The real shadow of a definably compact homogeneous space
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University of Notre Dame-June, 2016
The setting

We fix $\mathcal{M}$ an o-minimal expansion of a real closed field $R$, saturated “enough”.

We do not assume that $\mathcal{M}$ is (elementarily equivalent to) a structure over $R$. But some structures we meet will be such.

We consider a definable group $G$ in $\mathcal{M}$, and a definable action of $G$ on definable manifold $X$.

The action of $G$ will mostly be:

- continuous.
- transitive: $\forall x, y \in X \exists g \in G gx = y$.
- faithful: $\forall g \in G$, if $g \upharpoonright X = id$ then $g = e$. 
The “real shadow” of a definably compact group

The starting point

Pillay’s Conjecture- a theorem [HPP08]

If $G$ is a definably compact* group then there exists a smallest normal, type-definable, subgroup of bounded index** $G^{00}$ such that

- $G/G^{00}$, with the logic topology*** is a compact Lie group.
- $\dim G/G^{00}$, as a Lie group, equals $\dim G$, the dimension of the definable set $G$ in $\mathcal{M}$.

We call $G/G^{00}$ the real shadow of $G$ and denote it by $\mathbb{R}G$.

Justification for the name: Berarducci’s talk.
Two basic examples

Let $\mathcal{M} = \langle R, <, +, \cdot \rangle > \langle \mathbb{R}, <, +, \cdot \rangle$.

1. The simple case- $G^{00}$ is given by the infinitesimals of $\mathcal{M}$

Take $G$ a compact real algebraic subgroup of $GL(n, \mathbb{R})$ and consider $G(R)$.
Then

$$G^{00} = \mu_{\mathbb{R}}(G) = \bigcap_{n \in \mathbb{N}} \{ A \in G(R) : \|A - I\| < 1/n \}$$

(the infinitesimal neighborhood of $I$ in $\mathcal{M}$).
It is type-definable, of bounded index in $G$.
Then $RG = G/G^{00}$ is just $G(R)$.

2. The general case- $G^{00}$ is the “internal” infinitesimals

Let $\alpha >> \mathbb{R}$ and $G = [0, \alpha)$, with “$+$ mod $\alpha$”. A semi-algebraic group.
Here $G^{00} = \bigcap_{n \in \mathbb{N}} [0, \frac{\alpha}{n}) \cup (\alpha - \frac{\alpha}{n}, \alpha)$
Then $RG = G/G^{00}$ is a one-dim. compact connected Lie group.
Two subsequent questions

A. Is there a real shadow of definably compact homogeneous spaces?

Assume that $G$ is any definable group, $X$ a definably compact space and

$$G \times X \to X$$

a definable, transitive and continuous action on $X$. Does this action have a “real shadow”? (Since the action factors through a faithful action we will assume it is).

B. Groups which are not definably compact?

Is there “a real shadow” for arbitrary definable groups?
The simple case-groups over $\mathbb{R}$

**An example**

Let

$$G = SL_2(R) = \{ A \in GL_2(R) : \det A = 1 \},$$
real algebraic, over $\mathbb{Q}$.

The infinitesimals

$$\mu_R(G) = \bigcap_{n \in \mathbb{N}} \{ A \in G : ||A - I|| < 1/n \}$$
form a type-definable subgroup.

But it is neither normal nor of bounded index. Moreover, $G$ has NO normal subgroups.

How do we canonically recover $SL(2, \mathbb{R})$ from $G$?

Let $O_R(G) = \bigcup_{n \in \mathbb{N}} \{ A \in G : 1/n < ||A|| < n \}$, a $\bigvee$-definable group, and $\mu_R(G)$ is a type-definable normal subgroup of bounded index.

