded set in G, and in particular $C_i L_i^\Gamma \Gamma$ is closed in G.

Proof. Recall that for a coset $A = gH \subseteq G$, and a lattice Γ, we write A^Γ for gH^Γ. In particular, $\mathrm{cl}(A\Gamma) = A^\Gamma\Gamma$.

By Corollary 5.4,

$$\mathrm{cl}(X\Gamma) = \mathrm{st}(X^\sharp\Gamma^\sharp) = \bigcup_{A \in \mathcal{A}(X)} A^\Gamma\Gamma.$$

By Lemma 6.1, the family of cosets $\mathcal{A}(X)$ is definable in $\mathbb{R}_{\mathrm{om}}$. By Definable Choice, we may assume that the cosets in $\mathcal{A}(X)$ are pairwise distinct. As we already pointed out, the zero-dimensional cosets in this family are exactly the singletons of elements of X. Thus we restrict our attention to those cosets which have positive dimension and denote this definable sub-family by $\mathcal{A}(X)'$.

By Lemma 6.4, we can divide $\mathcal{A}(X)'$ into finitely many neat families of cosets, $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_m$. For each $i = 1, \dots, m$, the family $\mathcal{A}_i = \{g_t H_t : t \in T_i\}$ has an associated fixed group $L_i = \langle \bigcup_{t \in T_i} H_t \rangle_{alg}$. By Lemma 6.4, for every lattice $\Gamma \subseteq G$ and for each $i = 1, \dots, m$, we have

$$\bigcup_{A \in \mathcal{A}_i} \mathrm{cl}(A\Gamma) = \bigcup_{t \in T_i} \mathrm{cl}(g_t H_t^\Gamma)\Gamma = \bigcup_{t \in T_i} g_t L_i^\Gamma \Gamma.$$

For each $i = 1, \dots, m$ we consider the group L_i. By Lemma 2.11, for each $i = 1, \dots, m$, there exists a closed semi-algebraic "complement" $A_i \subseteq G$, to the group L_i. Namely, the map $(a, h) \to ah$ is a diffeomorphism of $A_i \times L_i$ and G. We let $(a_i, h_i) : G \to A_i \times L_i$ be its inverse map, so for every $g \in G$ we have $g = a_i(g)h_i(g)$. Notice that the map a_i is constant on left cosets of L_i.

Since the map $a_i \colon G \to A_i$ is continuous, we may replace the map $t \mapsto g_t$ on T_i by the continuous map $t \mapsto a_i(g_t)$ and thus assume, for each $I = 1, \dots, m$, that g_t takes value in A_i. By our choice of $\mathcal{A}(X)'$, it is also injective. We let $C_i = \mathrm{cl}(\{g_t : t \in T_i\})$ (there is no harm in taking closure since we are describing closed set $\mathrm{cl}(X\Gamma)$). So, $C_i \subseteq A_i$.

Thus,

$$\mathrm{cl}(X\Gamma) = \mathrm{cl}(X)\Gamma \cup \bigcup_{i=1}^m \bigcup_{A \in \mathcal{A}_i} \mathrm{cl}(A\Gamma) = (X \cup \bigcup_{i=1}^m C_i L_i^\Gamma)\Gamma.$$

This ends the proof of the main result.

Let us see that our sets C_i satisfy (1) and (2). It is sufficient to prove both for $C_i' = \{g_t : t \in T_i\}$ instead of $C_i = \mathrm{cl}(C_i')$. Indeed, by o-minimality $\dim C_i' = \dim C_i$ and clearly C_i is bounded if and only if C_i' is.

(1)We need to show that $\dim C_i' < \dim X$. By our choice of T_i and C_i', for each $g \in C_i'$ there exists $\alpha \in G^\sharp \setminus \mathcal{O}(G)$ such that $A_\alpha \subseteq gL_i$. In particular, the coset gL_i is near α.

Recall that G is a closed subset of $\mathbb{R}^{n^2}$ and $\mathcal{O}(G)$ is the collection of all elements of G which are $\mathbb{R}$-bounded. Given $g \in G$ we let $|g|$ be its Euclidean norm as an element of $\mathbb{R}^m$. As we noted in Section 2.3.1, for $\alpha \in G^\sharp$, $\alpha \in \mathcal{O}(G)$ if and only if $|\alpha| \in \mathcal{O}(\mathfrak{R})$.

