COMPLEX ANALYTIC GEOMETRY IN A NONSTANDARD SETTING

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Abstract. Given an arbitrary o-minimal expansion of a real closed field **R**, we develop the basic theory of definable manifolds and definable analytic sets, with respect to the algebraic closure of **R**, along the lines of classical complex analytic geometry. Because of the o-minimality assumption, we obtain strong theorems on removal of singularities and strong finiteness results in both the classical and the nonstandard settings.

We also use a theorem of Bianconi to characterize all complex analytic sets definable in $\mathbb{R}_{exp}$.

1. Introduction

Let **R** be a real closed field and **K** its algebraic closure, identified with $\mathbf{R}^2$ (after fixing a square-root of -1). In [13] we investigated the notion of a **K**-holomorphic function from (subsets of) **K** into **K** which are definable in o-minimal expansions of **R**. Examples of such functions are abundant, especially in the case when **R** is the field of real numbers and **K** is the complex field. In [14] we extended this investigation to functions of several variables and began examining the notions of a **K**-manifold and **K**-analytic set, modeled after the classical notions. Here we return to this last question, while modifying slightly the definitions of a **K**-manifold and a **K**-analytic set from [14].

The theorems in this paper are of different kinds. First, we give a rigorous treatment of the theory of complex analytic geometry in this nonstandard setting, along the lines of the classical theory. From this point of view the paper can be read as a basic textbook in complex analytic geometry, written from the point of view of a model theorist (note that since every germ of a holomorphic function is definable in the o-minimal structure $\mathbb{R}_{an}$, the results proved here cover parts of classical complex geometry as well). As is often the case, the loss of local compactness of the underlying fields **R** and **K**, which might be nonarchimedean, is compensated by o-minimality.

However, we do more than just recover analogues of the classical theory. In some cases o-minimality yields stronger theorems than the classical ones, even when the underlying field is that of the complex numbers. Indeed, in [16] we showed how, working over the real and complex fields, o-minimality implies strong closure

The first author thanks the Logic group at University of Illinois Urbana-Champaign for its warm hospitality during 2003-2004. The second author was partially supported by the NSF. Both thank the Newton Mathematical Institute for its hospitality during Spring 2005. Both authors thank the anonymous referee for a thorough reading of the paper.

theorems for locally definable complex analytic sets. The same theorems hold in the nonstandard settings as well.

We also prove here several finiteness results which were not treated in [16]. For example, it follows from our results that any definable locally analytic subset A of a definable complex manifold can be covered by finitely many definable open sets, on each of which A is the zero set of finitely many definable holomorphic functions (see Theorem 4.14). Similarly, we formulate and prove a finite version of the classical Coherence Theorem (see Theorem 11.1). Here again, one replaces compactness assumptions on the underlying manifolds with definability in an o-minimal structure.

The main topological tool for most of the theorems is a general result (see Theorem 2.14), interesting on its own right, which allows us to move from an arbitrary $2d$-dimensional definable set in $\mathbf{K}^n$ to a set whose projection on the first d $\mathbf{K}$-coordinates is "definably proper" over its image.

In the appendix to the paper we use a theorem of Bianconi [3] to characterize all definable locally analytic subsets of $\mathbb{C}^n$ in the structure $\mathbb{R}_{exp} = \langle \mathbb{R}, <, +, \cdot, e^x \rangle$. We also observe there that, given a holomorphic function f of n variables, definable in some o-minimal expansion of the real field, its real an imaginary parts can be extended to holomorphic functions (of $2n$ variables) which are definable in the same structure.

Although the theorems in [16] were formulated in the context of the real and complex fields, most of the proofs there were written with the nonstandard setting in mind and hence carry over almost verbatim to our setting. We therefore refer at times to [16] for proofs of theorems in our paper. Also, we let ourselves refer at times to proofs from Whitney's book [19], when we found that there was no advantage in copying them into this paper. This book, as well as Chirka's book [6] were of great help to us when we came to learn the basics of complex analytic geometry. For a reference on o-minimal structures we suggest van den Dries' book [7].

The structure of the paper is as follows: In Section 2 we consider the analogous notion in our setting to local compactness and proper maps and prove the result about a certain finite covering of definable locally closed sets. In Section 3 we discuss $\mathbf{K}$-manifolds and submanifolds. In section 4 we define $\mathbf{K}$-analytic subsets of $\mathbf{K}$-manifolds and establish their basic properties. In Section 5 we prove a strong version of Chow's Theorem. In Section 6 we show that the set of singular points of a $\mathbf{K}$- analytic set is $\mathbf{K}$-analytic itself. In Section 7 we prove a strong version of the Remmert Proper Mapping Theorem. In Section 8 we discuss the relationship to model theory and show, that just like Zil'ber's result in the classical case, every definably compact $\mathbf{K}$-manifold, equipped with all $\mathbf{K}$-analytic subsets of its cartesian products, is a structure of finite Morley Rank. In Section 9 we discuss $\mathbf{K}$-meromorphic maps. In Section 10 we formulate (and refer to a proof of) the analogue of the Campana-Fujiki Theorem in our nonstandard setting. In Section 11 we formulate and prove our finite version to the Coherence Theorem. Finally, in the

Appendix we prove the result about definable complex analytic sets in the structure $\mathbb{R}_{exp}$.

Throughout the paper we work in a fixed o-minimal expansion of a real closed field $\mathbf{R}$. *We use the term "definable" sets to mean definable in this fixed o-minimal structure, possibly with parameters.*

2. Topological preliminaries

2.1. **"Real" and "complex" dimensions.** As in complex analysis, we will sometimes prefer to view definable subsets of $\mathbf{K}^n$ as subsets of $\mathbf{R}^{2n}$. As such, every definable set $A \subseteq \mathbf{K}^n$ has its o-minimal dimension, which we denote by $\dim_{\mathbf{R}} A$. $\mathbf{K}$-analytic sets and $\mathbf{K}$-manifolds will also be associated a dimension with respect to $\mathbf{K}$, which we will denote by $\dim_{\mathbf{K}} A$. We say "L is a d-dimensional $\mathbf{K}$-linear subspace of $\mathbf{K}^n$" when the dimension of L, as a $\mathbf{K}$-vector space, equals d (i.e., $\dim_{\mathbf{K}} L = d$). When both make sense, it is immediate to see that $\dim_{\mathbf{R}} A = 2 \dim_{\mathbf{K}} A$.

2.2. **Locally closed sets and definably proper maps.**

Definition 2.1. Recall that *a definable* C^0 $\mathbf{R}$*–manifold of dimension* n*, with respect to* $\mathbf{R}$ is a set X, covered by finitely many nonempty sets $U_1, \ldots, U_k$, and for each $i = 1, \ldots, k$ there is a set-theoretic bijection $\phi_i : U_i \to V_i$, where V_i is definable and open in $\mathbf{R}^n$ and such that each $\phi_i(U_j \cap U_j)$ is definable an open and the transition maps are definable and continuous. Moreover, the topology induced on X by this covering is Hausdorff.

We call such a manifold *a definable* C^p $\mathbf{R}$*-manifold* if in addition the transition maps are C^p with respect to the field $\mathbf{R}$.

Although there is no a-priori assumption that X is a definable set it follows (see discussion in Section 4, [1]) that X, with its manifold topology, can be realized as a definable subset of $\mathbf{R}^k$ for some k, with the subspace topology.

Let X be a definable subset of $\mathbf{R}^n$. We recall that X is called *definably compact* if for every definable continuous $\gamma : (0,1) \to X$ the limit of $\gamma(t)$, as t tends to 0 in $\mathbf{R}$, exists in X. This is equivalent to X being closed and bounded in $\mathbf{R}^n$.

Definition 2.2. We say that a definable set $X \subseteq \mathbf{R}^n$ is *locally definably compact* if every $x \in X$ has a definable neighborhood $V \subseteq X$ (i.e, V contains an X-open set around x) which is definably compact.

$X \subseteq \mathbf{R}^n$ is *locally closed* if there is a (definable) open set $U \subseteq \mathbf{R}^n$ containing X such that X is relatively closed in U.

Let U be an open subset of $\mathbf{R}^n$. For $X \subseteq U$, the *frontier of* X *in* U is defined as $Fr_U(X) = Cl_U(X) \setminus X$, where $Cl_U(X)$ is the closure of X in U. If $U = \mathbf{R}^n$ then we write $Fr(X)$ instead of $Fr_{\mathbf{R}^n}$.

The following is easy to verify:

Lemma 2.3. *Let* X *be a definable subset of* $\mathbf{R}^n$. *Then the following are equivalent:*
(i) X *is locally definably compact.*

(ii) X *is locally closed in* $\mathbf{R}^n$.
(iii) $Fr(X)$ *is a closed subset of* $\mathbf{R}^n$.

Now, assume that $X \subseteq \mathbf{R}^n$ is locally closed in $\mathbf{R}^n$ and definably homoeomorphic to a set $Y \subseteq \mathbf{R}^m$. It follows from the lemma that Y is also locally closed in $\mathbf{R}^n$ (since the notion of "locally definably compact" is invariant under definable homeomorphism).

Definition 2.4. Let f be a definable continuous map from a definable $X \subseteq \mathbf{R}^n$ into $Y \subseteq \mathbf{R}^k$.

For $b \in Y$, we say that f *is definably proper over* b if for every definable curve $\gamma : (0,1) \to X$ such that $\lim_{t\to 0} f(\gamma(t)) = b$, $\gamma(t)$ tends to some limit in X as t tends to 0 (in [7] this is called "γ is completable"). If $f : X \to Y$ is definably proper over every $b \in Y$ then we say that f *is definably proper over* Y or just f *is definably proper.*

For $A \subseteq X$, we say that $f|A$ is *definably proper over its image* if $f|A : A \to f(A)$ is definably proper over $f(A)$.

We say that f is *bounded over* $b \in Y$ if there is a neighborhood $W \subseteq Y$ of b such that $f^{-1}(W)$ is a bounded subset of $\mathbf{R}^n$.

In [7], an equivalent definition for definable properness is given and it is shown (Section 6, Lemma 4.5) that a definable and continuous $f : X \to Y$ is definably proper (over Y) if and only if the pre-image of every closed and bounded set in $\mathbf{R}^k$ is closed and bounded in $\mathbf{R}^n$.

The following lemma, which is easy to verify, implies that the set of all $y \in Y$ such that f is definably proper over y is itself definable.

Lemma 2.5. *For* $f : X \to Y$ *a definable continuous map,* $X \subseteq \mathbf{R}^n$*, and* $y \in Y$*, the following are equivalent:*
(i) f *is definably proper over* y.
(ii) f *is bounded over* y *and the intersection of the closure of the graph of* f *in* $\mathbf{R}^n \times Y$ *with* $Fr(X) \times \{y\}$ *is empty.*

Lemma 2.6. *Let* $X \subseteq \mathbf{R}^n$ *be a definable, locally closed set,* $f : X \to \mathbf{R}^k$ *a definable continuous map. Then,*
(i) The set of all $y \in \mathbf{R}^k$ *such that* f *is definably proper over* y *is open in* R^k.
(ii) If f *is definably proper over* $f(X)$ *then* $f(X)$ *is a locally closed set.*

Proof. (i) is a corollary of Lemma 2.5.

(ii) Let $W = \{y \in \mathbf{R}^k : f \text{ is definably proper over } y\}$. By (i), W is a definable open set and by our assumption $f(X) \subseteq W$. It is easy to see that $f(X)$ is relatively closed in W. □

2.3. **Linear and affine subspaces.** We assume here that our structure is ω_1-saturated, but any statement which does not mention generic points holds in every elementarily equivalent structure.

Definition 2.7. *Let* $H \subseteq \mathbf{K}^n$ *be a* d*-dimensional* $\mathbf{K}$*-subspace of* $\mathbf{K}^n$*. We say that* H is generic over a set $C \subseteq \mathbf{R}$ *if the following holds: Let* $\{H_s : s \in S\}$ *be some* C*-definable parametrization of all* d*-dimensional* $\mathbf{K}$*-linear subspaces of* $\mathbf{K}^n$*. Then* $H = H_s$ *for some* s *generic in* S *over* C.

The following is easy to verify using the dimension formula.

Fact 2.8. *Let H be a d-dimensional $\mathbf{K}$-subspace of $\mathbf{K}^n$, $C \subseteq \mathbf{R}$. Then the following are equivalent:*

(1) *H is generic over C.*
(2) *H has a generic basis $\{v_1, \ldots, v_d\}$ over C. Namely, it is a $\mathbf{K}$-linear basis for H where for every $i = 1, \ldots, d$, $\dim_{\mathbf{R}}(v_i/Cv_1, \ldots, v_{i-1}) = 2n$.*
(3) *H has a generic-orthogonal basis $\{v_1, \ldots, v_d\}$ with respect to the standard dot product on $\mathbf{K}^n$ induced by $\mathbf{R}$. Namely, for every $i = 1, \ldots, d$, v_i is generic over $Cv_1 \ldots, v_{i-1}$ in the orthogonal complement of $sp_{\mathbf{K}}\{v_1, \ldots, v_{i-1}\}$ in $\mathbf{K}^n$.*

Lemma 2.9. *For $C \subseteq \mathbf{K}$, let $H \subseteq \mathbf{K}^n$ be a d-dimensional $\mathbf{K}$-subspace of $\mathbf{K}^n$ which is generic over C.*
(i) Let B be a generic basis for H over C. Then for every $0 \neq v \in H$,

$$\dim_{\mathbf{R}}(v/C) - (2n - 2d) \geqslant \dim_{\mathbf{R}}(v/BC).$$

(ii) For every $0 \neq v \in H$, $\dim_{\mathbf{R}}(v/C) \geqslant 2n - 2d$.

Proof. (i) Let $B = \{v_1, \ldots, v_d\} \subseteq \mathbf{K}^n$ be a generic basis for H over C (see 2.8).

Now, given a nonzero $v \in H$, we may assume, after reordering B, that $B_1 = \{v_1, \ldots, v_{d-1}, v\}$ is a basis for H. In particular, H is defined over B_1, hence, $\dim_{\mathbf{R}}(v_d/B_1C) \leqslant \dim_{\mathbf{R}} H \leqslant 2d$. By the dimension formula,

$$\dim_{\mathbf{R}}(v_d/B_1C) + \dim_{\mathbf{R}}(v/C, v_1, \ldots v_{d-1}) = \dim_{\mathbf{R}}(v/BC) + \dim_{\mathbf{R}}(v_d/C, v_1, \ldots, v_{d-1}).$$

Since $\dim_{\mathbf{R}}(v_d/C, v_1, \ldots, v_{d-1}) = 2n$, we have

$$\dim_{\mathbf{R}}(v/a) \geqslant \dim_{\mathbf{R}}(v/Cv_1, \ldots, v_{d-1}) \geqslant 2n - 2d + \dim_{\mathbf{R}}(v/BC).$$

(ii) This is immediate from (i). ☐

Corollary 2.10. *Given $a \in \mathbf{K}^n$, $H \subseteq \mathbf{K}^n$ a d-dimensional $\mathbf{K}$-subspace which is generic over a, and $A \subseteq \mathbf{K}^n$ an a-definable set, we have:*
(i) If $\dim_{\mathbf{R}}(A) \geqslant codim_{\mathbb{R}}(H) (= 2n-2d)$ then $\dim_{\mathbf{R}}(A \cap a + H) \leqslant \dim_{\mathbf{R}} A - (2n-2d)$.
(ii) If $\dim_{\mathbf{R}}(A) < codim_{\mathbb{R}}(H)$ then $A \cap (a + H) \subseteq \{a\}$.

Proof. Notice that if $A \cap a + H \; sub\{a\}$ then both (i) and (ii) hold (recall that the $\mathbf{R}$-dimension of the empty set is $-\infty$), so we assume that $A \cap a + H$ contains at least one element different than a.

Fix B a generic basis of H over a, and let $w \neq a$ be a generic element in $A \cap a + H$ over aB. The translated set $A - a$ is still definable over a, hence $(A - a) \cap H$ is defined over aB. By Lemma 2.9(i),

$$\dim_{\mathbf{R}}(w - a/aB) \leqslant \dim_{\mathbf{R}}(w - a/a) - (2n - 2d).$$

Since $\dim_{\mathbf{R}}((w - a)/aB) = \dim_{\mathbf{R}}(w/aB)$ and $\dim_{\mathbf{R}}((w - a)/a = \dim_{\mathbf{R}}(w/a)$, we have

$$\dim_{\mathbf{R}}(w/aB) \leqslant \dim_{\mathbf{R}}(w/a) - (2n - 2d) \leqslant \dim_{\mathbf{R}}(A) - (2n - 2d).$$

This implies both (i) and (ii). ☐

2.4. **Generic projections.** For H a d-dimensional $\mathbf{K}$-subspace of $\mathbf{K}^n$, we say that an orthogonal projection $\pi : \mathbf{K}^n \to H$ is *generic over* C if H is generic over C.