Now $O_R(G) / \mu_R(G)$, with the logic topology, is locally compact and isomorphic to $SL(2, \mathbb{R})$. This should be the “real shadow” of $G$. 
Fact- a prototype for a theorem

Let $\mathcal{M}$ be an o-minimal expansion of $\mathbb{R}$. Assume that $G = SL(2, \mathbb{R})$ acts continuously and transitively on a definably compact $X$. Then

- The orbits of $\mu_{\mathbb{R}}(G)$ induce on $X$ a type-definable equivalence relation $E$, of bounded index.
- $X/E$, with logic topology, is a real compact manifold, a homogeneous space for $SL(2, \mathbb{R})$ and we have:

\[ \mathcal{O}_{\mathbb{R}}(G) \times X \xrightarrow{\rho} X \]

\[ (\pi, \pi_X) \]

\[ SL(2, \mathbb{R}) \times X/E \xrightarrow{\rho} X/E \]
An $SL(2, R)$-definably compact homogeneous space

Fact

Let $\mathcal{M}$ be an o-minimal expansion of $R$. Assume that $G = SL(2, R)$ acts definably, continuously and transitively on a definably compact $X$. Then

- the orbits of $\mu_R(G)$ induce on $X$ a type definable equivalence relation $E$, of bounded index.
- $X/E$, with logic topology, is a real compact manifold, a homogeneous space for $SL(2, \mathbb{R})$ and we have:

$$\mathcal{O}_R(G) \times X \xrightarrow{\rho} X$$

The real shadow of the action

$$SL(2, \mathbb{R}) \times X/E \xrightarrow{\rho} X/E$$
Recall A group action is *faithful* if no element other than \( e \) acts as the identity.

Fact

For \( G \) a definable group, the following are equivalent:

1. There exists a definably compact \( X \) on which \( G \) acts faithfully, transitively and continuously.
2. \( G \) has no definable normal torsion-free subgroup.
Proof of Fact

(1) ⇒ (2)
Assume $G$ acts faithfully, transitively and continuously on definably compact $X$. Let $H < G$ be any definable torsion-free. By Conversano-Onshuus-Starchenko, $H$ has a fixed point $x_0 \in X$. If $H$ were normal in $G$ then $H$ would fix every element in $Gx_0$, so by transitivity, $H$ would fix $X$, contradicting faithfulness.

(2) ⇒ (1): Assume $G$ has no definable normal torsion-free s.g.
By Conversano, every $G$ has a definable torsion-free $H < G$ such that $G/H$ is definably compact.
Now $G$ acts faithfully on $G/H$: Assume $g aH = aH$ for all $a \in G$. Then $g \in \bigcap_{a \in G} H^a$.
The group on the right is definable, normal torsion-free so $g = e$. 
The main theorem

Let $G$ be a definable group with no normal, torsion-free definable subgroups. Then there exists a locally definable group $\hat{G} \subseteq G$ and a minimal type-definable subgroup of bounded index $\hat{G}^{00} \subseteq \hat{G}$ such that:

1. $\bar{G} = \hat{G}/\hat{G}^{00}$, with the logic topology, is a (locally compact) real Lie group, and $\dim \bar{G} = \dim G$.

2. For every definable continuous, transitive and faithful action of $G$ on a definably compact $X$, the $\hat{G}^{00}$-orbits induce on $X$ a type-definable equiv. relation of bounded index $E$. The quotient $\bar{X} = X/E$, with its logic topology, is a compact real manifold. The Lie group $\bar{G}$ acta continuously, transitively and faithfully on $\bar{X}$:
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\[
\begin{align*}
\bar{G} \times X & \xrightarrow{\rho} X \\
\end{align*}
\]

\[
\begin{align*}
\bar{G} \times X/E & \xrightarrow{\rho} X/E \\
\end{align*}
\]

The real shadow of action
Some steps in the proof

Since \( G \) has no definable normal torsion-free subgroup its center \( Z(G) \) is definably compact.

**A.** \( G/Z(G) \cong \) a linear real algebraic group \( G_1 \) over \( \mathbb{R}_{\text{alg}} \).

\[
1 \to Z(G) \overset{i}{\to} G \overset{\pi}{\to} G_1 \to 1.
\]

Let \( \hat{G}_1 = \mathcal{O}_R(G) = \bigcup_{n \in \mathbb{N}} \{ g \in G_1 : 1/n < ||g - I|| < n \} \) locally definable.