We define

$$X_i = \{(a_i(x), 1/|h_i(x)|) \in A_i \times \mathbb{R} : x \in X\}.$$

The set X_i is definable and there is clearly a definable surjection from X onto X_i, thus $\dim X \geq \dim X_i$.

Claim If $g \in C_i'$ then $(g, 0)$ is in $Fr(X_i) = \mathrm{cl}(X_i) \setminus X_i$.

Proof of Claim. Clearly, $(g, 0) \notin X_i$, so we need to see that it belongs to $\mathrm{cl}(X_i)$.

First note that since the map $(a_i, h) : G \to A_i \times L_i$ is a semialgebraic homeomorphism over $\mathbb{R}$, it sends $\mathcal{O}(G)$ onto $(\mathcal{O}(G) \cap A_i^\sharp) \times (\mathcal{O}(G) \cap L_i^\sharp)$. Next, as we noted above, there exists $\alpha \in X^\sharp \setminus \mathcal{O}(G)$ such that the coset gL_i is near α.

So, there exists $\epsilon \in \mu(G)$ such that $\alpha \in \epsilon g L_i^\sharp$. Since α and ϵg are in the same left coset of $L_i^\sharp$, we have $a_i(\epsilon g) = a_i(\alpha)$. Because $a_i(-)$ is a continuous map, and a_i is the identity on A_i, we have

$$\mathrm{st}(a_i(\epsilon g)) = a_i(g) = g,$$

and in particular, $a_i(\alpha) \in \mathcal{O}(G)$ and $\mathrm{st}(a_i(\alpha)) = g$.

We have $\alpha = a_i(\alpha)h_i(\alpha)$, and since $\alpha \notin \mathcal{O}(G)$ and $a_i(\alpha) \in \mathcal{O}(G)$, then $h_i(\alpha) \notin \mathcal{O}(G)$, so $|h_i(\alpha)| \notin \mathcal{O}(\mathfrak{R})$, hence $\mathrm{st}(1/|h_i(\alpha)|) = 0$. Thus, $(g, 0) = (\mathrm{st}(a_i(\alpha)), \mathrm{st}(1/|h_i(\alpha)|))$ is in $\mathrm{st}(X_i^\sharp)$, which by Fact 2.16, equals $\mathrm{cl}(X_i)$. □

By o-minimality, $\dim Fr(X_i) < \dim X_i \leq \dim X$, so it follows from our Claim that $\dim C_i' < \dim X$.

(2) We may assume that the groups $L_1, \ldots, L_r$ are maximal with respect to inclusion among $L_1, \ldots, L_m$ (note that we allow repetitions among the L_i's). We first prove:

Claim 7.2. *There is a definable closed bounded set $B \subseteq G$ such that*

$$X \subset BL_1 \cup \cdots \cup BL_r.$$

Proof of Claim. Our construction implies that for every $\alpha \in X^\sharp \setminus \mathcal{O}(G)$, if $A_\alpha = g_\alpha H_\alpha$ then there exists $i \in \{1, \dots, m\}$ and $g \in C_i'$ with $A_\alpha \subseteq gL_i$, hence $\alpha \in \mathcal{O}(G)L_i^\sharp$. Each L_i is contained in some L_j, with $1 \le j \le r$, and hence

$$X^\sharp \subseteq \mathcal{O}(G) \cup \bigcup_{i=1}^{r} \mathcal{O}(G)L_i^\sharp.$$

Writing $\mathcal{O}(G)$ as a countable union of definable closed bounded sets and using the Compactness Theorem (in Logic) we obtain that there is a definable closed bounded set $B \subseteq G$ with

$$X \subseteq B \cup \bigcup_{i=1}^{r} BL_i.$$

If X is bounded then $r = m = 0$ and then $X \subseteq B$ for some B. Otherwise, $B \subseteq BL_i$ for every i, and hence

$$X \subseteq \bigcup_{i=1}^{r} BL_i.$$

This proves Claim 7.2. □

We fix a set B as in Claim 7.2.