Definition 2.11. *A definable subset A of $\mathbf{R}^k$ is called* an m-dimensional C^p $\mathbf{R}$-submanifold of $\mathbf{R}^k$ *if for every $x \in A$ there is a definable open neighborhood U of x and definable C^p (with respect to to $\mathbf{R}$) map $f : U \to \mathbf{R}^{n-m}$ such that $A \cap U = f^{-1}(0)$ and the rank of the $\mathbf{R}$-differential df_x of f at x equals $n - m$.*

(We chose here a local definition of a submanifold. However, as in the semialgebraic case, every definable $\mathbf{R}$-submanifold can be realized as a definable $\mathbf{R}$-manifold with finitely many charts, see Proposition 9.3.10 in [2]. We will not make use of this fact in the text).

Lemma 2.12. *For $a \in \mathbf{K}^n$, let $H \subseteq \mathbf{K}^n$ be an s-dimensional $\mathbf{K}$-subspace of $\mathbf{K}^n$ which is generic over a and let $A \subseteq \mathbf{K}^n$ be an a-definable set such that $\dim_{\mathbf{R}} A = r = 2n - 2s$.*

If $x_0 \in (a + H) \cap A$, $x_0 \neq a$, then A is a C^1 $\mathbf{R}$-submanifold of $\mathbf{K}^n$ at x_0, of dimension r, and $a + H$ intersects A transversally at x_0.

Proof. By 2.10, the set $a + H \cap A$ is finite and every point $\neq a$ in this intersection is generic in A over a, thus A is a C^1 $\mathbf{R}$-submanifold of $\mathbf{K}^n$ of $\mathbf{R}$-dimension r, at every point of intersection. By restricting ourselves to a neighborhood $V \subseteq \mathbf{K}^n$ of x_0, we may assume that A is a C^1 $\mathbf{R}$-submanifold of $\mathbf{K}^n$.

The family of all $\mathbf{K}$-subspaces of $\mathbf{K}^n$ of $\mathbf{K}$-dimension s has $\mathbf{R}$-dimension $2s(2n - 2s)$. Using, say, the Grassmanian construction, there is a $\emptyset$-definable C^1 $\mathbf{R}$-manifold G whose dimension is $2s(2n - 2s)$ which parameterizes all these subspaces.

Let $\{H_g : g \in G\}$ be the family of all these subspaces. Then $H = H_{g_0}$ where g_0 is generic in G over a. Fix $H' \subseteq H_{g_0}$ a K-subspace of H_{g_0} of $\mathbf{K}$-dimension $s - 1$ which is generic over ag_0 among all such subspaces.

Claim If B is a generic basis for H' then $\dim_{\mathbf{R}}(g_0/aB) = 2n - 2s$.

This follows from the fact that H_{g_0}/H' is a one-dimensional subspace of $\mathbf{K}^n/H'$ which is generic over aB, and because the set of all 1-dimensional $\mathbf{K}$-subspaces of $\mathbf{K}^n/H'$ has dimension $2n - 2s$.

Let $G_1 = \{g \in G : H' \subseteq H_g\}$. It is B-definable, and by working in the quotient $\mathbf{K}^n/H'$ we can endow G_1 with a structure of a definable C^1 $\mathbf{R}$-manifold of dimension $2n - 2s$. We now have: for all $x \in \mathbf{K}^n \setminus H'$ there is a unique $g = h(x) \in G_1$ such that $x \in H_g$. Moreover, the $\mathbf{R}$-manifold structure on G_1 can be chosen so that $h : \mathbf{K}^n \setminus H' \to G_1$ is a C^1 map with respect to $\mathbf{R}$.

By 2.10, $a + H' \cap A$ contains at most the point a, hence there is an open neighborhood $U \subseteq \mathbf{K}^n$ of x_0 such that $U \cap a + H' = \emptyset$. We define $F : U \to G_1$ by $F(x) = h(x - a)$. Namely, $g = F(x)$ if $x \in a + H_g$. We now proceed just like in the proof of Lemma 3.6 in [13]:

Notice that each $H_g \cap U$ is a level set of F and that F is $\mathbf{R}$-differentiable on U. It follows that $ker(dF_x) = H_{F(x)}$ (where dF_x denotes the $\mathbf{R}$-differential of F at x).

Since $A \cap U$ is a C^1 $\mathbf{R}$-submanifold of $\mathbf{K}^n$ of dimension $r = 2n - 2s$, there is an a-definable open $V \subseteq \mathbf{R}^r$ and an a-definable $\sigma : V \to \mathbf{R}^{2n}$ such that $x_0 \in \sigma(V) \subseteq A \cap U$ and σ is an immersion. It follows that for every $y \in V$, $d\sigma_y$ gives an isomorphism between $\mathbf{R}^r$ and $T_{\sigma(y)}A$, and by the chain rule, $T_{\sigma(y)}A \cap H_{F(\sigma(y))}$ is nonzero if and only if $ker(d(F \circ \sigma)_y)$ is nonzero.

Assume now that $a + H_{g_0}$ intersects A at x_0 non-transversally. Then $T_{x_0}(A) \cap H_{g_0}$ is nonzero and since g_0 is generic in G_1 over aB, there is some neighborhood $W \subseteq G_1$ of g_0 such that for every $g \in W$, $a + H_g$ intersects A non-transversally. If we let $V_1 = \sigma^{-1}(F^{-1}(W))$ then for every $y \in V_1$, $ker(d(F \circ \sigma)_y)$ is nonzero. This implies that the rank of $d(F \circ \sigma)_y$ is less than r and hence $\dim_{\mathbf{R}}(F \circ \sigma(V_1)) < r$. Since $\dim V_1 = r$ it follows that for some $g \in W$, there are infinitely many $y \in V_1$ such that $F \circ \sigma(y) = g$, or said differently, $H_g \cap A$ is infinite. But, since $H_{g_0} \cap A$ is finite and g_0 generic in G_1 we could have chosen $W \subseteq G_1$ such that for every $g \in W$, $H_g \cap A$ is finite. Contradiction. □

For $A \subseteq \mathbf{K}^n$ definable, we write $Fr(A)$ for the frontier of A in $\mathbf{K}^n$, identified with $\mathbf{R}^{2n}$. By o-minimality, $\dim_{\mathbf{R}} Fr(A) \leqslant \dim_{\mathbf{R}}(A) - 1$.

Lemma 2.13. *Let $A \subseteq \mathbf{K}^n$ be a $\emptyset$-definable locally closed set of $\mathbf{R}$-dimension $2d < 2n$ and let L be a generic over $\emptyset$ d-dimensional $\mathbf{K}$-subspace of $\mathbf{K}^n$. Let $\pi : \mathbf{K}^n \to L$ be the orthogonal projection. Then:*
(i) For all $y \in L$, $\pi|A$ is bounded over y.
(ii) If $a \in \mathbf{K}^n \setminus Fr(A)$ and π is generic over a then $\pi|A$ is definably proper over $\pi(a)$.
(iii) If A is closed in $\mathbf{K}^n$ then $\pi|A$ is definably proper over all of L.

Proof. (i) This can be shown either by working in the projective space $\mathbb{P}^n(\mathbf{K})$, or directly as follows:

Let S^{2n-1} be the unit sphere in $\mathbf{R}^{2n}$, and let

$$A^* = \{x \in S^{2n-1} : \forall t > 0\, \exists x' \in A \left(||x'|| > t \& ||x - \frac{x'}{||x'||}|| < 1/t \right) \}.$$

Notice that A^* can also be obtained as follows: First intersect $Cl(\{(\frac{1}{||x'||}, \frac{x'}{||x'||}) : x' \in A\})$ with the set $\{0\} \times S^{2n-1}$ and then project onto the second coordinate. It follows from o-minimality that $\dim_{\mathbf{R}}(A^*) \leqslant 2d - 1$.

Let $H \subseteq \mathbf{K}^n$ be the orthogonal complement of L. It is not hard to see that if, for some $y \in L$, $\pi|A$ were not bounded over y then the intersection of H with A^* is nonempty. However, by Lemma 2.10 (ii), $H \cap A^* = \emptyset$, contradiction.

For (ii) we define,

$$I(\pi) = \{y \in L : \pi|A \text{ is not definably proper over } y\}.$$

By Lemma 2.6, $I(\pi)$ is a definable closed set.

Since A is locally closed, we have (see Lemma 2.5), for all $y \in L$, $y \in I(\pi)$ if and only if either $(y + H) \cap Fr(A) \neq \emptyset$ or $\pi|A$ is not bounded over y.

Lemma 2.10 (ii) implies that $\pi(a) + H \cap FrA = \emptyset$, and by (i), $\pi|A$ is bounded over $\pi(a)$ thus $\pi|A$ is definably proper over $\pi(a)$.

(iii) is an immediate corollary of (i) and (ii). □

2.5. **The main covering theorem.** In classical complex analytic geometry, given a complex analytic set A and $p \in A$, one moves to a local coordinates system, chosen properly, in order to ensure that the projection map from A onto (some of) the coordinates is a proper map in this neighborhood. The following theorem is probably the most significant advantage of the o-minimal setting in comparison with the general classical setting. It allows us to work "globally" rather than "locally".

Theorem 2.14. *Let A be a definable, locally closed subset of $\mathbf{K}^n$, such that $\dim_{\mathbf{R}} A = 2d < 2n$.*

Then there are definable open sets $U_1, \ldots, U_k \subseteq \mathbf{K}^n$ and d-dimensional $\mathbf{K}$-linear subspaces $L_1, \ldots, L_k$ such that

(1) *$\bigcup_{i=1}^k U_i = \mathbf{K}^n \setminus Fr_{\mathbf{K}^n}(A)$ (in particular A is contained in the union of the U_i's).*
(2) *For each $i = 1, \ldots, k$, if $\pi_i : \mathbf{K}^n \to L_i$ is the orthogonal projection, then $\pi_i | U_i \cap A$ is definably proper over $\pi_i(U_i)$.*
(3) *If furthermore A is a C^1 $\mathbf{R}$-submanifold of $\mathbf{K}^n$ then the U_i's and L_i's can be chosen to satisfy also: For every $i = 1. \ldots, k$ the map $\pi_i | A \cap U_i$ is a local diffeomorphism and there is $m = m_i$ such that every $y \in \pi_i(U_i)$ has exactly m pre-images under π_i in $A \cap U_i$.*

Proof. Note that since the statement of the theorem is first-order we may prove it in an ω_1-saturated elementary extension. We assume that A is $\emptyset$-definable.

Let $\pi_1, \ldots, \pi_{2n+1}$ be an independent sequence of generic orthogonal projections onto d-dimensional subspaces $L_1, \ldots, L_{2n+1}$, respectively (i.e, each π_k is generic over $\pi_1, \cdots, \pi_{k-1}$). We first claim that for every $a \in \mathbf{K}^n$ there is $i = 1, \ldots, 2n+1$ such that π_i is generic over a.

Indeed, if not then each π_i is not generic over a, which implies, by the dimension formula,

$$\dim_{\mathbf{R}}(a/\emptyset) > \dim_{\mathbf{R}}(a/\pi_1) > \dim_{\mathbf{R}}(a/\pi_1, \pi_2) > \cdots > \dim_{\mathbf{R}}(a/\pi_1, \cdots, \pi_{2n+1}).$$

This is clearly impossible because $\dim_{\mathbf{R}}(a/\emptyset) \leqslant 2n$ (by $\dim(a/\pi_1, \cdots, \pi_t)$ we mean $\dim(a/g_1, \ldots, g_t)$ where g_i is a parameter for L_i in some 0-definable parameterization of all d-dimensional $\mathbf{K}$-subspaces of $\mathbf{K}^n$).

For every $i = 1, \ldots, 2n+1$, let U_i be the definable open set $U_i = \pi_i^{-1}(L_i \setminus I(\pi_i))$ (see the proof of 2.13 for the definition of $I(\pi)$). By Lemma 2.6 (ii), U_i is an open set.

We claim that $\mathbf{K}^n \setminus Fr(A) = \bigcup_i U_i$.

For one inclusion, notice that if $x \in Fr(A)$ then no π_i is definably proper over $\pi(x)$ and hence x is not in any of the U_i's. For the opposite inclusion, consider $x \in \mathbf{K}^n \setminus Fr(A)$. By the above observations, there is an $i = 1, \ldots, n$ such that π_i is generic over x. But then, by Lemma 2.13 (ii), $\pi_i | A$ is definably proper over $\pi_i(x)$ and hence $x \in U_i$.

By definition of the U_i's, each $\pi_i | (A \cap U_i)$ is definably proper over $\pi_i(U_i)$, thus proving (2).

For (3), assume that A is a C^1 $\mathbf{R}$-submanifold of $\mathbf{K}^n$ and choose the π_i's as above. For every $i = 1, \ldots, 2n+1$, let

$$B_i' = I(\pi_i) = \{y \in L_i : \ \pi_i|A \text{ not definably proper over } y\},$$

$$B_i'' = \{y \in L_i : \exists x \in A \ \ (\pi_i(x) = y \,\&\ \ \pi_i|A \text{ not a local diffeomorphism at } x)\,\},$$

and let $B_i = B_i' \cup B_i''$.

Claim B_i is a closed subset of L_i.

Indeed, take $y \in L_i \setminus B_i$. We want to show that there is a neighborhood of y that is disjoint from B_i. Since $y \notin I(\pi_i)$, $\pi_i|A$ is definably proper over y. But then either $y \in \pi_i(A)$ or there is a neighborhood V of y which is disjoint from $\pi_i(A)$, and in particular then $V \cap B_i = \emptyset$. We assume then that $y \in \pi_i(A)$. Since $y \notin B_i$, for all $x \in \pi_i^{-1}(y) \cap A$, $\pi_i|A$ is a local diffeomorphism near x. In particular, $\pi^{-1}(y)$ is finite, and by the properness of $\pi_i|A$ over y (and hence over a neighborhood of y), there is an open neighborhood $V \subseteq L_i$ of y such that the restriction of π_i to $A \cap \pi_i^{-1}(V)$ is a finite-to-one surjection onto V, which is also a local diffeomorphism. We have then $V \cap B_i = \emptyset$, completing the proof that B_i is a closed subset of L_i.

For every $i = 1, \ldots, 2n+1$, let $U_i = \pi_i^{-1}(L_i \setminus B_i)$.

Claim $\bigcup_i U_i = \mathbf{K}^n \setminus Fr(A)$.

As in (2), we need to see that if π_i is generic over $a \in \mathbf{K}^n \setminus Fr(A)$ then $a \in U_i$, or equivalently, $\pi_i(a) \notin B_i$.

By Lemma 2.13, π_i is definably proper over $\pi_i(a)$ and hence $\pi_i(a) \notin I(\pi_i)$. It is left to verify that for all $x \in A \cap \pi_i^{-1}\pi_i(a)$, $\pi_i|A$ is a local diffeomorphism at x.

It follows from Lemma 2.12 that for every $x_0 \neq a$ such that $\pi_i(x_0) = \pi_i(a)$, $\pi_i|A$ is a local diffeomorphism at x_0. This is sufficient to ensure that $a \in U_i$ in case that a itself is not in A. If $a \in A$, then, since A is a C^1 $\mathbf{R}$-submanifold at a, and π_i is generic over a it is easy to see that π_i is a local diffeomorphism at a as well and hence $a \in U_i$, thus proving the claim.

After possibly partitioning each U_i to its definably connected components, we may assume that each U_i is definably connected. Notice that $\pi_i|(A \cap U_i)$ is still a local diffeomorphism at every point of these components and that its restriction to each component is still definably proper over $\pi_i(U_i)$.

It is left to show the following:

Claim Assume that A is a relatively closed subset of U, a definable open subset of $\mathbf{K}^n$, such that the projection $\pi : U \to \mathbf{K}^d$ satisfies:
(i) $\pi(U)$ is definably connected, and
(ii) $\pi|A : A \to \pi(U)$ is a local diffeomorphism which is moreover definably proper over $\pi(U)$.

Then there is an m such every element in $\pi(U)$ has exactly m pre-images under π in $A \cap U$.

Since π is a local homeomorphism, it is everywhere a finite-to-one map. By o-minimality, there is an r such that it is at most r-to-1. For $i = 1, \ldots, r$, let W_i be the set of y in $\pi(U)$ such that $\pi^{-1}(y) \cap A$ contains exactly i elements. The sets $W_1, \ldots, W_r$ form a partition of $\pi(U)$. Definable properness implies that each W_i is an open set and therefore closed as well. By the definable connectedness of $\pi(U)$, there is exactly one such W_i. □

Remark We presume that a similar theorem to 2.14 can be proved with respect to projection onto $\mathbf{R}$-subspaces of $\mathbf{R}^n$. However, since generic $\mathbf{K}$-subspaces of $\mathbf{K}^n$ are not generic as $\mathbf{R}$-subspaces of $\mathbf{R}^{2n}$, such a theorem will not directly imply Theorem 2.14.

