The real shadow of a definably compact homogeneous space

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The setting

We fix $\mathcal{M}$ an o-minimal expansion of a real closed field R, saturated "enough".

We do not assume that $\mathcal{M}$ is (elementarily equivalent to) a structure over $\mathbb{R}$. But some structures we meet will be such.

We consider a definable group G in $\mathcal{M}$, and a definable action of G on definable manifold X.
The action of G will mostly be:

- continuous.
- transitive: $\forall x, y \in X \; \exists g \in G \; gx = y$.
- faithful: $\forall g \in G$, if $g \restriction X = id$ then $g = e$.

The "real shadow" of a definably compact group

The starting point

Pillay's Conjecture- a theorem [HPP08]

If G is a definably compact* group then there exists a smallest normal, type-definable, subgroup of bounded index** G^{00} such that

- G/G^{00}, with the logic topology*** is a compact Lie group.
- $\dim G/G^{00}$, as a Lie group, equals $\dim G$, the dimension of the definable set G in $\mathcal{M}$.

We call G/G^{00} **the real shadow of** G and denote it by $\mathbb{R}G$.

Justification for the name: Berarducci's talk.

Two basic examples

Let $\mathcal{M} = \langle R, <, +, \cdot \rangle \succ \langle \mathbb{R}, <, +, \cdot \rangle$.

1. The simple case- G^{00} is given by the infinitesimals of $\mathcal{M}$

Take G a compact real algebraic subgroup of $GL(n, \mathbb{R})$ and consider $G(R)$.
Then

$G^{00} = \mu_{\mathbb{R}}(G) = \bigcap_{n \in \mathbb{N}} \{A \in G(R) : ||A - \mathrm{I}|| < 1/n\}$

(the infinitesimal neighborhood of I in $\mathcal{M}$).
It is type-definable, of bounded index in G.
Then $\mathbb{R}G = G/G^{00}$ is just $G(\mathbb{R})$.

2. The general case-G^{00} is the "internal" infinitesimals

Let $\alpha >> \mathbb{R}$ and $G = [0, \alpha)$, with "$+$ mod α". A semi-algebraic group.

Here $G^{00} = \bigcap_{n \in \mathbb{N}} [0, \frac{\alpha}{n}) \cup (\alpha - \frac{\alpha}{n}, \alpha)$

Then $\mathbb{R}G = G/G^{00}$ is a one-dim. compact connected Lie group.

Two subsequent questions

A. Is there a real shadow of definably compact homogeneous spaces?

Assume that G is **any** definable group, X a definably compact space and

$G \times X \to X$

a definable, transitive and continuous action on X.
Does this action have a "real shadow"?
(Since the action factors through a faithful action we will assume it is).

B. Groups which are not definably compact?

Is there "a real shadow" for arbitrary definable groups?

An example

Let

$G = SL_2(R) = \{A \in GL_2(R) : det\, A = 1\}$, real algebraic, over $\mathbb{Q}$.

The infinitesimals

$\mu_{\mathbb{R}}(G) = \bigcap_{n \in \mathbb{N}} \{A \in G : ||A - \mathrm{I}|| < 1/n\}$ form a type-definable subgroup.

But it is neither normal nor of bounded index. Moreover, G has NO normal subgroups.

How do we canonically recover $SL(2, \mathbb{R})$ from G?

Let $\mathcal{O}_{\mathbb{R}}(G) = \bigcup_{n \in \mathbb{N}} \{A \in G : 1/n < ||A|| < n\}$, **a $\bigvee$-definable** group, and $\mu_{\mathbb{R}}(G)$ **is** a type-definable normal subgroup of bounded index.

Now $\mathcal{O}_{\mathbb{R}}(G)/\mu_{\mathbb{R}}(G)$, with the logic topology, is **locally compact** and isomorphic to $SL(2, \mathbb{R})$. This should be the "real shadow" of G.

An $SL(2, R)$-definably compact homogeneous space

Fact- a prototype for a theorem

Let $\mathcal{M}$ be an o-minimal expansion of R. Assume that $G = SL(2, R)$ acts continuously and transitively on a definably compact X. Then
• the orbits of $\mu_{\mathbb{R}}(G)$ induce on X a type-definable equivalence relation E, of bounded index.
• X/E, with logic topology, is a real compact manifold, a homogeneous space for $SL(2, \mathbb{R})$ and we have:

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{R}}(G) \times X & \xrightarrow{\rho} & X \\ \big\downarrow{\scriptstyle (\pi,\pi_X)} & & \big\downarrow{\scriptstyle \pi} \\ SL(2, \mathbb{R}) \times X/E & \xrightarrow{\rho} & X/E \end{array}$$

An $SL(2, R)$-definably compact homogeneous space

Fact

Let $\mathcal{M}$ be an o-minimal expansion of R. Assume that $G = SL(2, R)$ acts definably, continuously and transitively on a definably compact X. Then

• the orbits of $\mu_{\mathbb{R}}(G)$ induce on X a type definable equivalence relation E, of bounded index.

• X/E, with logic topology, is a real compact manifold, a homogeneous space for $SL(2, \mathbb{R})$ and we have:

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{R}}(G) \times X & \xrightarrow{\ \rho\ } & X \\ \big\downarrow{\scriptstyle (\pi, \pi_X)} & & \big\downarrow{\scriptstyle \pi} \\ SL(2, \mathbb{R}) \times X/E & \xrightarrow{\ \rho\ } & X/E \end{array}$$

The real shadow of the action

Faithful actions on definably compact sets?