Let \( \hat{G} = \pi^{-1}(\hat{G}_1) \), locally definable subgroup of \( G \).

**B.** \( \hat{G} \) has a minimal type-definable subgroup of bounded index \( \hat{G}^{00} \) (this is not always true for locally definable groups!)

**C.** \( \bar{G} = \hat{G}/\hat{G}^{00} \) is a real Lie group of dimension \( = \dim G \).

(The main difficulty: \( \hat{G}^{00} \cap Z(G) = Z(G)^{00} \), using that \( \hat{G}^{00} \) has f.s.g.)
How canonical are $\hat{G}$ and the real shadow $\bar{G}$?

The choice of $\hat{G}$ depends on that of $\hat{G}_1$. This depends on a choice of a maximal definably compact subgroup $K_1$ of $G_1$:

Once $K_1$ is fixed, its preimage $K = \pi^{-1}(K_1)$ is a maximal definably compact subgroup of $G$. An action of $G$ on a definably compact $X$ induces on $X$, via $K^{00}$, a type-definable equiv. relation $E$ of bounded index.

Lemma

The collection of elements in $G$ which leave $E$ invariant is exactly $\hat{G}$. The elements which fix all $E$-classes are exactly $\hat{G}^{00}$.

Note: Different $K_1$'s may lead to different loc. def. subgps of $G$ but all are conjugate, so all real Lie groups obtained this way are isomorphic.

Open question

Assume that some other locally definable $\hat{G}' \subseteq G$ satisfies the result of the Main Theorem. Is $\hat{G}' / (\hat{G}')^{00}$ isomorphic to $\hat{G} / \hat{G}^{00}$?
Question: Is the main theorem trivial?

Namely, is every group with no normal torsion-free subgroup isomorphic to a group over $\mathbb{R}$, so we are always in the “simple case” like in the example of $SL(2, \mathbb{R})$?

Answer: No!

Fact

There exists a semi-algebraic group $G$ over a real closed $\mathbb{R}$, with no definable normal torsion-free subgroup, such that $G$ is not definably isomorphic (in any structure) to a group defined over $\mathbb{R}$. 
Proof:

This is a (small) variation of an example of Conversano-Pillay:

1. Start from the universal cover of $\text{SL}(2, \mathbb{R})$:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z} & \overset{i}{\longrightarrow} & \tilde{\text{SL}}(2, \mathbb{R}) & \overset{\pi}{\longrightarrow} & \text{SL}(2, \mathbb{R}) & \longrightarrow & 1 .
\end{array}
$$

Here $\tilde{\text{SL}}(2, \mathbb{R})$ is locally definable, on the set $\mathbb{Z} \times \text{SL}(2, \mathbb{R})$, via a definable cocycle $h : \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \rightarrow \mathbb{Z}$.

2. Pick an infinitesimal element $g \in SO(2, R)^{00}$ and identify $\mathbb{Z}$ with $\langle g \rangle$.

3. Using the cocycle $h$ obtain a **definable** extension $G$ living on $SO(2, R) \times \text{SL}(2, R)$,

$$
\begin{array}{cccccc}
1 & \longrightarrow & SO(2, R) & \overset{i}{\longrightarrow} & G & \overset{\pi}{\longrightarrow} & \text{SL}(2, \mathbb{R}) & \longrightarrow & 1 .
\end{array}
$$

4. $G$ has no definable normal torsion-free subgroup.

And since $\langle g \rangle = [G, G] \cap SO(2, R) \subseteq \mathbb{Z}(G)^{00}$, the group $G$ is not definably isomorphic to any group over $\mathbb{R}$. 
So what is the real shadow of the above group?

(Probably) $SO(2, \mathbb{R}) \times SL(2, \mathbb{R})$. 