3. K-manifolds and submanifolds

The notions of a $\mathbf{K}$-manifold and its $\mathbf{K}$-analytic subsets were first treated in [14]. However, here we simplify the definitions. Instead of requiring uniform definability of the data which constitutes the $\mathbf{K}$-manifold or the $\mathbf{K}$-analytic subset, we require all data to be finite. Lemma 3.3 and Corollary 4.14 justify the changes.

We first recall the basic notions of $\mathbf{K}$-holomorphicity from [13] and [14].

For $U \subseteq \mathbf{K}$ a definable open set and $z_0 \in U$, we say that a definable function $f : U \to \mathbf{K}$ is $\mathbf{K}$-*differentiable at* z_0 if $\lim_{h \to 0}(f(z_0 + h) - f(z_0))/h$ exists in $\mathbf{K}$. If f is $\mathbf{K}$-differentiable at every point on U then we say that f is $\mathbf{K}$-*holomorphic in* U. If U is a definable open subset of $\mathbf{K}^n$ and $f : U \to \mathbf{K}$ is a definable function then f is called $\mathbf{K}$-*holomorphic in* U if it is continuous and in addition $\mathbf{K}$-holomorphic in each of the variables separately. A $\mathbf{K}$-*holomorphic map* $f : U \to \mathbf{K}^m$ is a definable map, each of whose coordinate maps is a $\mathbf{K}$-holomorphic function.

Definition 3.1. *A d-dimensional* $\mathbf{K}$-*manifold* is a set X, together with a finite cover $X = \bigcup_{i=1}^r U_i$, and for each $i = 1, \ldots, r$ a set-theoretic bijection $\phi_i : U_i \to V_i$, with V_i a definable subset open of $\mathbf{K}^d$, such that for each i, j, the set $\phi_i(U_i \cap U_j)$ is a definable open subset of V_i and the transition maps $\phi_j \phi_i^{-1}$ are K-holomorphic (in particular definable). Moreover, the naturally induced topology is Hausdorff.

The notion of *a* $\mathbf{K}$-*holomorphic map between* $\mathbf{K}$-*manifolds* is defined using the transition maps and charts just like in classical differential topology.

Notice that every $\mathbf{K}$-manifold is in particular a definable C^1 $\mathbf{R}$-manifold. Therefore, as was discussed earlier, we may assume that the underlying set X, with its manifold topology, is a definable subset of $\mathbf{R}^k$ for some k. Also, when $\mathbf{K}$ is the field of complex numbers then "$\mathbf{K}$-holomorphic" just means "holomorphic and definable in some o-minimal expansion of the real field".

Definition 3.2. Let N be a $\mathbf{K}$-manifold. A definable $M \subseteq N$ is called *a d-dimensional* $\mathbf{K}$-*submanifold of* N if for every $a \in M$ there is a definable open set $U \subseteq N$ containing a and a $\mathbf{K}$-holomorphic $f : U \to \mathbf{K}^{n-d}$ such that $M \cap U = f^{-1}(0)$ and $Rank_{\mathbf{K}}(Df_a) = n - d$ (where Df_x is the $\mathbf{K}$-differential of f at x.

It is not clear a-priori that a $\mathbf{K}$-submanifold can itself be covered by a finite atlas, but just like definable $\mathbf{R}$-submanifolds, it turns out to be true:

Lemma 3.3. *Let N be a $\mathbf{K}$-manifold, $M \subseteq N$ a $\mathbf{K}$-submanifold. Then M is a $\mathbf{K}$-manifold as well. More precisely, there is a $\mathbf{K}$-manifold M_1 and a $\mathbf{K}$-holomorphic embedding $f : M_1 \to N$ such that $f(M_1) = M$.*

Proof. Since N is covered by finitely many charts we may assume that $N = U$ a definable open subset of $\mathbf{K}^n$, and that $M \subseteq U$ is a $\mathbf{K}$-submanifold of dimension d. Moreover, after considering all the possible projections on d of the $\mathbf{K}$-coordinates, we may assume that the projection π onto the first d coordinates is a local homeomorphism. We now use Proposition 9.3.9 from [2] and conclude that there exists an finite open definable cover $M = \bigcup_{i=1}^{r} U_i$ such that for every $i = 1, \ldots, r$, the projection map $\pi|U_i$ is a homeomorphism onto $\pi(U_i)$ (the result from [2] is stated for semialgebraic sets but the proof just uses the triangulation and trivialization theorems, both true in our o-minimal setting, see [7]). We therefore may assume that the projection of M onto the first d coordinates is a homeomorphism.

Hence M is the graph of a continuous function ϕ from $\pi(M)$ into $\mathbf{K}^{n-d}$. By the implicit function theorem, ϕ is $\mathbf{K}$-holomorphic at generic points and since it is everywhere continuous, it follows from Theorem 2.14 in [14] that ϕ is $\mathbf{K}$-holomorphic. This is sufficient in order to obtain the necessary charts □

Lemma 3.4. *Let $\mathcal{M}$ be a definably connected d-dimensional $\mathbf{K}$-manifold and $f : M \to \mathbf{K}$ a $\mathbf{K}$-holomorphic function. Then the zero set of f is either the whole of M or a subset of M whose $\mathbf{R}$-dimension is $2d-2$.*

Proof. We may assume that M is an open subset of $\mathbf{K}^d$. Fix an arbitrary $a = (a_1, \ldots, a_d)$ in Z the zero set of f . If Z is not the whole of M then using a change of coordinates we may assume that f is regular in the last variable, and hence we may apply the Weierstrass Preparation Theorem (see Theorem 2.20 in [14] and the definitions preceding it). It follows that in a neighborhood of a the set Z is the zero set of a definable $\mathbf{K}$-holomorphic function $h(z_1, \ldots, z_{d-1}, y)$ which is a monic polynomial in $(y - a_d)$ and coefficients which are $\mathbf{K}$-holomorphic functions of $z_1, \ldots, z_{d-1}$. By Theorem 2.3 (5) in [14], there is a neighborhood U_1 of $(a_1, \ldots, a_{d-1})$ such that for every $(z_1, \ldots, z_{d-1})$ in U_1 the function h has a fixed finite number of zeroes, counted with multiplicity, near a_d. It follows that $\dim_{\mathbf{R}}(Z) = 2d-2$ in a neighborhood of a. □

Lemma 3.5. *Let M, N be definably connected $\mathbf{K}$-manifolds of dimension d and assume that $f : M \to N$ is a $\mathbf{K}$-holomorphic function, such that $\dim_{\mathbf{R}} f(M) = 2d$.*

Let

$$M_f = \{x \in M : f \textit{ is a local } \mathbf{K}\textit{-biholomorphism at } x\}.$$

Then

$$\dim_{\mathbf{R}}(M \setminus M_f) \leqslant 2d-2.$$

In particular, M_f is definably connected.

Proof. By working locally, we may assume that M is an open, definably connected subset of $\mathbf{K}^d$, which we call U, and that $N = \mathbf{K}^d$. We write U_f for M_f. Notice that $U \setminus U_f$ is contained in the zero set of the function $|Df| : U \to \mathbf{K}$ (where $|Df_x|$ is the determinant of the corresponding $\mathbf{K}$-linear function at x). Since this last function is $\mathbf{K}$-holomorphic in U its zero set, by 3.4, is either the whole of U or of $\mathbf{R}$-dimension $2d-2$. If $|Df_x|$ vanished on the whole of U then the dimension of $f(M)$ would be smaller than $2d$. It thus follows from our assumption that the latter must hold, proving the lemma. □

Lemma 3.6. *Let M be a definably connected d-dimensional $\mathbf{K}$-submanifold of $\mathbf{K}^n$, $\pi : \mathbf{K}^n \to \mathbf{K}^d$ the projection on the first d coordinates, and assume that $\dim_{\mathbf{R}} \pi(M) = 2d$. Then*

(1) *Outside a definable subset of M of $\mathbf{R}$-dimension $2d-2$, π is a local homeomorphism.*
(2) *There are definable open, pairwise disjoint $V_1, \ldots, V_r \subseteq \pi(M)$, such that $\dim_{\mathbf{R}}(\pi(M) \setminus \bigcup_i V_i) \leqslant 2d-1$ and for each i, there are definable $\mathbf{K}$-holomorphic functions $\phi_{i,1}, \ldots, \phi_{i,i} : V_i \to \mathbf{K}^{n-d}$, taking distinct values at every $x \in V_i$, such that for all $x \in V_i$ and $y \in \mathbf{K}^{n-d}$, $(x,y) \in M$ iff $y = \phi_{i,j}(x)$ for some $j = 1, \ldots, i$.*
(3) *If moreover $\pi|M$ is definably proper over $\pi(M)$ then there is a definable closed $S \subseteq \mathbf{K}^d$ of dimension $2d-2$ and $m \in \mathbb{N}$ such that for every $x \in \pi(M) \setminus S$, $\pi^{-1}(x) \cap M$ contains exactly m distinct elements, at each of which π_M is a local $\mathbf{K}$-biholomorphism.*

Proof. The first part of the lemma follows easily from Lemma 3.5.

For (2), note that the existence of such sets and definable *continuous* functions follows from o-minimality. The functions are $\mathbf{K}$-holomorphic at generic points by the implicit function theorem, and by continuity they are $\mathbf{K}$-holomorphic on their whole domain (see Theorem 2.14 in [14]).

(3) Assume now that $\pi|M$ is definably proper over $\pi(M)$, and let M_π be the set of all points in M where π is a local $\mathbf{K}$-biholomorphism.

Let $S = Cl_{\mathbf{K}^d}(\pi(M \setminus M_\pi))$ and $V = \pi(M) \setminus S$. By Lemma 3.5 and o-minimality, $\dim_{\mathbf{R}} S \leqslant 2d-2$, and therefore V is a definably connected open set. Moreover, every element in V has a finite number of pre-images in M, at each of which π is a local homeomorphism.

Thus, if we let $A = \pi^{-1}(V) \cap M$ then A satisfies the assumptions of the last claim from the proof of Theorem 2.14 (3). It follows that there is an m such that every point in V has exactly m pre-images in M. □

The following technical lemma will play an important role in later results.

Lemma 3.7. *Let M be a definably connected $\mathbf{K}$-submanifold of $\mathbf{K}^n$, and let $f : M \to \mathbf{K}$ be a $\mathbf{K}$-holomorphic function. Assume that Z is the set of all $z_0 \in Cl_{\mathbf{K}}(M)$ such that the limit of $f(z)$ exists and equals 0 as z approaches z_0 in M. If $\dim_{\mathbf{R}} Z \geqslant \dim_{\mathbf{R}} M - 1$ then f vanishes on all of M.*

Proof. This is the exact analogue of Theorem 3.1 from [16]. The proof there (as well as lemma 2.10 which it uses) goes through word-for-word in the current setting.

4. **K**-analytic sets

Definition 4.1. If A is a definable subset of a **K**-manifold M, then A is *a locally* **K**-*analytic subset of* M if for every $a \in A$ there is a definable open $V \subseteq M$ containing a and a **K**-holomorphic map $f : V \to \mathbf{K}^m$, for some m, such that $A \cap V = f^{-1}(0)$. A is called *a* **K**-*analytic subset of* M if in addition to the above A is closed in M.

A definable closed $A \subseteq M$ is *finitely* **K**-*analytic subset of* M if it can be covered by finitely many definable open subsets of M, on each of which A is the zero set of some **K**-holomorphic map.

A **K**-analytic subset A of M is called *reducible* if it can be written as $A = A_1 \cup A_2$, where A_1 and A_2 are **K**-analytic in M and none of the A_i's is contained in the other. When A is not reducible it is called *irreducible.*

Remark Because a **K**-holomorphic map remains **K**-holomorphic in every elementary extension, the notions of a **K**-manifold and a finitely **K**-analytic subset are invariant under elementary extensions. It is not so clear a-priori that a **K**-analytic subset of a **K**-manifold remains so in every elementary extension (this was the reason that we originally defined a **K**-analytic subset in [14], using uniformly definable data). However, as we eventually show, every **K**-analytic set is necessarily finitely **K**-analytic thus making this notion first-order as well. Moreover, we will give an explicit first-order characterization of **K**-analytic sets (see 4.14 below).

Note that, by definition, every (locally) **K**-analytic set is definable in our ambient o-minimal structure.

Examples

1. Any **K**-algebraic subset of $\mathbf{K}^n$ is clearly a **K**-analytic subset of $\mathbf{K}^n$. More generally, the intersection of any such algebraic set with a semialgebraic open set $U \subseteq \mathbf{K}^n$ is a **K**-analytic subset of U. The same is true for projective algebraic subsets of $\mathbb{P}^n(\mathbf{K})$ (which is itself a **K**-manifold).

2. Consider $\mathbb{R}_{an}$, the expansion of the real field by all real analytic functions on the closed unit cubes. Every compact complex manifold M can be realized as a **K**-manifold in this structure (with $\mathbf{K} = \mathbb{C}$) and every analytic subset of M is a definable **K**-analytic subset (see discussion in Section 2.2, p.8, of [14]). Moreover, as we pointed in [14], elementary extensions of $\mathbb{R}_{an}$ give rise to new and interesting **K**-analytic subsets of such manifolds (see 3.3 there).

3. Consider the structure $\mathbb{R}_{exp} = \langle \mathbb{R}, <, +, \cdot, e^x \rangle$, which is the expansion of the real field by the real exponential function. This is an o-minimal structure ([20]) but, as is shown in [3], every germ of a holomorphic map which is definable in $\mathbb{R}_{exp}$ is semialgebraic. Using this fact we prove in the appendix (see Theorem 12.6):

Theorem 4.2. *Let $G \subseteq \mathbb{C}^n$ be an open set and let X be a $\mathbb{C}$-analytic subset of G such that both X and G are definable in $\mathbb{R}_{exp}$. Then there is a complex algebraic set $A \subseteq \mathbb{C}^n$ such that X is the union of several irreducible components of $A \cap G$.*

Remark Note that it is not true, under the above assumptions, that there is an algebraic $A \subseteq \mathbf{K}^n$ such that $X = A \cap G$. Take for instance X to be the algebraic irreducible set $\{(x,y) \in \mathbb{C}^2 : y^2 = x^2(x+1)\}$. Locally, in a small neighborhood U of $(0,0)$, this analytic set has two irreducible component, each given as the vanishing set of one of branches of $y \pm x\sqrt{x+1}$. Both components are semialgebraic but there is no algebraic subset of $\mathbb{C}^2$ whose intersection with U gives only one of these components.

Before we continue developing the theory of $\mathbf{K}$-analytic sets we prove a corollary to Lemma 3.7:

Lemma 4.3. *Let M be a definably connected $\mathbf{K}$-submanifold of some definable open $U \subseteq \mathbf{K}^n$ and let A be a $\mathbf{K}$-analytic subset of U. If $\dim_{\mathbf{R}}(A \cap Cl(M)) \geqslant \dim_{\mathbf{R}} M - 1$ then M is contained in A.*

Proof. Pick first a generic z_0 in $A \cap Cl(M)$ and consider a $\mathbf{K}$-holomorphic function f, in some neighborhood W of z_0, which vanishes on A. We may choose W so that every definably connected component of $M \cap W$ has z_0 in its closure. The function f, considered as a function on each component of $W \cap M$, satisfies the assumptions of Lemma 3.7 and therefore vanishes on $W \cap M$. The same is true for all the functions which define A near z_0 and hence M is contained in A, in some neighborhood of z_0. Consider now the set M' of all points in M where M is locally contained in A. M' is clearly open in M and it is easily seen to be also closed thus $M \subseteq A$. □

Definition 4.4. Let N be a $\mathbf{K}$-manifold. If $A \subseteq N$ is an arbitrary definable set then $Reg_{\mathbf{K}}A$ is the set of all points $z \in A$ such that in some neighborhood of a, the set A is a $\mathbf{K}$-submanifold of N. We let $Sing_{\mathbf{K}}A = A \setminus Reg_{\mathbf{K}}A$.

Later on we will prove that $Sing_{\mathbf{K}}A$ is itself a $\mathbf{K}$-analytic set. At this stage we can prove the following approximation.

Lemma 4.5. *If N is a $\mathbf{K}$-manifold and A is a $\mathbf{K}$-analytic subset of N then $Reg_{\mathbf{K}}A$ is dense in A and*

$$\dim_{\mathbf{R}}(Sing_{\mathbf{K}}A) \leqslant \dim_{\mathbf{R}} A - 2.$$

Proof. We prove the theorem by induction on $n = \dim_{\mathbf{K}} N$. We may assume that N is an open definable subset of $\mathbf{K}^n$, which we call U. Since we prove the theorem for an arbitrary such U the density of $Reg_{\mathbf{K}}A$ will follow from the dimension inequality in the lemma.