Claim 7.3. *For every $\alpha \in X^\sharp$ there is $b \in B$ and $i \in \{1, \dots, r\}$ such that $A_\alpha \subseteq bL_i$, and in particular, $H_\alpha \subset L_i$*

Proof of Claim. Let $\alpha \in X^\sharp$. It follows from Claim 7.2 that there is $b \in B$ and $i \in \{1, \dots, r\}$ such that α is near the coset bL_i. (If $\alpha \in B^\sharp$, then α is near the coset bL_1, where $b = \operatorname{st}(\alpha) \in B$). This proves Claim 7.3. □

We now proceed with the proof of (2) and fix a maximal L_i. Without loss of generality, $i = 1$.

We need to show that C_1' is bounded. So assume towards getting a contradiction that C_1' is unbounded.

It is not hard to see that there is a bounded closed definable set $B_1 \subseteq A_1$ (recall A_1 is the complement of L_1) such that $B \subseteq B_1L_1$, hence $BL_1 \subseteq B_1L_1$. Because C_1' is unbounded subset of A_1, we have $C_1' \not\subseteq B_1$.

Thus, by our choice of C_1' and L_1, there is a neat family $\mathcal{F} = \{g_tH_t \colon t \in T_1\}$ (with g_t taking values in A_1), such that: (i) $H_{\mathcal{F}} = L_1$, (ii) for every $t \in T_1$ there is $\alpha \in X^\sharp$ with $A_\alpha = g_tH_t$ and (iii) for some $t_0 \in T_1$, $g_{t_0} \notin B_1$.

By the continuity of g_t, there exists an open $U \subseteq T_1$ containing t_0 such that for all $t \in U$, $g_t \notin B_1$. It follows that for all $t \in U$, $g_tL_1 \not\subseteq$

B_1L_1 (here we use the fact that A_1 contains a single representative for each left coset of L_1), and since $BL_1 \subseteq B_1L_1$, we also have $g_tL_1 \nsubseteq BL_1$.

By Claim 7.3, the set U is covered by definable sets S_i, $i = 1, \ldots, m$, where $S_i = \{t \in U : g_tH_t \subseteq BL_i\}$. However, by what we just showed, $U \cap S_1 = \emptyset$, so we have

$$U \subseteq \bigcup_{L_i \neq L_1} S_i.$$

It follows from o-minimality that there exists i_0, with $L_{i_0} \neq L_1$, such that S_{i_0} contains nonempty open set $U_{i_0} \subseteq U$. Thus, $U_{i_0} \subseteq T_1 \cap S_{i_0}$, so for every $t \in U_{i_0}$, H_t is contained in $L_1 \cap L_{i_0}$. By the maximality of L_1, and since $L_1 \neq L_{i_0}$, the group $L_1 \cap L_{i_0}$ is a proper subgroup of L_1. Hence

$$\langle \bigcup_{t \in U_{i_0}} H_t \rangle_{alg}$$

is a proper subgroup of L_1, contradicting the neatness of the family $\mathcal{F}$. Thus C_1' and therefore C_1 is bounded.

This ends the proof of the clause (2) and Theorem 7.1. □

8. On uniform distribution

In this section we make some observations related to a uniform distribution. Similar observations were made by A. Wilkie in [18].

As above we work on an o-minimal expansion $\mathbb{R}_{\mathrm{om}}$ of the real field $\mathbb{R}$. All definability is taken in $\mathbb{R}_{\mathrm{om}}$.

We consider only the abelian case $G = (\mathbb{R}^n, +)$ and we fix the lattice $\Gamma = \mathbb{Z}^n$.

Let $\mathbf{T}_n = \mathbb{R}^n/\mathbb{Z}^n$ and $\pi \colon \mathbb{R}^n \to \mathbf{T}_n$ be the projection. We will denote by μ_n the normalized Haar measure on $\mathbf{T}^n$.

By *a definable curve in* $\mathbb{R}^n$ we mean the image of a definable continuous map $\gamma(t) \colon \mathbb{R}^{\geq 0} \to \mathbb{R}^n$. Slightly abusing notations, if $\gamma(t) \colon \mathbb{R}^{\geq 0} \to \mathbb{R}^n$ is a definable continuous map then we use γ to denote its image $\gamma \subseteq \mathbb{R}^n$, i.e. the corresponding curve, and call the map $\gamma(t)$ *a definable parametrization of* γ.