Take $a \in A$ and assume that $g_1, \ldots, g_r : W \to K$ are $\mathbf{K}$-holomorphic functions on a definably connected open $W \subseteq U$, $a \in W$, such that $A \cap W$ is the intersection of the zero-sets of the g_i's. It is sufficient to show that $\dim_{\mathbf{R}}(Sing_{\mathbf{K}}(A \cap W)) \leqslant \dim_{\mathbf{R}}(A \cap W) - 2$, thus we may assume that $A \subseteq W$. Notice that $A \cap W$ remains $\mathbf{K}$-analytic in elementary extensions thus we may use generic points. We assume that A and the g_i's are definable over $\emptyset$, and that $A \neq W$.

Let z_0 be a generic point in $Sing_{\mathbf{K}}A$. We claim that there is a 0-definable $\mathbf{K}$-holomorphic function $f : W \to \mathbf{K}$ which vanishes on A, such that one of its partial first derivatives $\partial f/\partial z_i$ is not identically zero on any relatively open subset of A containing z_0.

Indeed, we may take f to be a partial derivative of sufficiently large order of one of the g_i's. If no such f satisfies the above property then in particular the partial derivatives of all g_i's, of any order, vanish at a, and then (see Theorem 2.13 in [14]) all the g_i's are zero functions, thus $A = W$.

Let $B = \{z \in W : \partial f/\partial z_i(z) = 0\}$, $N' = Z(f) \setminus B$ and $A' = A \setminus B$. By the implicit function theorem, N' is a submanifold of W of dimension $n-1$, and A' is a $\mathbf{K}$-analytic subset of N'. By our assumptions on f, z_0 is in the closure of A'. Notice that because B is a closed set,

$$Reg_{\mathbf{K}}A' = Reg_{\mathbf{K}}(A \setminus B) = Reg_{\mathbf{K}}(A) \setminus B$$

and

$$Sing_{\mathbf{K}}A' = Sing_{\mathbf{K}}(A) \setminus B.$$

By induction, $Reg_{\mathbf{K}}A'$ is dense in A' and $\dim_{\mathbf{R}}(Sing_{\mathbf{K}}A') \leqslant \dim_{\mathbf{R}} A' - 2$.

Case 1 $z_0 \notin B$.

In this case z_0 is in $Sing_{\mathbf{K}}A'$ and therefore $\dim_{\mathbf{R}}(z_0/\emptyset) \leqslant \dim_{\mathbf{R}}(Sing_{\mathbf{K}}A')$ (since A' is $\emptyset$-definable). Because z_0 was taken generic in $Sing_{\mathbf{K}}A$ we have that

$$\dim_{\mathbf{R}}(Sing_{\mathbf{K}}A) \leqslant \dim_{\mathbf{R}}(Sing_K A') \leqslant \dim_{\mathbf{R}} A' - 2 \leqslant \dim_{\mathbf{R}} A - 2.$$

Case 2 $z_0 \in B$.

Because $Reg_{\mathbf{K}}A'$ is dense in A', we have $z_0 \in B \cap Cl_U(Reg_{\mathbf{K}}A')$ and hence $z_0 \in Cl_U(B \cap (Reg_{\mathbf{K}}(A) \setminus B))$. We may now apply Lemma 4.3 as follows:

Denote the submanifold $Reg_{\mathbf{K}}(A) \setminus B$ by M'. We claim that $\dim_{\mathbf{R}}(B \cap Cl_U(M')) \leqslant \dim_{\mathbf{R}} A - 2$.

Indeed, if not then

$$\dim_{\mathbf{R}}(B \cap Cl_U(M')) \geqslant \dim_{\mathbf{R}} A - 1 \geqslant \dim_{\mathbf{R}} M' - 1,$$

and, by Lemma 4.3, there is some (nonempty) definably connected component of M' which is contained in B. This is absurd because M' is disjoint from B.

We thus showed that $\dim_{\mathbf{R}}(B \cap Cl_U(M')) \leqslant \dim_{\mathbf{R}} A - 2$. Since z_0 belongs to this last intersection and is generic in $Sing_{\mathbf{K}}A$ we have $\dim_{\mathbf{R}}(SingA) \leqslant \dim_{\mathbf{R}} A - 2$. □

Lemma 4.6. *If A is a $\mathbf{K}$-analytic subset of a $\mathbf{K}$-manifold N and $Reg_{\mathbf{K}}A$ is definably connected then A is irreducible. In particular, the number of irreducible components of A is finite.*

Proof. This follows from the density of $Reg_{\mathbf{K}}A$ in A, together with the fact that a $\mathbf{K}$-holomorphic function which vanishes on some open subset of its domain must vanish on a whole definably connected component of the domain (we identify for this purpose the submanifold $Reg_{\mathbf{K}}A$ with an open subset of some $\mathbf{K}^d$). □

Our main technical result is the following.

Lemma 4.7. *Let U be a definable open subset of $\mathbf{K}^n$, A a definable relatively closed subset of U whose $\mathbf{R}$-dimension is $2d$. Assume that $Reg_{\mathbf{K}}A$ is definably connected, dense in A, and that $\dim_{\mathbf{R}}(Sing_{\mathbf{K}}A) \leqslant 2d-2$. Assume also that the projection map* *onto the first d coordinates, $\pi : A \to \mathbf{K}^d$, is finite-to-one and definably proper over its image.*

Then, $\pi(A)$ is open in $\mathbf{K}^d$ and there is an $r \in \mathbb{N}$ and a $\mathbf{K}$-holomorphic map $\Psi : \pi(A) \times \mathbf{K}^{n-d} \to \mathbf{K}^r$ such that A is the zero set of Ψ.

Moreover, the map Ψ can be definably recovered in the structure $\langle \mathbf{R}, <, +, \cdot, A\rangle$.

Proof. We first prove that $\pi(A)$ is open in $\mathbf{K}^d$.

We assume that A and U are definable over the empty set. Since A is a locally closed, definably connected set then, by Lemma 2.6 (ii), $\pi(A)$ is relatively closed in some definably open set W. Since $\dim_{\mathbf{R}} \pi(A) = 2d$, then either $\pi(A) = W$ or the boundary of $\pi(A)$ in W has $\mathbf{R}$-dimension $2d-1$ (see for example Proposition 2 in [10]). If the latter holds then there is a point y on the boundary of $\pi(A)$ in W, such that $\dim_{\mathbf{R}}(y/\emptyset) = 2d-1$. But, since $\pi(A)$ is relatively closed in W, there is $x \in A$, $\dim_{\mathbf{R}}(x/\emptyset) \geqslant 2d-1$, such that $\pi(x) = y$. Because of the assumption on $Sing_{\mathbf{K}}A$, x must be in $Reg_{\mathbf{K}}A$ and furthermore, by Lemma 3.5, $\pi|Reg_{\mathbf{K}}A$ is a local $\mathbf{K}$-biholomorphism near x, which implies that y is in the interior of $\pi(A)$. Contradiction. This implies that $\pi(A)$ is open.

Let $S_1 = \pi(Sing_{\mathbf{K}}A)$. By assumption, $\dim_{\mathbf{R}}(Cl(S_1)) \leqslant 2n-2$, and since $\pi|A$ is finite-to-one, we also have $\dim_{\mathbf{R}}(\pi^{-1}(Cl(S_1)) \leqslant 2n-2$.

Let $M = A \cap (\mathbf{K}^n \setminus \pi^{-1}(Cl(S_1)))$. Notice that M is a dense, relatively open subset of $Reg_{\mathbf{K}}A$ and that $\dim_{\mathbf{R}}(A \setminus M) \leqslant 2n-2$. Moreover, $\pi|M$ is definably proper over its image.

Let $S \subseteq \mathbf{K}^d$ be the set from Lemma 3.6 (3) and let $V = \pi(M) \setminus S$. Namely, every $x \in V$ has precisely m pre-images in M.

Choose $\phi_1(x), \ldots, \phi_m(x)$ definable (but not necessarily continuous) maps from V into $\mathbf{K}^{n-d}$, which for every $x \in V$ give all the y's in $\mathbf{K}^{n-d}$ such that $(x, y) \in M$. We denote $(\phi_1, \ldots, \phi_m)$ by Φ.

We use the following fact, which appears in different forms in any basic book on complex analytic or algebraic geometry[1]:

Fact. There is a polynomial map $F(\bar{y}_1, \ldots, \bar{y}_m, \bar{x})$, from $\mathbf{K}^{(n-d)(m+1)}$ into $\mathbf{K}^r$, for some r, such that

(i) F is symmetric under any permutation of $\{\bar{y}_1, \ldots, \bar{y}_m\}$.

(ii) For every $(\bar{b}_1, \ldots, \bar{b}_m, \bar{c}) \in \mathbf{K}^{(n-d)(m+1)}$,

$$F(\bar{b}_1, \ldots, \bar{b}_m, \bar{c}) = 0 \Leftrightarrow \bigvee_{i=1}^{m} \bar{c} = \bar{b}_i. \tag{1}$$

Notice that each coordinate function of F can be viewed as a polynomial in $\bar{x}$ whose coefficients, which we denote by $a_i(\bar{y}_1, \ldots, \bar{y}_m)$, are symmetric functions of the $\bar{y}_i$'s.

[1] One way to obtain F is as follows ([6], p.44 for a similar argument): Choose $\bar{u}_1, \ldots, \bar{u}_{(n-d)m}$ to be a sequence of vectors in $\mathbf{K}^{n-d}$, each $n-d$ of them are linearly independent over $\mathbf{K}$, and for any $i = 1, \ldots, (n-d)m$, consider the function $f_i(\bar{Y}, \bar{x}) = \Pi_{j=1}^{m} \langle \bar{u}_i, (\bar{x} - \bar{y}_j)\rangle$. Now $F = (f_1, \ldots, f_{(n-d)m})$ will do the job.

Consider the map, defined on $V \times \mathbf{K}^{n-d}$,

$$\Psi(\bar{x}', \bar{x}) = F(\phi_1(\bar{x}'), \ldots, \phi_m(x'), \bar{x})$$

The ϕ_i's are $\mathbf{K}$-holomorphic on V outside a set S' of $\mathbf{R}$-dimension $2d-1$ (see the proof of Lemma 3.6 (2)). Each coordinate function of Ψ is a polynomial in $\bar{x}$ whose coefficients, which we write as $a_i(\Phi)$, are symmetric functions of $(\phi_1, \ldots, \phi_m)$. It follows that the $a_i(\Phi)$'s are $\mathbf{K}$-holomorphic outside S' and invariant under permutations of the ϕ_i's. We claim that these coefficients are continuous on V and hence $\mathbf{K}$-holomorphic .

Indeed, since $\pi|M$ is definably proper over V and a local homeomorphism on M, we can, for every $\bar{x}' \in S'$, re-define the ϕ_i's in a neighborhood of $\bar{x}'$ so they become continuous there, without changing the value that the a_i's take there. Therefore, each $a_i(\Phi)$ is continuous on V.

It follows from the theorem on the removal of singularities (see [14]) that the $\bar{a}_i$'s are $\mathbf{K}$-holomorphic on V, and hence Ψ is $\mathbf{K}-$*holomorphic* on $V \times \mathbf{K}^{n-d}$. By our choice of F, for every $(\bar{a}', \bar{a}) \in V \times \mathbf{K}^{n-d}$, $\Psi(\bar{a}', \bar{a}) = 0$, if and only if $\bar{a} = \phi_i(\bar{a}')$ for some $i = 1, \ldots m$ (see the proof of Claim 2.25 in [14] for a similar argument).

We now extend Ψ to $W \times \mathbf{K}^{n-d}$, where $W = \pi(Cl_U(M)) = \pi(A)$. We only need to show that each $a_i(\Phi)$, as functions on V, can be extended to W. Since $\dim_{\mathbf{R}}(W \setminus V) \leqslant 2d-2$, it is enough to show that they are bounded at all points of W (see Theorem 2.15 in [14]).

Take $\bar{x}' \in W$ then, since $\pi|A$ is definably proper, the functions ϕ_i, $i = 1, \ldots, m$, are bounded in a neighborhood of $\bar{x}'$, and therefore $a_i(\Phi)$ is bounded there as well. It follows that Ψ can be extended $\mathbf{K}$-holomorphically to $W \times \mathbf{K}^{n-d}$.

It is left to show that $\Psi^{-1}(0) = A$.

By our choice of Ψ, the set $\Psi^{-1}(0)$ is a $\mathbf{K}$-analytic subset of $W \times \mathbf{K}^{n-d}$, containing A. We need to show that the opposite inclusion is true as well.

Take $(\bar{a}', \bar{a}) \in \Psi^{-1}(0) \cap U \subseteq W \times \mathbf{K}^{n-d}$. Because $\bar{a}'$ is in the closure of V there is a curve $\gamma : (0,1) \to V$, which approaches $\bar{a}'$ as t tends to 0. Consider

$$\Psi(\gamma(t), \bar{a}) = F(\phi_1(\gamma(t)), \ldots, \phi_m(\gamma(t)), \bar{a}).$$

By the continuity of F, as $\gamma(t)$ tends to $\bar{a}'$ the function $\Psi(\gamma(t), \bar{a})$ tends to

$$F(lim_{t\to 0}\phi_1(\gamma(t)), \ldots, lim_{t\to 0}\phi_m(\gamma(t)), \bar{a}).$$

(these limits exist since $\pi|A$ is definably proper over $\bar{a}'$).

But Ψ itself is continuous hence,

$$F(lim_{t\to 0}\phi_1(\gamma(t)), \ldots, lim_{t\to 0}\phi_m(\gamma(t)), \bar{a}) = 0.$$

However, our choice of F now implies that for some $i = 1, \ldots, m$, $\bar{a} = \lim_{t\to 0} \phi_i(\gamma(t))$ and therefore $(\bar{a}', \bar{a}) = lim_{t\to 0}(\gamma(t), \phi_i(\gamma(t)))$.

Since for every t we have $(\gamma(t), \phi_1(t), \ldots, \phi_m(t)) \in M$, it follows that (a', a) is in $Cl(M \cap V \times \mathbf{K}^{n-d})$. But this last set is clearly contained in A, and hence we showed that $\Psi^{-1}(0) = A$. □

Remark 4.8. (1) *We will later need the following observation regarding the above proof (we stick to the same notation):*

If $z = (\bar{x}', \bar{x}) \in M$ and $\bar{x}' \in V$, for V as above, then $Rank(D\Psi)_z = n - d$, where $D\Psi$ is the differential with respect to $\mathbf{K}$.

To see that, first notice that if $\bar{x}' \in V$ then, as in the proof above, we may assume that $\phi_1, \dots, \phi_m$ are $\mathbf{K}$-holomorphic in a neighborhood of $\bar{x}'$.

Take F as in the footnote of the previous page. We now can show that the matrix $(\partial\Psi_i/\partial x_j), j = 1, \dots, n-d$, has rank $n-d$ at z over $\mathbf{K}$, as required (similar proofs can be found in most literature on complex analytic geometry).

(2) *The proof of the last lemma implies that, under the assumptions of the lemma, there is a definable set $S \subseteq \pi(A)$, with $\dim_{\mathbf{R}} S \leqslant 2d-2$, such that every $y \in \pi(A) \setminus S$ has exactly m pre-images in A under π. Consider now an arbitrary $y_0 \in \pi(A)$, and let $z_1, \dots, z_t \in A$ be its preimages under π. By definable properness, there are pairwise-dsjoint neighborhoods $U_1, \dots, U_t$ of $z_1, \dots, z_t$, respectively, and a neighborhood $G \subseteq \pi(A)$ of y_0 such that*

$$\pi^{-1}(G) \cap A = \bigcup_i U_i \cap A.$$

By applying the last lemma to each $U_i \cap A$ (and possibly shrinking G) we may assume $\pi(U_i \cap A) = G$ for every $i = 1, \dots, t$. By considering $\pi^{-1}(y) \cap A$ for a generic $y \in G$, it is easy to deduce that $t \leqslant m$. We thus showed that under the assumptions of the last lemma, a generic fiber in A has maximal number of elements.

The following theorem, in the classical setting, is a corollary of Shiffman's Theorem (see [6]). A definable set A is called of *pure dimension d* if the dimension of every nonempty relatively open subset of A is d.

Corollary 4.9. *Let M be a $\mathbf{K}$-manifold, $F \subseteq M$ a definable closed set and A a $\mathbf{K}$-analytic subset of $M \setminus F$ of pure dimension d. If $\dim_{\mathbf{R}} F \leqslant 2d-2$ then $Cl(A)$ is a finitely $\mathbf{K}$-analytic subset of M.*

Proof. Consider each definably connected component of $Reg_{\mathbf{K}}A$. It is sufficient to show that the closure in M of each such component is $\mathbf{K}$-analytic in M. Let A' be such a component and let A'' be the closure of A' in M. Notice that the set of $\mathbf{K}$-singular points of A'' is contained in $Sing_{\mathbf{K}}(A) \cup F$ and therefore, by Lemma 4.5, its $\mathbf{R}$-dimension is at most $2d-2 = \dim_{\mathbf{R}} A - 2$.

After partitioning M further, working in charts, and using Theorem 2.14, we may assume that A'' satisfies the assumptions of Theorem 4.7 thus concluding that A'' (and hence also A) is a finitely $\mathbf{K}$-analytic subset of M. □

One advantage of the o-minimal setting over the general one is our ability to handle well the intersection of one $\mathbf{K}$-analytic set with the closure of another. For that we need the following lemma, which is an analogue of Theorem 3.2 from [16].

Corollary 4.10. *Let M be a $\mathbf{K}$-manifold and let $A_1 \subseteq M$ be a definable, locally $\mathbf{K}$-analytic subset of M such that $Reg_{\mathbf{K}}A_1$ is definably connected and $\dim_{\mathbf{K}} A_1 = d$. Assume that $A_2 \subseteq M$ is another definable, locally $\mathbf{K}$-analytic subset of M. Then either $A_1 \subseteq A_2$ or $\dim_{\mathbf{R}}(Cl(A_1) \cap A_2) \leqslant 2d-2$.*

Proof. It is sufficient to prove the result locally, in a neighborhood of every point in M thus we may assume that A_1 and A_2 are $\mathbf{K}$-analytic subset of M. We apply Lemma 4.3 with A_2 playing the role of A and $Reg_{\mathbf{K}}A_1$ playing the role of M in that lemma (notice that $Reg_{\mathbf{K}}A_1$ is dense in A_1). □

We can now prove a generalization of of Theorem 4.9, with the purity assumption omitted. As was pointed out in [16], the result is not true outside the o-minimal setting.

Theorem 4.11. *Let M be a $\mathbf{K}$-manifold and A a definable closed subset of M. Assume that for every open $U \subseteq M$, $\dim_{\mathbf{R}}(Sing_{\mathbf{K}}(A \cap U)) \leqslant \dim_{\mathbf{R}}(A \cap U) - 2$. Then A is a finitely $\mathbf{K}$-analytic subset of M.*

In particular, every $\mathbf{K}$-analytic subset of M is finitely $\mathbf{K}$-analytic.

Proof. This is an analogue of Corollary 4.2, from [16], which is itself based on Theorem 4.1 there. The proof of 4.1 goes through in our setting, with Theorem 3.2 there replaced in the current paper by Corollary 4.10. The finiteness result is obtained since in the very last step, where originally Shiffman's Theorem was used, we are now using Corollary 4.9 to obtain a finitely $\mathbf{K}$-analytic set. □

Actually, the proof of Theorem 4.1 in [16] shows in particular that the closure in M of every definably connected component of $Reg_{\mathbf{K}}A$ is itself a $\mathbf{K}$-analytic subset of M. Thus, just as in the classical case, we have:

Lemma 4.12. *If A is a $\mathbf{K}$-analytic subset of M then its irreducible components are exactly the closures of the definably connected components of $Reg_{\mathbf{K}}A$.*

The following strong version of the Remmert-Stein Theorem is a direct analogue of Theorem 4.4 from [16]. The proof there works here as well, using Lemma 3.7 and Theorem 4.11.

Theorem 4.13. *Let M be a $\mathbf{K}$-manifold and $E \subseteq M$ a $\mathbf{K}$-analytic subset of M (of arbitrary dimension). If A is a $\mathbf{K}$-analytic subset of $M \setminus E$ then its closure in M is a $\mathbf{K}$-analytic subset of M.*

To sum-up the main results so far, we have:

Corollary 4.14. *If M is a $\mathbf{K}$-manifold and A a closed definable subset of M then the following are equivalent:*
(i) For every open $W \in \mathbf{K}^n$, $\dim_{\mathbf{R}}(Sing_{\mathbf{K}}(A \cap W)) \leqslant \dim_{\mathbf{R}}(A \cap W) - 2$.
(ii) A is $\mathbf{K}$-analytic subset of M.
(iii) A is finitely $\mathbf{K}$-analytic subset of M.

Moreover, in (iii) the open sets and $\mathbf{K}$-holomorphic functions which carve A in each of them can be chosen be definable in the structure $\langle R, <, +, \cdot, A \rangle$.

Proof. The proof of (i) $\Rightarrow$ (iii) follows from Theorem 4.11. (ii) $\Rightarrow$ (i) is Lemma 4.5. We only need to note why the definability clause is true. This follows from 4.7. □

Note that both Clause (i) and Clause (iii) guarantee that $\mathbf{K}$-analytic sets remain $\mathbf{K}$-analytic in every elementary extension.

5. $\mathbf{K}$-analytic subsets of $\mathbf{K}^n$; Chow's Theorem

We now present an o-minimal proof of a strong version of Chow's Theorem. Note that since every analytic subset of $\mathbb{P}(\mathbb{C})$ is definable in the o-minimal structure $\mathbb{R}_{an}$, the classical Chow's Theorem is an immediate corollary.

Theorem 5.1. *Let A be a definable $\mathbf{K}$-analytic subset of $\mathbf{K}^n$. Then A is an algebraic set over $\mathbf{K}$.*

Proof. We may assume that A is irreducible of $\mathbf{K}$-dimension d. We may also assume that A is definable over $\emptyset$. Consider a generic (over $\emptyset$) orthogonal projection π of $\mathbf{K}^n$ onto a d-dimensional $\mathbf{K}$-linear subspace L.

Since A is a closed subset of $\mathbf{K}^n$, it follows from Lemma 2.13 that $\pi|A$ is definably proper over the whole of L, and in particular, $\pi(A)$ is a closed subset of L. By Lemma 4.5 and Lemma 4.7, (this last lemma applied to $M = Reg_{\mathbf{K}}A$), $\pi(A)$ is also open and therefore $\pi(A) = L$.

To simplify notation we will assume now that $L = \mathbf{K}^d$ and π is the projection onto the first d coordinates. By Lemma 4.7, there is a $\mathbf{K}$-holomorphic map $\Psi : \mathbf{K}^n \to \mathbf{K}^m$ such that $A = \Psi^{-1}(0)$. By Theorem 2.17 of [14], every $\mathbf{K}$-holomorphic function on $\mathbf{K}^n$ is a polynomial, therefore Ψ is a polynomial map, and A must be algebraic. □

The following corollary is a very important ingredient in the development of the theory of analytic sets.

Corollary 5.2. *Every bounded $\mathbf{K}$-analytic subset of $\mathbf{K}^n$ is finite.*

Proof. One can prove the result directly but instead, we may use our version of Chow's theorem and then transfer this fact for algebraic sets from the complex and real fields to $\mathbf{K}^n$. □

6. The set of singular points

As in the classical case, we are now ready to prove a strong version of Lemma 4.5.

Theorem 6.1. *Let M be a $\mathbf{K}$-manifold and let A be a $\mathbf{K}$-analytic subset of M. Then $Sing_{\mathbf{K}}A$ is a $\mathbf{K}$-analytic subset of M.*

Proof. Let $A = \bigcup A_i$ be the decomposition of A into its irreducible components. Notice that $z \in Sing_{\mathbf{K}}A$ if and only if $z \in A_i \cap A_j$ for $i \neq j$, or $z \in Sing_{\mathbf{K}}A_i$ for some i. Therefore we may assume that A is irreducible of dimension d. Also, we may assume that $M = U$ a definable open subset of $\mathbf{K}^n$.

Since $Sing_{\mathbf{K}}A$ is closed in A, it is sufficient to prove that it is locally $\mathbf{K}$-analytic. Fix $a \in Sing_{\mathbf{K}}A$ and let $L_1, \ldots, L_k$ be a sequence of independent and generic over a, d-dimensional $\mathbf{K}$-linear subspaces of $\mathbf{K}^n$ and let $U_1, \ldots, U_k$ be the corresponding open sets as in Theorem 2.14. Because the L_i's are generic over a, the point a belongs to the intersection of the U_i's. By Lemma 4.7 for every $i = 1, \ldots, k$ there is a $\mathbf{K}$-holomorphic map $\Psi_i : U_i \to \mathbf{K}^{d_i}$ whose zero set is $A \cap U_i$. Moreover, each L_i has a definable open subset V_i such that the following properties hold, for each $i = 1, \ldots, k$:

(1) $\pi_i^{-1}(V_i) \subseteq Reg_{\mathbf{K}}A$.

(2) For all $z \in A \cap \pi_i^{-1}(V_i)$, $Rank_{\mathbf{K}}(D\Psi_i)_z = n - d$.
(3) For every $z \in Reg_{\mathbf{K}} A$ there is an $i = 1, \ldots, k$ such that $z \in \pi^{-1}(V_i)$.

Indeed, the existence of V_i satisfying (1) and (2) above can be read-off from the proof of Lemma 4.7 and Remark 4.8 which follows it (V_i is the set of points $z' \in Ł_i$ such that A is a $\mathbf{K}$-submanifold at every point of the set $X = \pi^{-1}(z')$ and $\pi_i|A$ is a local $\mathbf{K}$-biholomoprhism at every point of X). As for (3), as was pointed out in the proof of the lemma, for each $z \in Reg_{\mathbf{K}} A$ there is an i such that π_i is generic over z. Let H_i be the orthogonal complement to L_i. Then, by Lemma 2.12, $z + H_i$ intersects A transversally and in particular, every point of intersection is in $Reg_{\mathbf{K}} A$. It follows that $z \in \pi^{-1}(V_i)$.

For every $i = 1, \ldots, k$, let

$$Z_i = \{z \in U_i : Rank_{\mathbf{K}}(D\Psi_i)_z < n - d\}.$$

Claim $(\bigcap_{i=1}^{k} U_i) \cap Sing_{\mathbf{K}} A = \bigcap_{i=1}^{k} Z_I$.

Proof Assume that $z \in (\bigcap_i U_i) \cap Reg_{\mathbf{K}} A$. Then, by (3) above, there is an i such that $z \in V_i$. By (2), $Rank_{\mathbf{K}}(D\Psi_i)_z = n - d$, and so $z \notin Z_i$.

For the opposite inclusion, assume that $Rank_{\mathbf{K}}(D\Psi_i)_z = n - d$ for some $i = 1, \ldots, n$. Then, by the Implicit Functions Theorem, there is a d-dimensional $\mathbf{K}$-submanifold M containing z which is contained in A. Since $\dim_{\mathbf{K}} A = d$, it easily follows that $M = A$ in some neighborhood of z, and thus $z \in Reg_{\mathbf{K}} A$. $\square$

7. Dimension, rank and Remmert's Theorem

Once we know that every bounded analytic subset of $\mathbf{K}^n$ is finite (see Corollary 5.2), the following three results on dimension are standard. We include their proofs for the sake of completeness.

Theorem 7.1. *Let $A \subseteq U \subseteq \mathbf{K}^n$ be an $\mathbf{K}$-analytic subset of a definable open set, $a \in A$. Assume that H is a p-dimensional affine $\mathbf{K}$-subspace of $\mathbf{K}^n$ such that a is an isolated point of $A \cap H$, and let $H^{\perp}$ be the orthogonal complement of H. Then*
(i) the projection π of A onto $H^{\perp}$ is at most finite-to-one, in some neighborhood of a.
(ii) $\dim_a A \leqslant n - p$.

Proof. (i) By continuity of π, there is a neighborhood $V \subseteq \mathbf{K}^n$ of a such that for every $y \in \pi(V)$, $\pi^{-1}(y) \cap A \cap V$ is closed and bounded near a. Since each such fiber is $\mathbf{K}$-analytic in V it is also $\mathbf{K}$-analytic in $\mathbf{K}^n$ and so, by Corollary 5.2, it is either empty or finite. (ii) follows immediately from (i). $\square$

Theorem 7.2. *Assume that $A \subseteq U \subseteq \mathbf{K}^n$ is a $\mathbf{K}$-analytic set and $f : U \to \mathbf{K}^d$ is $\mathbf{K}$-holomorphic . Then the map $x \mapsto \dim_{\mathbf{K}}(A \cap f^{-1}(f(x)))_x$ is upper semicontinuous on A.*

Proof. We need to show that for every natural number $p \geqslant 0$, the set $A_0 = \{x \in A : \dim_{\mathbf{K}}(A \cap f^{-1}(f(x)))_x \geqslant p\}$ is closed in U.

Take $x_0 \in Cl(A_0)$, and denote $A \cap f^{-1}(f(x_0))$ by B. Assume that $\dim_{\mathbf{K}} B_{x_0} = q$ and let H be a generic $\mathbf{K}$-space of K-dimension $n - q$. By Lemma 2.10, x_0 is an isolated point in $B \cap x_0 + H$. Let $V \subseteq \mathbf{K}^n$ be a small ball around x_0 such that $V \cap B \cap x_0 + H = \{x_0\}$. By the continuity of f and since A is closed in U, there is a neighborhood W of x_0 such that for every $x \in A \cap W$, the set $A \cap f^{-1}(f(x)) \cap x + H$ is closed in $\mathbf{K}^n$ and contained in V. By Theorem 7.1, it must be also finite. It follows that every $x \in A \cap W$, is an isolated point of $A \cap f^{-1}(f(x)) \cap x + H$. By 7.1, $\dim_{\mathbf{K}}(A \cap f^{-1}(f(x)))_x \leqslant q$ for all $x \in W$. Since $x_0 \in Cl(A_0)$, it follows that $p \leqslant q$. ☐

Theorem 7.3. *(***The dimension Theorem** *) Let M be an n-dimensional* **K**-*manifold, X, Y definable, irreducible* **K**-*analytic subsets of M. Then every irreducible component of $X \cap Y$ has* **K**-*dimension not less than* $\dim_{\mathbf{K}} X + \dim_{\mathbf{K}} Y - n$.

Proof. We may clearly assume that $M = \mathbf{K}^n$ and A is 0-definable. Since $X \cap Y$ is **K**-biholomorphic with $X \times Y \cap \Delta$, where Δ is the diagonal in $\mathbf{K}^n \times \mathbf{K}^n$ it is sufficient to prove the following:

Given an irreducible d-dimensional **K**-analytic set $A \subseteq \mathbf{K}^n$ and given $a \in A$, if H_0 is an $n-1$-dimensional **K**-linear subspace of $\mathbf{K}^n$, then

$$\dim_{\mathbf{K}}(A \cap a + H_0)_a \geqslant d - 1.$$

Assume to the contrary that $\dim_{\mathbf{K}}(A \cap a + H_0)_a \leqslant d - 2$. By 2.10, there is a **K**-linear space $H \subseteq H_0$, with $\dim_{\mathbf{K}}(H) = n - 1 - (d-2)$ such that a is an isolated point of $A \cap a + H$. We now apply Theorem 7.1 to A, H and $\mathbf{K}^n$, and conclude that $\dim_{\mathbf{K}}(A)_a \leqslant n - (n+1-d) = d - 1$. Contradiction. ☐

☐

We can now prove the following version of Remmert's proper mapping theorem.

Theorem 7.4. *Let $f : M \to N$ be a* **K**-*holomorphic map between definable* **K**-*manifolds. Let $A \subseteq M$ be a* **K**-*analytic subset. Assume that $f(A)$ is a closed subset of N. Then $f(A)$ is a* **K**-*analytic subset of N.*

Proof. The proof is a much simplified variant of the proof of Theorem 6.1 in [16]. Namely, instead of A and $f(A)$ being only locally definable we now have both of these sets definable. The rest of the argument is identical, using Theorem 4.11. The upper semi-continuity of the function $x \mapsto \dim_{\mathbf{K}} f^{-1}(f(x))_x$ is given by Theorem 7.2. ☐

Note that if $f : M \to N$ is a **K**-holomorphic map which is definably proper over N then for every **K**-analytic $A \subseteq M$, $f(A)$ is closed in N and therefore, by the above, it is **K**-analytic in N. This is more or less the precise content of Remmert's original theorem.

Example 7.4.1. *Let A be the following analytic subset of $\mathbb{C}^3$:*

$$A = \mathbb{C} \times \{(0,0)\} \cup \{(0,0)\} \times \mathbb{C} \cup \bigcup_{n=1}^{\infty} \{(z, 1/nz, n) : z \in \mathbb{C}\}.$$

Let π be the projection of A on the first two coordinates. Then

$$\pi(A) = \mathbb{C} \times \{0\} \cup \bigcup_{i=1}^{\infty} \{(z, 1/nz) : z \in \mathbb{C}\},$$

which is a closed set but not analytic.

Although A above has infinitely many irreducible components we expect that a similar example can be obtained with A irreducible.

8. **K**-manifolds as Zariski structures

The initial motivation to consider, model theoretically, compact complex manifolds and their analytic subsets is due to the following theorem:

Theorem 8.1. *[21] Consider the structure whose universe is a compact complex manifold, and whose atomic relations are all the complex analytic subsets of M and of its cartesian products. Then the structure eliminates quantifiers, it is stable of finite Morley Rank and moreover, it is a Zariski structure (with the closed sets taken as the* **K***-analytic ones).*

We are almost ready to prove a generalization of that theorem. But first we need one more definability result.