We say that a definable curve γ is *bounded* if it is contained in a compact subset of $\mathbb{R}^n$.

Remark 8.1. If $\gamma(t) \colon \mathbb{R}^{\geq 0} \to \mathbb{R}^n$ is a definable map then, by o-minimality, the function $t \mapsto \|\gamma(t)\|$ is eventually monotone. In particular, if γ is unbounded then there is $T > 0$ such that $\gamma(t)$ is injective on $[T, \infty)$.

Let $\gamma \subseteq \mathbb{R}^n$ be a curve and $R > 0$. We let $\gamma_R = \gamma \cap B^n(R, 0)$, where $B^n(R, 0)$ is the closed ball in $\mathbb{R}^n$ of radius R centered at the origin. We define $\mu_{\gamma,R}$ to be the probability measure on $\mathbf{T}_n$ that is the pushforward of the probability measure on $\mathbb{R}^n$ obtained by averaging over γ_R. More precisely, for a set $X \subseteq \mathbf{T}_n$ we define $\mu_{\gamma,R}(X)$ to be the ratio

$$\mu_{\gamma,R}(X) = \frac{\text{the arc length of } \gamma_R \cap \pi^{-1}(X)}{\text{the arc length of } \gamma_R}.$$

Since, by o-minimality, γ is a finite unions of smooth sub-manifolds of $\mathbb{R}^n$, for any Borel subset $X \subseteq \mathbf{T}_n$, $\mu_{\gamma,R}(X)$ is well defined, and $\mu_{\gamma,R}$ is a Borel probability measure on $\mathbf{T}_n$.

Definition 8.2. We say that a definable curve $\gamma \subseteq \mathbb{R}^n$ is *continuously uniformly distributed mod $\mathbb{Z}^n$ (c.u.d. mod $\mathbb{Z}^n$ for short)* if the family $\mu_{\gamma,R}$ weakly converges, as R goes to infinity, to the normalized Haar measure μ_n on $\mathbf{T}_n$. In other words γ is c.u.d. mod $\mathbb{Z}^n$ if for any continuous function $f\colon \mathbf{T}_n \to \mathbb{R}$ we have

$$\lim_{R\to\infty} \int_{\mathbf{T}_n} f d\mu_{\gamma,R} = \int_{\mathbf{T}_n} f d\mu_n.$$

Recall that the structure $\mathbb{R}_{\text{om}}$ on the real field is called *polynomially bounded* if for any definable function $f(t)\colon \mathbb{R}^{\geq 0} \to \mathbb{R}$ there is $n \in \mathbb{N}$ and $C > 0$ with $f(t)\mathfrak{g} \leq Ct^n$ for all sufficiently large t.

We have the following.

Theorem 8.3. *Assume the structure $\mathbb{R}_{\text{om}}$ is polynomially bounded. The following conditions are equivalent.*

(1) $\pi(\gamma)$ is dense in $\mathbf{T}_n$.
(2) γ is c.u.d. mod $\mathbb{Z}^n$.

8.1. **Proof of Theorem 8.3.** In this section we prove Theorem 8.3.

Obviously (2) $\Longrightarrow$ (1) for any definable curve γ. Thus we need to consider only the direction (1) $\Longrightarrow$ (2).

We are going to use the following version of Weyl's criterion that follows from the density of trigonometric polynomials in the space of continuous $\mathbb{Z}^n$-invariant functions on $\mathbb{R}^n$.

Fact 8.4. *(Weyl's criterion) Let $\mu_R, R \geq 0$, be a family of probability measures on $\mathbf{T}_n$. Then μ_R converges weakly to a probability measure μ if and only if*

$$\lim_{R\to\infty} \int_{\mathbf{T}_n} \chi d\mu_R = \int_{\mathbf{T}_n} \chi d\mu,$$

for every non-trivial multiplicative character (i.e. a non-trivial continuous group homomorphism) $\chi\colon \mathbf{T}^n \to \mathbb{C}^*$.