Lemma 8.2. *Let $f : M \to N$ be a* **K***-holomorphic map between* **K***-manifolds and let A be a* **K***-analytic subset of M. If $f|A$ is definably proper over N then for every $k \in \mathbb{N}$, the set $B = \{y \in N : \dim_{\mathbf{K}} f^{-1}(y) \geqslant k\}$ is a* **K***-analytic subset of N.*

Proof. Consider the set

$$A(k) = \{z \in A : \dim_z f^{-1}f(z) \geqslant k\}.$$

The upper semi-continuity of dimension implies that this set is closed, but moreover it is a **K**-analytic subset of M. The argument for the latter statement is described in details in Whitney's book, [19], in the proof of Theorem 9F on page 240 there. (The two theorems used in this argument, from Chapters 2 and 4, are easily seen to hold in our setting as well. Notice that we only need the analyticity result there).

Since $f|A$ is definably proper, the image of $A(k)$ under f is closed in N and therefore, by Theorem 7.4, it is **K**-analytic in N. This is precisely the set B. □

Together with the dimension Theorem, Remmert mapping theorem, and the decomposition of **K**-analytic sets into finitely many irreducible components we obtain, exactly as in theorem 8.1:

Theorem 8.3. *Let M be a definably compact* **K***-manifold, equipped with all* **K***-analytic subsets of cartesian products of M. Then M eliminates quantifiers, it is stable of finite Morley rank and a (complete) Zariski structure (with the closed sets taken as the* **K***-analytic ones).*

Remarks

1. Notice that not every **K**-manifold in an o-minimal structure, when equipped with all its **K**-analytic subsets, is stable. Indeed, work in the field of real numbers and take for example M to be the open unit disc in **K**, equipped with all semialgebraic analytic subsets of its cartesian products. One of the analytic subsets of $M \times M$ is the graph of the function $z \mapsto \frac{1}{2}z$ but its image in M is an open disc of radius 1/2. This is easily seen to contradict stability.

2. It is not necessary for a **K**-manifold to be compact, or definably compact, in order for the induced structure to be stable. E.g., by Theorem 5.1, if we take $M = \mathbf{K}^n$, (in any o-minimal structure) then the structure induced by all the **K**-analytic sets is just that of an algebraically closed field.

Question Let M be a **K**-manifold and assume that M, when equipped with all **K**-analytic subsets of its cartesian products, is stable. Is it possible to definably "compactify" M? I.e., is M a Zariski open subset of a definably compact **K**-manifold (either in the same structure or in some o-minimal expansion)?

In [14] we discusses several examples of definably compact **K**-manifolds in this nonstandard o-minimal setting and showed how one obtains in this manner new objects, which do not arise in nonstandard models of the theory of compact complex spaces. We now return to one of these examples.

8.1. **Locally modular elliptic curves.** In [14] and in [15] we showed how to view the family of all one-dimensional complex tori as definable in the structure $\langle \mathbb{R}, +, \cdot \rangle$ with the upper half plane $\mathbb{H} \subseteq \mathbf{K}$ as the parameter set. We briefly recall this definition here:

Given τ in $\mathbb{H}$, we consider the half-open parallelogram E_τ in $\mathbb{H}$ whose sides are determined by $1, \tau \in \mathbf{K}$. By covering $Cl(E_\tau)$ with finitely many definable open sets we can endow E_τ with a **K**-manifold structure which in fact "glues" the two opposite sides of the parallelogram.

We thus obtain in every o-minimal expansion of a real closed field a definably compact one dimensional **K**-manifold, call it $\mathcal{E}_\tau$, for every $\tau \in \mathbb{H}(\mathbf{K})$. Moreover, the natural group structure on $\mathcal{E}_\tau$ is a definable **K**-holomorphic map, making $\mathcal{E}_\tau$ into a **K**-group. Several basic results on such **K**-groups are proved in [15], section 5.1.

Assume now that our structure is an o-minimal expansion of a structure which is itself elementarily equivalent to $\mathbb{R}_{an,exp}$. In [15] we gave a characterization of all those $\tau \in \mathbb{H}(\mathbf{K})$ such that $\mathcal{E}_\tau$ is definably **K**-biholomorphic with a nonsingular algebraic projective cubic curve. Let us call $\mathcal{E}_\tau$ a *nonstandard elliptic curve* if it is not such a curve. For example, if the real part of τ, as an element of **K**, is infinitely large, then $\mathcal{E}_\tau$ is nonstandard. Or, if the imaginary part of τ is infinitesimally small (and positive) and the real part is infinitesimally close to an irrational number then $\mathcal{E}_\tau$ is nonstandard.

Consider now $\mathcal{E}_\tau$ with the induced **K**-analytic structure, as above. Since it is definably compact the structure we get is a Zariski structure of finite Morley Rank. Moreover, since it is definably connected of **K**-dimension one, it is also strongly minimal. The proof of the following theorem will appear elsewhere.

Theorem 8.4. *Let $\mathcal{E}_\tau$ be a nonstandard elliptic curve as above. Then, when equipped with all* **K***-analytic subsets of the cartesian products, $\mathcal{E}_\tau$ is a locally modular, strongly minimal structure.*

9. Meromorphic maps

Let M, N be **K**-manifolds, U a definable open subset of M whose complement is a **K**-analytic subset of M. We say that a **K**-holomorphic map $f : U \to N$ is **K**-*meromorphic* if the closure of its graph in $M \times N$ is a **K**-analytic subset of $M \times N$. (Although we will not use it here, the above definition generalizes Definition 2.29 from [14] of a *definably meromorphic function* from a **K**-manifold M into **K**. Namely, f is a **K**-meromorphic map, in our sense, from M into **K** if and only if it can be written locally, at every point of M, as the quotient of two **K**-holomorphic functions).

As the following theorem shows, the requirement about the closure of the graph of f in the above definition is obtained for free in the o-minimal setting.

Corollary 9.1. *Let M, N be **K**-manifolds, $S \subseteq M$ an irreducible **K**-analytic subset of M, and assume that L is a closed definable subset of S which contains $Sing_{\mathbf{K}}S$. If $f : S \setminus L \to N$ is a **K**-holomorphic map and $\dim_{\mathbf{R}} L \leqslant \dim_{\mathbf{R}} S - 2$ then the closure of the graph of f in $M \times N$ is a **K**-analytic subset of $M \times N$.*

Proof. This is an analogue of Theorem 6.4 from [16]. With the obvious adjustments the proof goes through almost in full in our setting. One exception however is Proposition 6.5 from that paper which we use in the proof. There we originally quoted a result of Kurdyka and Parusinski, but it is not difficult to see that a corresponding result can be proved in our setting as well. □

10. Campana-Fujiki

One of the applications of the theory developed thus far, in the classical case, is a generalization of a theorem by Campana and Fujiki (see [5], [9]) which is discussed in [16] (see Theorem 9.2 there). The theorem of Campana and Fujiki has attracted the attention of people in model theory when it was noticed that in some cases one could replace the "Zariski structure machinery" with the geometric tool provided by the theorem, in order to establish connections between certain definable objects and algebraic varieties (see [18], [12], [17]). In [16] we gave a slightly different proof of the theorem and generalized the result from compact complex manifolds to arbitrary definable complex manifolds. Here we will just formulate the generalization to the nonstandard setting and refer to the proof in [16].

We first need the definition of a **K**-holomorphic -vector-bundle (we did not choose the most general one but the one sufficient for our purposes):

Definition 10.1. Let M be a **K**-manifold, *a **K**-holomorphic vector-bundle over M, of dimension d* consists of a **K**-manifold X, a **K**-holomorphic map $\rho : X \to M$, such that:

There is a finite cover of M by definable open sets $M = \bigcup_{i=1}^{k} V_i$, and for each $i = 1, \ldots, k$ there is a **K**-biholomorphism $\phi_i : V_i \times \mathbf{K}^d \to \rho^{-1}(V_i)$ such that $\rho\phi_i(x, y) = x$. Furthermore, if $x \in V_i \cap V_j \neq \emptyset$ then $\phi_i^{-1}\phi_j$ induces a **K**-linear automorphism $g_{i,j}(x)$ of the vector space $\mathbf{K}^d$, in the fiber above x. The map $x \mapsto g_{i,j}(x)$ is **K**-holomorphic from $V_i \cap V_j$ into $GL(d, \mathbf{K})$.

In the theorem below, we use the term *a Zariski open subset* of a $\mathbf{K}$-analytic set S, to mean the complement of another $\mathbf{K}$-analytic set in S. For $S \subseteq M \times N$, we let π_M denote the projection onto M.

Theorem 10.2. *Assume that N, M are $\mathbf{K}$- manifolds, and S is an irreducible $\mathbf{K}$-analytic subset of $N \times M$. Then there is a $\mathbf{K}$-holomorphic vector-bundle $\pi : V \to M$, a $\mathbf{K}$-meromorphic map $\sigma : S \to \mathbb{P}(V)$, and a Zariski open subset S^0 of S such that for all $(b, a), (b', a) \in S^0$, $\sigma(b, a) = \sigma(b', a)$ if and only if $S_b = S_{b'}$ near a, and the following diagram of $\mathbf{K}$-meromorphic maps is commutative (we view π as a $\mathbf{K}$-meromorphic map from the projectivization*
$P(V)$ of V into M)

$$\begin{array}{ccc} S^0 & \xrightarrow{\ \sigma\ } & \mathbb{P}(V) \\ & {\scriptstyle \pi_M} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & M & \end{array}$$

11. A finite version of the coherence theorem

The coherence of analytic sheaves, due to Cartan and Oka, is one of the most important results in the theory of complex analytic spaces. Our goal in this section is to formulate the essential ingredients of the theorem in our setting, and show how o-minimality yields here once again a strong finiteness result.

Notation Let M be a $\mathbf{K}$-manifold of dimension n. For $U \subseteq M$ a definable open set, we let $\mathcal{O}(U)$ denote the ring of $\mathbf{K}$-holomorphic functions on U.

For $p \in M$, we will denote by $\mathcal{O}_p$ the ring of germs at p of $\mathbf{K}$-holomorphic functions. When $M = \mathbf{K}^n$, we sometimes write $\mathcal{O}_{n,p}$. If A is a $\mathbf{K}$-analytic subset of M, and $p \in M$ then we denote by $\mathcal{I}(A)_p$ the ideal in $\mathcal{O}_p$ of all germs at p which vanish on A in some neighborhood of p. For a definable open $V \subseteq M$, we let $\mathcal{I}_V(A)$ be the set of all $\mathbf{K}$-holomorphic functions on V which vanish on $A \cap V$; this is a module over $\mathcal{O}(V)$. When A is a $\mathbf{K}$-analytic subset of V we just write $\mathcal{I}(A)$ for $\mathcal{I}(A)_V$.

A function $f : A \to \mathbf{K}$ is called $\mathbf{K}$-holomorphic if there is an open definable set V, $A \subseteq V \subseteq U$, and a $\mathbf{K}$-holomorphic function on V which extends f. For p in A, we denote by $\mathcal{O}(A)_p$ the ring of of germs at p of $\mathbf{K}$-holomorphic functions on $A \cap W$ for some open neighborhood W of p. The ring $\mathcal{O}(A)_p$ can be identified with the quotient ring $\mathcal{O}_p/\mathcal{I}(A)_p$.

Our formulation of the Coherence Theorem is as follows:

Theorem 11.1. *Let M be a $\mathbf{K}$-manifold and $A \subseteq M$ a $\mathbf{K}$-analytic subset of M. Then there are finitely many open sets $V_1, \dots, V_k$ whose union covers M and for each $i = 1 \dots, k$ there are finitely many $\mathbf{K}-$holomorphic functions $f_{i,1}, \dots, f_{i,m_i}$ in $\mathcal{I}_{V_i}(A)$, such that for every $p \in V_i$ the functions $f_{i,1}, \dots, f_{i,m_i}$ generate the ideal $\mathcal{I}(A)_p$ in $\mathcal{O}_p$.*

Moreover, the V_i's and the $f_{i,j}$'s are all definable over the same parameters defining M and A.

Remarks

1. The fact that at each point $p \in A$ the ideal $\mathcal{I}(A)_p$ is finitely generated was already established in [14]. The content of the Coherence Theorem as well as Theorem 11.1 is of course much stronger than that.

2. If we eliminate in the theorem the requirement for finitely many V_i's then what we have is the coherence of a certain sheaf of $\mathbf{K}$-holomorphic functions, just as in the classical case.

3. If M is a compact complex manifold then the theorem above follows immediately from the classical Coherence Theorem, together with the compactness of M. Once again, o-minimality plays a similar role to compactness and yields a finiteness result even when the M is not compact (or even definably compact).

4. Because of the "moreover" clause, it is sufficient to prove the theorem in a sufficiently saturated elementary extension of the original structure, and the same V_i's and $f_{i,j}$'s will work for all elementarily equivalent structures. Indeed, assume that we already proved the theorem for a saturated structure, with the V_i's and the $f_{i,j}$'s all $\emptyset$-definable, and fix V_i. It follows that for every definable family of functions $\{h_{a,p} : p \in A \cap V_i, a \in S\}$, where each $h_{a,p}$ is in $\mathcal{I}(A)_p$, there exist m_i definable families

$$\{\{g_{a,p,j} : p \in A \cap V_i, a \in S\} : i = 1, \dots, m_i\},$$

each $g_{a,p,j}$ in $\mathcal{O}_p$, such that for all $a \in S$ and $p \in A \cap V_i$,

$$h_{a,p} = \Sigma_{j=1}^{m_i} g_{a,p,j} f_{i,j},$$

in $\mathcal{O}_p$. When quantifying over a and p this last equality carries over to any elementarily equivalent structure.

The proof of Theorem 11.1 follows closely ideas from Sections 8,9 in Chapter 8 of [19]. The only novelty is the fact that we are trying to obtain a finite covering of M. The proof requires the following two subtheorems.

We first state the two subtheorems, and before proving them we will show how indeed the two imply Theorem 11.1.

Theorem 11.2. *Assume that $U \subseteq \mathbf{K}^n$ is a definable open set and $A \subseteq U$ an irreducible $\mathbf{K}$-analytic subset of U of dimension d. Assume also:*
(i) The projection π of A on the first d coordinates is definably proper over its image, and $\pi(A)$ is open in $\mathbf{K}^d$.
(ii) There is a definable set $S \subseteq \mathbf{K}^d$, of $\mathbf{R}$-dimension $\leqslant 2d-2$ and a natural number m, such that $\pi|A$ is m-to-1 outside the set $A \cap \pi^{-1}(S)$, π is a local homeomorphism outside of the set $\pi^{-1}(S)$, and $A \setminus \pi^{-1}(S)$ is dense in A.
(iii) The coordinate function $z \mapsto z_{d+1}$ is injective on $A \cap \pi^{-1}(x')$ for every $\mathbf{R}$-generic $x' \in \pi(A)$. Namely, for all $z, w \in \pi^{-1}(x')$, if $z_{d+1} = w_{d+1}$ then $z = w$.

Then, there is a definable open set $U' \subseteq U$ containing A, a natural number s and $\mathbf{K}$-holomorphic functions $G_1, \dots G_r, D : U' \to \mathbf{K}$, such that for every $p \in A$ and $f \in \mathcal{O}_p$, if we let $g_1, \dots, g_r, \delta$ be the germs at p of $G_1, \dots, G_r, D$, respectively, then:

$$f \in \mathcal{I}(A)_p \Leftrightarrow \exists f_1 \dots, f_r \in \mathcal{O}_p \ (\delta^s f = f_1 g_1 + \dots + f_r g_r).$$

For the next theorem we need one more definition. Let $p \in \mathbf{K}^n$ and $g_1, \ldots, g_t$ germs at p of $\mathbf{K}$-holomorphic maps into $\mathbf{K}^N$. The *module of relations associated to* $g_1, \ldots, g_t$ *at* p is the module over $\mathcal{O}_p$ defined as

$$R_p(g_1, \ldots, g_t) = \{(f_1, \ldots, f_t) \in \mathcal{O}_p^t : f_1 g_1 + \cdots + f_t g_t = 0\}.$$

Theorem 11.3. *Assume that A is a $\mathbf{K}$-analytic subset of $U \subseteq \mathbf{K}^n$ and assume that $G_1, \ldots, G_t$ are $\mathbf{K}$-holomorphic maps from A into $\mathbf{K}^N$. Then we can write A as a union of finitely many relatively open sets $A_1, \ldots, A_m$ such that on each A_i the following holds:*

There are finitely many tuples of $\mathbf{K}$-holomorphic functions on A_i, $\{(H_{j,1}, \ldots, H_{j,t}) : j = 1, \ldots, k\}$, $k = k(i)$, with the property that for every $p \in A_i$, the module $R_p(g_1, \ldots, g_t)$ equals its submodule generated by $\{(h_{j,1}, \ldots, h_{j,t}) : j = 1, \ldots, k\}$ over $\mathcal{O}_p$ (where g_i and $h_{i,j}$ are the germs of G_i and $H_{i,j}$ at p, respectively).

Let us see first how the two subtheorems above, taken together, imply Theorem 11.1:

Our intention, of course, is to use Theorem 2.14, and Theorem 3.6. For that, we need to treat each of the irreducible components of A separately. The missing ingredient is the following:

Claim Assume that $A = A_1 \cup \cdots \cup A_r$, where each of the A_i's is $\mathbf{K}$-analytic in U. If Theorem 11.1 is true for each of the A_i's then it holds for A as well.

To prove the claim, we basically repeat the argument of Theorem 8C, p.279 from [19]: We may assume that $r = 2$.

By our assumptions on each of the A_i's we may assume, after replacing U by an open subset, that there are $\mathbf{K}$-holomorphic Φ_i, $i = 1, \ldots, s$ in $\mathcal{I}(A_1)$ which at every point $p_1 \in A_1$ generate $\mathcal{I}(A_1)_{p_1}$ in $\mathcal{O}_{p_1}$. Similarly, there are Ψ_j, $j = 1, \ldots, t$ in $I(A_2)$.

Consider the relation module associated to the tuple $(\Phi, \Psi) = (\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t)$, at every point $p \in U$. By Theorem 11.3, we may assume, possibly after replacing U by an open subset, that there are $(\xi^i, \eta^i) = (\xi_1^i, \ldots, \xi_s^i, \eta_1^i, \ldots, \eta_t^i)$, $i = 1, \ldots, r$, which, at every point $p \in U$, generate the relation module $R_p(\phi, \psi)$. Let

$$\Theta^i = \Sigma_{j=1}^s \xi_j^i \phi_j, \ \ i = 1, \ldots, r.$$

As is shown in [19], the germs of these functions generate $\mathcal{I}(A_1 \cup A_2)_p$ in $\mathcal{O}_p$ at every $p \in A_1 \cap A_2$. All other points of $A_1 \cup A_2$ are handled by Φ and Ψ separately. We thus finished proving the claim.

Using the claim, we may assume that A is of pure $\mathbf{K}$-dimension d. As in the proof or Theorem 2.14, we consider a sequence of generic independent $\mathbf{K}$-subspaces of $\mathbf{K}^n$ of dimension d, $L_1, \ldots, L_{2n+1}$. For each i choose a generic orthogonal basis $B = \{v_1, \ldots, v_n\}$ for $\mathbf{K}^n$ such that $v_1, \ldots, v_d$ form a basis for L_i. By Fact 2.8, every m-subset of B generate a generic m-dimensional subspace. In particular, if $m = d+1$ and L is such a $d+1$-dimensional subspace then for every $x \in A$ which is generic over the parameters defining the L_i's, the space L is generic over x. Hence, by Theorem 2.10 (ii), $x + L^{\perp}$ intersects A at exactly x.

We may now replace U by each one of the U_i's as in the proof of Theorem 2.14 and after performing a suitable linear transformation, we may assume that the orthogonal projection onto L_i is actually the projection of $\mathbf{K}^d$ onto the first d coordinates. Therefore, assumptions (i), (ii) and (iii), from Theorem 11.2 now hold, because of Theorems 2.14, 3.6 and the above discussion. We may therefore apply Theorem 11.2 and obtain $\mathbf{K}$-holomorphic functions on A, $G_1, \ldots, G_r$ and D which satisfy the statement in that theorem.

Take at each point in A the relation module associated to $(G_1, \ldots, G_r, D^s)$, as given by 11.2, with $N = 1$. Next, replace A by an A_i as given by Theorem 11.3 and let $\{(H_{j,1}, \ldots, H_{j,r+1}) : j = 1, \ldots, k\}$ be as in that theorem. We claim that $H_{1,r+1}, \ldots, H_{k,r+1}$ generate, at each $p \in A_i$, the ideal $\mathcal{I}(A_i)_p$ in $\mathcal{O}_p$.

If $f \in \mathcal{I}(A_i)_p$ then, by Theorem 11.2, there exist $f_1, \ldots, f_r \in \mathcal{O}_p$ such that $F = (f_1, \ldots, f_r, f) \in R_p(g_1, \ldots, g_r, \delta^s)$. By our assumptions, F is in the submodule of $\mathcal{O}_p^{r+1}$ generated by the germs at p of $\{(H_{j,1}, \ldots, H_{j,r+1}) : j = 1, \ldots, k\}$ over $\mathcal{O}_p$ and in particular, f is in the ideal generated by the germs at p of $\{H_{j,r+1} : j = 1, \ldots, k\}$. As for the opposite inclusion, it follows from Theorem 11.3 that this last ideal is contained in $\mathcal{I}(A_i)_p$.

We therefore showed how Theorem 11.1 follows from Theorems 11.2 and 11.3.

Proof of Theorem 11.2

We follow the proof of Theorem 9B on p. 280 of [19].

For $\ell \leqslant n$, we will denote by $\mathbf{K}^\ell$ the subspace $z_{\ell+1} = \ldots = z_n = 0$, and by $\pi_\ell : \mathbf{K}^n \to \mathbf{K}^\ell$ the natural projection. For $p \in \mathbf{K}^n$, $\ell \leqslant n$ and $p' = \pi_\ell(p)$, the ring $\mathcal{O}_{\ell,p'}$ has a natural embedding into $\mathcal{O}_{n,p}$, and hence we consider it as a subring of $\mathcal{O}_{n,p}$ and denote it also by $\mathcal{O}_{\ell,p}$. In other words, $\mathcal{O}_{\ell,p}$ is the subring of $\mathcal{O}_{n,p}$ consisting of functions depending on variables $z_1, \ldots, z_\ell$ only. We denote by $\mathcal{O}(A)_{\ell,p}$ the ring $\mathcal{O}(A)_p \cap \mathcal{O}_{\ell,p}$.

By working in charts, we may assume that $M = U$ is a definable open subset of $\mathbf{K}^n$ and $A \subseteq U$ is $\mathbf{K}$-analytic in U. For $p \in \mathbf{K}^n$, we denote by p' the projection $\pi(p)$. We also denote by z the tuple of variables $(z_1, \ldots, z_n)$ and by z' the tuple $(z_1, \ldots, z_d)$.

Claim 11.4. *Let f be a $\mathbf{K}$-holomorphic function on A. Then there is a monic polynomial $P_f(z', u) \in \mathcal{O}(V)[u]$ of degree m such that $P_f(z', f(z))$ vanishes on A (P might be a reducible polynomial).*

Proof. For every $z' \in V \setminus S$ (S as in Clause (ii) of the theorem), let $\phi_1(z'), \ldots, \phi_m(z')$, be all points in $\mathbf{K}^{n-d}$ such that $(z', \phi_i(z')) \in A$ and write $\phi_i = (\phi_{i,d+1}, \ldots, \phi_{i,n})$. As we saw, the ϕ_i's are $\mathbf{K}$-holomorphic outside a set of $\mathbf{R}$-dimension $2d-1$.

Now, take an arbitrary $f \in \mathcal{O}(A)$, and consider the polynomial $P(X_1, \ldots, X_m, Y) = \Pi_{i=1}^m (Y - X_i)$. Since P is symmetric in $X_1, \ldots, X_m$, it follows that the function

$$G(z', Y) = \Pi_{i=1}^m (Y - f(z', \phi_i(z')))$$

is $\mathbf{K}$-holomorphic in $V \times \mathbf{K}$ and can be written as a polynomial in Y over the ring $\mathcal{O}(V)$ (see Lemma 4.7 for a similar argument). We claim that $G(z', f(z)) = 0$ for every $z = (z', z'') \in A$.

Indeed, for every $z' \in \mathbf{K}^d \setminus S$ and $z = (z', z'') \in A$, we have $z'' = \phi_i(z')$ for some $i = 1, \dots, m$ and hence $G(z', f(z)) = 0$. But $\pi^{-1}(\mathbf{K}^d \setminus S) \cap A$ is dense in A therefore $G(z', f(z)) = 0$. □

For each $i = d+1, \dots, n$, we denote by Z_i the coordinate function from $\mathbf{K}^n$ into $\mathbf{K}$ that assigns to each $a \in \mathbf{K}^n$ its i-th coordinate. We denote by $P_i(z', u)$ the polynomial $P_{Z_i}(z', u)$ from Claim 11.4. Each $P_i(z', u)$ is a monic polynomial in u, of degree m, over $\mathcal{O}(V)$, and $P_i(z', z_i)$ vanishes on A.

We denote by $D(z')$ the discriminant of the polynomial $P_{d+1}(z', -)$. Namely,

$$D(z') = \Pi_{1 \leqslant i < j \leqslant m}(\phi_{i,d+1}(z') - \phi_{j,d+1}(z')).$$

By the symmetric nature of D, it is $\mathbf{K}$-holomorphic on V.

Claim 11.5. *The zero set of $D(z')$ is nowhere dense in V.*

Proof. From Assumptions (ii) and (iii) it follows that there is a definable open dense subset $V_0 \subseteq V$ such that for every $a' \in V_0$ the pre-image $\pi^{-1}(a')$ contains exactly m points and the $d+1$-coordinate of these points are all distinct. Since $P_{d+1}(a', u)$ has degree m in the u-variable and $P_{d+1}(a', z_{d+1})$ vanishes on A we obtain that for all $a' \in V_0$, the function Z_{d+1} is a bijection between $\pi^{-1}(a')$ and the zero set of $P_{d+1}(a', u)$. Thus, for $a' \in V_0$, $P_{d+1}(a', z_{d+1})$ has only simple roots and $D(a') \neq 0$. Since $D(z')$ does not have zeroes in V_0, the zero set of $D(z')$ is nowhere dense in V □

The main ingredient in the proof of the theorem is the following:

Claim 11.6. *For each $i = d+2, \dots, n$ there is a polynomial $R_i(z', u) \in \mathcal{O}(V)[u]$ such that $D(z')Z_i - R_i(z', z_{d+1})$ vanishes on A.*

Proof. See Theorem 4A, p.84 in [19], and take P_i to be g_i from that theorem. The proof there works in our setting as well. □

Let $G_1, \dots, G_r$ be a listing of the functions $P_{d+1}, \dots, P_n$ and $D(z')z_i - R_i(z', z_{d+1})$.

We fix $p \in A$. and for $i = d+1, \dots, n$ we denote by $p_i(z', z_i)$ the germs of the functions $P_i(z', z_i)$ in the ring $\mathcal{O}(A)_p$. Notice that all $p_i(z', z_i)$ are polynomials in z_i over $\mathcal{O}(A)_{d,p}$ of degree at most m. We denote by δ the germ of $D(z')$ in $\mathcal{O}(A)_p$, and, for $i = d+2, \dots, n$, we denote by $r_i(z', z_{d+1})$ the germ of the function $R_i(z', z_{d+1})$ in the ring $\mathcal{O}(A)_p$. Again we have that each $r_i(z', z_{d+1})$ is a polynomial in z_{d+1} over $\mathcal{O}(A)_{d,p}$.

Let J_p be the ideal of $\mathcal{O}_p$ generated by the germs of $G_1, \dots, G_r$ at p and let $s = (m-1)(n-(d+1))$. In order to prove the theorem we need to prove:

For every $f \in \mathcal{O}_{n,p}$, $f \in \mathcal{I}(A)_p$ if and only if $\delta^s f \in J_p$.

Assume first that $\delta^s f \in J_p$. By Claim 11.4 and Claim 11.6, $J_p \subseteq \mathcal{I}(A)_p$, hence $\delta^s f \in \mathcal{I}(A)_p$. Since $\delta^s f$ vanishes on A near p, the germ f vanishes at all points of A

near p where $\delta \neq 0$. It follows from Claim 11.5 that the latter set is dense in A and hence $f \in \mathcal{I}(A)_p$.

The opposite inclusion will again be proved via a sequence of claims.

Claim 11.7. *For every $h(z', z_{d+1}) \in \mathcal{O}_{d+1,p}$, if $h \in \mathcal{I}(A)_p$ then h is divisible by $p_{d+1}(z', z_{d+1})$.*

Proof. Consider a small open definable set $U_1 = V_1 \times W_1 \subseteq \mathbf{K}^d \times \mathbf{K}^{n-d}$ containing p such that h is defined on U and such that $A \cap (V_1 \times \partial W_1) = \emptyset$. By our choice of p_{d+1}, for every $z' \in V_1$, every zero of $p_{d+1}(z', -)$ (in the $d+1$ variable) is also a zero of $h(z', -)$, of the same multiplicity. It follows that p_{d+1} divides $f(z', z_{d+1})$ in $\pi_{d+1}(U_1)$ (see p.17 in [14] for a similar argument). $\square$

Claim 11.8. *For every $g(z) \in \mathcal{O}_{n,p}$ there is $h(z', u_{d+1}, \ldots, u_n) \in \mathcal{O}_{d,p}[\bar{u}]$ of degree less than m in each variable $u_{d+1}, \ldots, u_n$ such that $g(z)$ is equivalent to $h(z', z_{d+1}, \ldots, z_n)$ modulo J_p.*

Proof. Given $g(z)$, we first divide g by $p_n(z_n)$, using Weierstrass Division Theorem (see Theorem 2.23 in [14]). The germ $g(z)$ is then equivalent, modulo J_p, to a polynomial $r(z_n)$ over $\mathcal{O}_{n-1,p}$ whose degree is smaller than $m(= deg(p_n))$. We next consider each of the coefficients of $r(z_n)$, and replace it, after dividing by $p_{n-1}(z_{n-1})$, with a polynomial in z_{n-1}, over $\mathcal{O}_{n-2,p}$, of degree at most m. We continue until we get, modulo J_p, a polynomial in $z_{d+1}, \ldots, z_n$, over $\mathcal{O}_{d,p}$. $\square$

Claim 11.9. *For every $g(z) \in \mathcal{O}_{n,p}$ there is $q(z', u) \in \mathcal{O}_{d,p}[u]$ such that $\delta^s g(z)$ is equivalent modulo J_p to $q(z', z_{d+1})$.*

Proof. Let $h(z', u_{d+1}, \ldots, u_n) \in \mathcal{O}_{d,p}[\bar{u}]$ be as in Claim 11.8. Each monomial appearing in h has total degree at most s. Hence $\delta^s h(z) = h_1(z', z_{d+1}, \delta z_{d+2}, \ldots, \delta z_n)$, where $h_1(z', u_{d+1}, \ldots, u_n) \in \mathcal{O}_{d,p}[\bar{u}]$. Since, for $i = d+2, \ldots, d_n$, each δz_i is equivalent (modulo J_p) to $r_i(z', z_{d+1})$, we obtain that $\delta^s h$ is equivalent to a polynomial over $\mathcal{O}_{d,p}$ in one variable z_{d+1}. $\square$

We can now finish the proof of Theorem 11.2. If $f \in \mathcal{I}(A)_p$ then, by Claim 11.9, $\delta^s f$ is equivalent (modulo J_p) to some $q(z', z_{d+1})$, where $q(z', u) \in \mathcal{O}_{d,p}[u]$. Since $J_p \subseteq \mathcal{I}(A)_p$, we have $q(z', z_{d+1}) \in \mathcal{I}(A)_p$. By Claim 11.7, $q(z', z_{d+1})$ is divisible by $p_{d+1}(z', z_{d+1})$ and therefore it belongs to J_p. $\square$

Proof of Theorem 11.3.

The proof we use here is almost identical to the proof of the classical corresponding theorem in [19] (see p. 275 Theorem 8B in Chapter 8).

We first use induction on n, the dimension of the domain space. The case $n = 0$ is just about vector spaces and is easy to verify (see [19]).

Take $n > 0$ and assume first that the dimension of the target space is $N = 1$. Take $f = (f_1, \ldots, f_s)$ a $\mathbf{K}$-holomorphic map on A.

Claim 1 Let $V = \pi(U) \subseteq \mathbf{K}^{n-1}$ be the projection of U onto the first $n-1$ coordinates. We may assume that V is open and that $f_1, \ldots, f_s$ are polynomials in the

n-th variable with coefficients in $\mathcal{O}(V)$.

We first take a sequence of $2n+1$ generic and independent $n-1$-dimensional $\mathbf{K}$-subspaces and cover U by finitely many open sets, as in Theorem 2.14, with respect, simultaneously, to the sets $Z(f_1), \dots, Z(f_s)$. We therefore may assume that the projection map onto the first $n-1$ coordinates, when restricted to the zero set for each f_i is definably proper over its image, finite-to-one and its image is open in $\mathbf{K}^{n-1}$. We now want to use the following strong form of the Weierstrass Preparation Theorem:

Under the above assumptions, for each i, there is a Weierstrass polynomial $\omega_i(z', z_n)$ in the variable z_n, whose coefficients are $\mathbf{K}$-holomorphic functions on $V = \pi(U)$, and there is a $\mathbf{K}$-holomorphic nonvanishing function $u_i(z', z_n)$ on $V \times \mathbf{K}$ such that

$$\forall (z', z_n) \in V \times \mathbf{K} \;\; f_i(z', z_n) = u_i(z', z_n)\omega_i(z', z_n).$$

This is a modified, "global" version of Theorem 2.20 in [14]. The assumption on the zero set of $f_i(z', z_n)$ allows for this modification (notice that $V \times \mathbf{K}$ here satisfies the assumptions on the open set W, at the top of p.16, in [14]).