Since $\int_{\mathbf{T}_n} \chi d\mu_n = 0$ for every non-trivial multiplicative character $\chi\colon \mathbf{T}_n \to \mathbb{C}^*$ we have that a definable curve γ is c.u.d. mod $\mathbb{Z}^n$ if and only if $\lim_{R\to\infty} \int_{\mathbf{T}_n} \chi d\mu_{\gamma,R} = 0$ for for every non-trivial multiplicative character $\chi\colon \mathbf{T}_n \to \mathbb{C}^*$.

We fix a definable curve $\gamma \subseteq \mathbb{R}^n$. It is easy to see that (1) fails for any bounded definable curve γ, hence we may assume that γ is unbounded.

Also it is not hard to see that both (1) and (2) of the theorem do not depend on an initial segment of γ, i.e. if for two definable curve γ_1 and γ_2 the difference $\gamma_1 \Delta \gamma_2$ is contained in a bounded set of $\mathbb{R}^n$ then (1) holds for γ_1 if and only if it holds for γ_2 and the same is true for the clause (2). Thus, using Remark 8.1, we may assume that for any $R \geq 0$ there is a unique point P_R on γ whose distance to the origin is R, and we have a definable parametrization $\gamma(t)$ of γ with $\|\gamma(t)\| = t$ for all $t \geq 0$. Also, using o-minimality, we may assume that the parametrization $\gamma(t)$ is smooth. With this parametrization, for a continuous function $f\colon \mathbf{T}_n \to \mathbb{R}$ we have

$$\int_{\mathbf{T}_n} f d\mu_{\gamma,R} = \frac{\int_0^R (f\circ\pi\circ\gamma)(t)\|\dot\gamma(t)\|dt}{\int_0^R \|\dot\gamma(t)\|\|dt}.$$

Since $\|\gamma(t)\| = t$, using o-minimality, we have that $\lim_{R\to\infty} \|\dot\gamma(t)\| = 1$. Thus the definable curve γ is c.u.d. mod $\mathbb{Z}^n$ if and only if

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R (\chi\circ\pi\circ\gamma)(t)dt = 0 \tag{8.1}$$

for any non-trivial multiplicative character $\chi\colon \mathbf{T}^n \to \mathbb{C}^*$.

For every multiplicative character $\chi\colon \mathbf{T}^n \to \mathbb{C}^*$ there is $l \in \mathbb{Z}^n$ with $\chi\circ\pi(x) = e^{2\pi i(l\cdot x)}$, where $(\ \cdot\ )$ is the standard dot product on $\mathbb{R}^n$; and vice versa, for every $l \in \mathbb{Z}^n$ we have $e^{2\pi i(l\cdot x)} = \chi\circ\pi(x)$ for some multiplicative character χ. Hence the curve γ is c.u.d. mod $\mathbb{Z}^n$ if and only if

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R e^{2\pi i(l\cdot\gamma(t))}dt = 0 \tag{8.2}$$

for every nonzero $l \in \mathbb{Z}^n$.

Theorem 8.3 will follow from the following proposition.

Proposition 8.5. *Assume $\mathcal{R}$ is polynomially bounded. If $f(t)\colon \mathbb{R}^{\geq 0} \to \mathbb{R}$ is a smooth definable unbounded function with bounded first derivative then*

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R e^{if(t)}dt = 0.$$

Proof of Proposition. Notice it is sufficient to consider only the tail of f. Changing f to $-f$, if needed, and considering the tail of f we may assume that $f(t)$ is strictly increasing and in particular $f'(t) > 0$.

For $R > 0$ let

$$I_R = \frac{1}{R}\left|\int_0^R e^{if(t)}\right| dt.$$

Using substitution $t = Ru$, we obtain

$$I_R = \frac{1}{R}\left|\int_0^1 e^{if(Ru)} R du\right| = \left|\int_0^1 e^{if(Ru)} du\right|.$$

Thus we need to show that $\lim_{R\to\infty}\left|\int_0^1 e^{if(Ru)} du\right| = 0.$

We use van der Corput lemma (see [14, Proposition VII.1.2]).