It is now easy to verify that it is sufficient to prove the theorem for the relation module associated to the germs of $\omega_1, \dots, \omega_s$.

Let $\omega = (\omega_1, \dots, \omega_s)$ be polynomials in z_n over $\mathcal{O}(V)$, where V open in $\mathbf{K}^{n-1}$. Assume that the degree of all of these polynomials is at most m. Let $\pi = \pi_{n-1}$ be the projection onto the first $n-1$ coordinates, and for $z \in \mathbf{K}^n$, let $z' = \pi(z)$. The following local claim will allow us to treat only tuples of polynomials in z_n, $P = (P_1, \dots, P_s)$, in $R_p(\omega)$.

Claim 2 For $p \in U$, let $\phi = (\phi_1, \dots, \phi_s)$ be a tuple of $\mathbf{K}$-holomorphic functions, $\phi \in R_p(\omega)$. Then there are tuples $\{\psi^i = (\psi^i_1, \dots, \psi^i_s) : i = 1, \dots, t\}$, where each ψ^i_j is a polynomial in z_n of degree $\leqslant m$ over $\mathcal{O}_{p'}$, such that ϕ is in the module generated by $\psi^1, \dots, \psi^t$ over $\mathcal{O}_p$.

We omit the proof of Claim 2 since the one given in Whitney's book goes through in our setting as well (see the proof of statement (b) on p. 276, from [19]).

The remainder of the argument is very close to Whitney's proof. However, because of our special formulation of the theorem and because Whitney's proof contains a small error (involving indices, see bottom of p. 277) of we repeat the proof in almost full details.

Claim 2 allows us to consider, for each $p \in U$, only tuples $\phi = (\phi_1, \dots, \phi_s) \in R_p(\omega)$ where each ϕ_i is polynomial in z_n of degree $\leqslant m$ over some neighborhood of $\pi(p)$.

For each $i = 1, \dots, s$, we write $\omega_i = \Sigma_{j=0}^m A_{i,j} x_n^j$. Notice that if $P = (P_1, \dots, P_s)$ is a tuple of $\mathbf{K}$-holomorphic polynomials in the variable z_n, all over $\mathcal{O}_{p'}$, and $P_i = \Sigma_{i=0}^m B_{i,j} z_n^j$ then

$$P \in R_p(\omega) \Leftrightarrow \text{ for every } k = 0, \dots, 2m \;\; \Sigma_{i=1}^s \Sigma_{j=0}^m B_{i,j} A_{i,k-j} = 0.$$

(where $A_{i,k-j}$ is taken to be zero when $k-j < 0$ or $k-j > m$).

We want to translate this last condition to another relation module, of **K**-holomorphic maps over open subsets of $\mathbf{K}^{n-1}$. We do it as follows: For each $k = 0, \ldots, 2m$ and for each $i = 1, \ldots, s$, we consider the tuple $A_i^k = (a_{i,j})_{j=0,\ldots,m}$, defined by

$$a_{i,j} = \begin{cases} A_{i,k-j} & \text{if } 0 \leqslant k-j \leqslant m \\ 0 & \text{otherwise} \end{cases}.$$

We then let, for $k = 0, \ldots, 2m$, $G^k = (G_0^k, \ldots, G_{s(m+1)}^k)$ be the $s(m+1)$- tuple obtained by the concatenation $A_1^{k\frown} \cdots {}^\frown A_s^k$.

Finally, we consider an $s(m+1)$-tuple $H = (H_0, \ldots, H_{s(m+1)})$ of **K**-holomorphic maps from $V = \pi(U) \subseteq \mathbf{K}^{n-1}$ into $\mathbf{K}^{2m+1}$, defined by

$$H_i = \begin{pmatrix} G_i^0 \\ \cdot \\ \cdot \\ \cdot \\ G_i^{2m} \end{pmatrix}.$$

By induction on n, the theorem holds for the relation modules of the form $R_q(H_1, \ldots, H_{s(m+1)})$, for $q \in V$. Therefore, after replacing V by an open subset, we may assume that there is a sequence of **K**-holomorphic maps $B^1, \ldots, B^t$, where each B^j is an $s(m+1)$-tuple of **K**-holomorphic functions on V, such that at every point $q \in V$, the relation module $R_q(H)$ is generated by the germs of the B^j's over $\mathcal{O}_q$. Now, each B^j stores the coefficients of s polynomials $P^j = (P_1^j, \ldots, P_s^j)$ of degree $\leqslant m$, in the variable z_n. We claim that $P^1, \ldots, P^t$ generate, at every point $p \in U$, the relation module $R_p(\omega)$.

By Claim 2, it is sufficient to check $(\phi_1, \ldots, \phi_s) \in R_p(\omega)$, where the ϕ_i's are polynomial maps of degree $\leqslant m$ in z_n, over $\mathcal{O}_{\pi(p)}$. Our construction implies the result for such polynomials.

We now finished the proof for the case $N = 1$. For the general case, see the argument on p.278 in [19]. □

The proof of Theorem 11.1 is thus finished, except for the "moreover" clause. For that, notice that the only external parameters which were used in the argument were the ones needed for the linear subspaces and projections from Theorem 2.14. It is true that in the proof we referred to generic subspaces, thus requiring possibly external parameters, but the properties satisfied by these subspaces (e.g. clauses (i), (ii) and (iii) of Theorem 11.3) can all be expressed in a first order way and hence, by "definable choice" in o-minimal structures, these subspaces can be chosen to be definable over the original parameters. □.

12. Appendix

We are going to prove Theorem 4.2. We start with several lemmas, interesting on their own right.

For $W \subseteq \mathbf{K}^n$ a definable open set and $f : W \to \mathbf{K}$, we denote by $W_{\mathbf{R}}$ the set W, when viewed as a subset of $\mathbf{R}^{2n}$, and by $f_{\mathbf{R}}(x, y)$ the associated **R**-function from $W_{\mathbf{R}}$

into $\mathbf{R}^2$. Using the real and imaginary parts of f, we write, for every $x+iy \in W$,

$$f_{\mathbf{R}}(x,y) = f(x+iy) = u_f(x,y) + iv_f(x,y).$$

Recall that $\mathbf{R}^n$ is viewed as a subset of $\mathbf{K}^n$ by identifying it with the set $\{(x,0) \in \mathbf{R}^{2n}\}$.

Lemma 12.1. *Assume that $W \subseteq \mathbf{K}^m$ is a definably connected open set and $G \subseteq W \cap \mathbf{R}^m$ is a definable, nonempty, relatively open subset of $\mathbf{R}^m$. Assume that $f, g : W \to \mathbf{K}$ are $\mathbf{K}$-holomorphic functions such that $f|G = g|G$. Then $f(z) = g(z)$ for all $z \in W$.*

Proof. Because W is definably connected it is sufficient to find an open subset of W where the two functions agree. Equivalently, (see Theorem 2.13 (2) in [14]) it is sufficient to find a point in W where f and g have the same partial $\mathbf{K}$-derivatives, of every order.

For any point $a \in G \subseteq W$, the partial $\mathbf{K}$-derivatives of f and g at a of first order, can be calculated along $W \cap \mathbf{R}^m$. Since f and g agree on G it follows that the partial $\mathbf{K}$-derivatives of f and g, of first order, agree on G. We may proceed, and thus show that all the $\mathbf{K}$-derivatives of f and g, of every order, agree at every point in G. It follows that f and g agree on all of W. $\square$

In the language of the complex numbers the theorem below says that if f is a definable (in some o-minimal structure) holomorphic function of n-variables then $Re(f)$ and $Im(f)$ can be extended to definable holomorphic functions of $2n$-variables.

Lemma 12.2. *Let $f : G \to K$ be a $\mathbf{K}$-holomorphic function on some definable open $G \subseteq \mathbf{K}^n$. Then the functions $u_f(x,y)$ and $v_f(x,y)$, of $2n$ $\mathbf{R}$-variables, can be extended to $\mathbf{K}$-holomorphic (in particular definable) functions of $2n$ $\mathbf{K}$-variables, U_f and V_f, respectively.*

U_f and V_f are unique in the sense that any other two such extensions, U', V' must agree with U_f and V_f, respectively, in some open neighborhood of $G_{\mathbf{R}}$ in $\mathbf{K}^{2n}$.

Proof. The uniqueness of U_f and V_f follows from Lemma 12.1. Thus, if we can, for every $(x,y) \in G_{\mathbf{R}}$, find an open neighborhood $W_{f,(x,y)} \subseteq \mathbf{K}^{2n}$ and definable $U_{f,(x,y)}, V_{f,(x,y)} : W_{f,(x,y)} \to \mathbf{K}$ as needed, and if furthermore $U_{f,(x,y)}$, $V_{f,(x,y)}$ and $W_{f,(x,y)}$ are definable uniformly in (x,y), then, by the uniqueness statement, the union of the $U_{f,(x,y)}$'s gives a function U_f on some open set in $\mathbf{K}^{2n}$ which contains $G_{\mathbf{R}}$, and the same is true for the union of the $V_{f,(x,y)}$'s.

Consider $a = x+iy \in G$ (so $(x,y) \in G_{\mathbf{R}}$). Without loss of generality, $a = 0$, for if not, replace f with $f(z+a)$. We now define $U_f(z,w)$ in an open neighborhood of $(0,0) \in \mathbf{K}^{2n}$ as follows (this is a variation on the so-called "Halmos trick"):

For $z = x+iy \in \mathbf{K}$, we let $\overline{z} = x - iy$ be the $\mathbf{K}$-conjugate of z, and for $z = (z_1, \ldots, z_n) \in \mathbf{K}^n$ we let $\overline{z} = (\overline{z_1}, \ldots, \overline{z_n})$ be the n-tuple of K-conjugates. For $(z,w) \in \mathbf{K}^{2n}$ near $(0,0)$, let

$$U_f(z,w) = 1/2(f(z+iw) + \overline{f(\overline{z} + i\overline{w})}).$$

It is immediate to see that the restriction of U_f to $\mathbf{R}^{2n}$ equals u (because $Re(z) = 1/2(z+\overline{z})$. To check that U_f is $\mathbf{K}$-holomorphic in each variable one has to consider the corresponding limits and verify that they indeed exist. The continuity and therefore the $\mathbf{K}$-holomorphicity of U_f in all $2n$ variables is immediate (notice that the definition of U_f makes sense only for $(z,w) \in \mathbf{K}^{2n}$ such that $z+iw$ and $\overline{z}+i\overline{w}$ belong to W).

To see that $V_f(z,w)$ is also definable just note that V_f is the real part of $-if$. The uniformity of U_f and V_f in (x,y) is easy to verify. □

The idea of the following lemma is taken from [3] (Theorem 4), but since converging power series are not available in our nonstandard setting we replace them by definable functions.

Lemma 12.3. *Let f be a $\mathbf{K}$-holomorphic function on some open subset of $\mathbf{K}^n$ and write $f(x+iy) = u_f(x,y) + iv_f(x,y)$. Assume that either the function u_f or the function v_f (as functions of $2n$ real variables) are $\mathbf{R}$-algebraic in some neighborhood of $(a,b) \in \mathbf{R}^{2n}$. Then the function f is $\mathbf{K}$-algebraic in some open neighborhood of $a+ib$.*

Proof. We assume that u_f is $\mathbf{R}$-algebraic. Therefore, there is a polynomial $P(x,y,w)$ over $\mathbf{R}$, in $2n+1$ variables, such that in some neighborhood $G \subseteq \mathbf{R}^{2n}$ of (a,b) we have $P(x,y,u_f(x,y)) = 0$. We now view P as a polynomial over $\mathbf{K}$, take $U_f(z,w)$ to be as in Lemma 12.2 and consider the function $h(z,w) = P(z,w,U_f(z,w))$. This function is defined in some open set $W \subseteq \mathbf{K}^{2n}$ containing G but since its restriction to G is zero, it follows from Lemma 12.1 that it vanishes everywhere, and hence $U_f(z,w)$ is $\mathbf{K}$-algebraic as well.

By replacing f with $f(z+a+ib)$ we may assume that $(a,b) = (0,0)$. It follows from the definition of $U_f(z,w)$ in the proof of Lemma 12.2 that for z near 0 we have

$$U_f(z/2, z/2i) = f(z) + \overline{f(0)}.$$

Since $U_f(z,w)$ is $\mathbf{K}$-algebraic it follows that $f(z)$ is also algebraic.

The case where $v_f(x,y)$ is $\mathbf{R}$-algebraic is treated similarly, using the fact that v is the real part of $-if$. □

We now restrict ourselves to the classical setting, where our o-minimal structure is assumed to be an expansion of the real field $\langle \mathbb{R}, <, +, \cdot \rangle$.

The following theorem is taken from [4] (see Theorem 6 on p. 202). Its proof makes use of the Baire Category Theorem and we do not know whether it is true for o-minimal structures in general (with the notion of "holomorphic" replaced by "$\mathbf{K}$-holomorphic ").

Theorem 12.4. *Assume that $f(z,w)$ is a holomorphic function defined on $G \times W \subseteq \mathbb{C}^{n+m}$. Assume that for each $z \in G$, the function $f(z,w)$ is complex-algebraic in w and for each $w \in W$, $f(z,w)$ is complex-algebraic in z. Then $f(z,w)$ is complex-algebraic as a function of (z,w).*

We are now ready to prove the main theorem of this appendix.

Theorem 12.5. *Assume that $\mathcal{M}$ is an o-minimal structure expanding the field of real numbers such that every definable holomorphic function of 1-variable is locally semialgebraic.*

Let $G \subseteq \mathbb{C}^n$ be a definable open set and X a definable irreducible complex analytic subset of G. Then there is a complex algebraic set $A \subseteq \mathbb{C}^n$ such that X is one of the irreducible components of $A \cap G$.

Proof. We first prove the theorem for $n = 2$. Fix $a \in Reg_{\mathbb{C}} X$. After a change of coordinates we may assume that near a the set X is the graph of a definable holomorphic map h from some open subset of $\mathbb{C}$ into $\mathbb{C}$. By our assumption, h is locally semialgebraic and therefore there is an open set $W \subseteq \mathbb{C}$ such that $h|W$ is real algebraic and in particular its real and imaginary parts are real algebraic. By Lemma 12.3, h is an algebraic function on some open subset of W containing a.

It follows that there is an algebraic set $A \subseteq \mathbb{C}^2$ and an open set $W \subseteq \mathbb{C}^2$ such that $X \cap W = A \cap W$. If we now define $B = \{z \in Reg_{\mathbb{C}} X : \exists \text{ open } W \ (X \cap W = A \cap W)\}$ then B is non-empty, open an closed in $Reg_{\mathbb{C}} X$. Because $Reg_{\mathbb{C}} X$ is connected, it follows that $B = Reg_{\mathbb{C}} X$ and hence $Reg_{\mathbb{C}} X \subseteq A \cap G$. Since $Reg_{\mathbb{C}} X$ is dense in X, we have $X \subseteq A \cap G$. By dimension considerations, X is an irreducible component of $A \cap G$.

Consider now the general case $X \subseteq G \subseteq \mathbb{C}^n$, $\dim_{\mathbb{C}} X = d$, and again take $a \in Reg_{\mathbb{C}} X$. After a change in coordinates we may assume that near a, the set X is the graph of a holomorphic map Φ from an open set in $\mathbb{C}^d$ into $\mathbb{C}^{n-d}$. Let h be one of the coordinate functions of this map. Namely, $h(z_1, \ldots, z_d)$ is a definable holomorphic function from an open subset of $\mathbb{C}^d$ into $\mathbb{C}$. By the earlier paragraph, if we fix any of the $d - 1$ variables then the function we obtain in the remaining variable is algebraic, in the sense that it satisfies a nontrivial algebraic equation. It follows from Theorem 12.4 that $h(z_1, \ldots, z_d)$ satisfies an algebraic equation. Since this is true for any one of the coordinate functions of Φ, there is some algebraic set $A \subseteq \mathbb{C}^n$ which agrees with X on some open set. Just as before, X is an irreducible component of $A \cap G$. $\square$

The following corollary answers a question of Chris Miller which was posed in [3]. (The original conjecture implies that X below must definable in $\langle \mathbb{C}, +, \cdot \rangle$ but this is clearly false because G may not be definable there).

Corollary 12.6. *Let X be an analytic subset of an open set $G \subseteq \mathbb{C}^n$. Assume that G and X are definable in $\mathbb{R}_{exp}$. Then there is a complex algebraic set $A \subseteq \mathbb{C}^n$ such that X is an irreducible component of $A \cap G$.*

Proof. This is an immediate corollary of Bianconi's theorem (see [3]) that every $\mathbb{R}_{exp}$-definable holomorphic function of 1-variable is semialgebraic, together with Theorem 12.5 above. $\square$

We do not know whether the same theorem remains true in structures which are elementarily equivalent to $\mathbb{R}_{exp}$.

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