Lemma 8.6 (van der Corput)**.** *Suppose φ is real valued smooth on $(0,1)$ with $\varphi'(x)$ monotonic and $|\varphi'(x)| \geq 1$ on $(0,1)$. Then for any $\lambda \in \mathbb{R}$ we have*

$$\left|\int_0^1 e^{i\lambda\varphi(x)} dx\right| \leq c\lambda^{-1}.$$

for some absolute constant c.

Case 1: $\lim_{t\to\infty} f'(t) \neq 0$. In this case there is $k > 0$ such that $f'(t) \geq k > 0$ for all t.

Notice that the derivative of the function $f(Ru)$ on $(0,1)$ is bounded from below by $\lambda = kR$. For $\varphi(x) = \frac{1}{\lambda} f(Ru)$ we can apply van der Corput Lemma and conclude that $I_R \leq c(kR)^{-1}$. Proposition follows.

Case 2: $\lim_{t\to\infty} f'(t) = 0$.

Since $f'(t)$ is positive, considering a tail of f if needed we may assume that f' is decreasing.

Since $\mathbb{R}_{\text{om}}$ is polynomially bounded, $f'(t) \geq \alpha t^{-\varepsilon}$ for some $\alpha > 0$ and $\varepsilon > 0$. Since $f(t)$ is unbounded, there is alo such ε with $\varepsilon < 1$.

Since f' is decreasing we have $|f'(t)| \geq \alpha R^{-\varepsilon}$ on $(0,R)$. Notice that the derivative of the function $f(Ru)$ on $(0,1)$ is bounded from below by $\lambda = \alpha R^{1-\varepsilon}$. For $\varphi(x) = \frac{1}{\lambda} f(Rx)$ we can apply van der Corput Lemma and conclude that

$$I_R \leq c\lambda^{-1} = c\alpha^{-1} R^{\varepsilon - 1}.$$

Since $\varepsilon < 1$, $\lim_{R\to\infty} I_R = 0$. It proves the proposition. □

To derive the direction (1) $\Longrightarrow$ (2) in Theorem 8.3, first notice that the curve γ has a unique unbounded type p (the type of $\gamma(t)$ at $t = +\infty$). Let $A_p = g_p + H_p$ be the nearest coset of p. By Theorem 5.2, $\pi(\gamma)$ is dense in $\mathbf{T}_n$ if and only if $\mathrm{cl}(H_p + \mathbb{Z}^n) = \mathbb{R}^n$. Thus if (1) holds then p is not near any rational $\mathbb{R}$-linear subspace of of $\mathbb{R}^n$, equivalently $(l{\cdot}\gamma(t))$ is unbounded for every nonzero $l \in \mathbb{Z}^n$.

For nonzero $l \in \mathbb{Z}^n$, we apply Proposition 8.5 to the function $f(t) = (l{\cdot}\gamma(t))$ (by our assumption on $\|\dot\gamma(t)\|$ the derivative of f is bounded) and conclude that γ is c.u.d. mod $\mathbb{Z}^n$. This finishes the proof of Theorem 8.3. □

Notice that the assumption of polynomial boundedness can not be omitted. Indeed, working in the structure $\mathbb{R}_{\exp}$ let γ be the curve $\{(t, \ln(t)) \colon t \geq 1\}$. It is not hard to see, by direct computations, that in this case for the multiplicative character $\chi \colon \mathbf{T}_2 \to \mathbb{C}^*$ induced by the map $x \mapsto (l{\cdot}x)$ with $l = (0,1) \in \mathbb{Z}^2$, the limit $\lim_{R\to\infty} \int_{\mathbf{T}_2} \chi d\mu_{\gamma,R}$ does not exists. In particular, the family $\mu_{\gamma,R}$ does not weakly converge, as R goes to infinity, so the curve γ is not continuously uniformly distributed mod $\mathbb{Z}^2$.

Note however, that the nearest coset to the type of $(\alpha, \ln(\alpha))$, for $\alpha >> 0$, is the whole $\mathbb{R}^2$. Thus, for every lattice $\Gamma \subseteq \mathbb{R}^2$, $\pi_\Gamma(\gamma)$ is dense in $\mathbb{R}^2/\Gamma$.

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