Recall A group action is *faithful* if no element other than e acts as the identity.

Fact

For G a definable group, the following are equivalent:

1. There exists a definably compact X on which G acts faithfully, transitively and continuously.
2. G has no definable normal torsion-free subgroup.

Proof of Fact

(1) $\Rightarrow$ (2)

Assume G acts faithfully, transitively and continuously on definably compact X.Let $H < G$ be any definable torsion-free. By Conversano-Onshuus-Starchenko, H has a fixed point $x_0 \in X$.

If H were normal in G then H would fix every element in Gx_0, so by transitivity, H would fix X, contradicting faithfulness.

(2) $\Rightarrow$ (1): Assume G has no definable normal torsion-free s.g

By Conversano, every G has a definable torsion-free $H < G$ such that G/H is definably compact.

Now G acts faithfully on G/H: Assume $g\,aH = aH$ for all $a \in G$. Then $g \in \bigcap_{a \in G} H^a$.
The group on the right is definable, normal torsion-free so $g = e$.

The main theorem

Let G be a definable group with no normal, torsion-free definable subgroups. Then there exists a **locally definable** group $\hat{G} \subseteq G$ and a minimal type-definable subgroup of bounded index $\hat{G}^{00} \subseteq \hat{G}$ such that:

1. $\bar{G} = \hat{G}/\hat{G}^{00}$, with the logic topology, is a (**locally** compact) real Lie group, and $\dim \bar{G} = \dim G$.
2. For every definable continuous, transitive and faithful action of G on a definably compact X, the $\hat{G}^{00}$-orbits induce on X a type-definable equiv. relation of bounded index E. The quotient $\bar{X} = X/E$, with its logic topology, is a compact real manifold. The Lie group $\bar{G}$ acta continuously, transitively and faithfully on $\bar{X}$:

$$\begin{array}{ccc} \hat{G} \times X & \xrightarrow{\rho} & X \\ \Big\downarrow{\scriptstyle (\pi,\pi_X)} & & \Big\downarrow{\scriptstyle \pi} \\ \bar{G} \times X/E & \xrightarrow{\rho} & X/E \end{array}$$

The main theorem

Let G be a definable group with no normal, torsion-free definable subgroups. Then there exists a locally definable group $\hat{G} \subseteq G$ and a minimal type-definable subgroup of bounded index $\hat{G}^{00} \subseteq \hat{G}$ such that:

1. $\bar{G} = \hat{G}/\hat{G}^{00}$, with the logic topology, is a (**locally** compact) real Lie group, and $\dim \bar{G} = \dim G$.
2. For every definable continuous, transitive and faithful action of G on a definably compact X, the $\hat{G}^{00}$-orbits induce on X a type-definable equiv. relation of bounded index E. The quotient $\bar{X} = X/E$, with its logic topology, is a compact real manifold, with $\bar{G}$ acting continuously, transitively and faithfully on $\bar{X}$:

$$\begin{array}{ccc} \hat{G} \times X & \xrightarrow{\rho} & X \\ \downarrow{\scriptstyle (\pi, \pi_X)} & & \downarrow{\scriptstyle \pi} \\ \bar{G} \times X/E & \xrightarrow{\rho} & X/E \end{array}$$

The real shadow of action

Some steps in the proof

Since G has no definable normal torsion-free subgroup its center $Z(G)$ is definably compact.

A. $G/Z(G) \cong$ a linear real algebraic group G_1 over $\mathbb{R}_{alg}$.

$$1 \longrightarrow Z(G) \xrightarrow{i} G \xrightarrow{\pi} G_1 \longrightarrow 1 \ .$$

Let $\hat{G}_1 = \mathcal{O}_{\mathbb{R}}(G) = \bigcup_{n \in \mathbb{N}} \{g \in G_1 : 1/n < ||g - \mathrm{I}|| < n\}$ locally definable.
Let $\hat{G} = \pi^{-1}(\hat{G}_1)$, locally definable subgroup of G.

B. $\hat{G}$ has a minimal type-definable subgroup of bounded index $\hat{G}^{00}$ (**this is not always true for locally definable groups!**)

C. $\bar{G} = \hat{G}/\hat{G}^{00}$ is a real Lie group of dimension $= \dim G$.
(The main difficulty: $\hat{G}^{00} \cap Z(G) = Z(G)^{00}$, using that $\hat{G}^{00}$ has f.s.g.)

How canonical are $\hat{G}$ and the real shadow $\bar{G}$?

The choice of $\hat{G}$ depends on that of $\hat{G}_1$. This depends on a choice of a maximal definably compact subgroup K_1 of G_1:
Once K_1 is fixed, its preimage $K = \pi^{-1}(K_1)$ is a maximal definably compact subgroup of G. An action of G on a definably compact X induces on X, via K^{00}, a type-definable equiv. relation E of bounded index.

Lemma

The collection of elements in G which leave E invariant is exactly $\hat{G}$. The elements which fix all E-classes are exactly $\hat{G}^{00}$.

Note: Different K_1's may lead to different loc. def. subgps of G but all are conjugate, so all real Lie groups obtained **this way** are isomorphic.

Open question

Assume that some other locally definable $\hat{G}' \subseteq G$ satisfies the result of the Main Theorem. Is $\hat{G}'/(\hat{G}')^{00}$ isomorphic to $\hat{G}/\hat{G}^{00}$?

Question: Is the main theorem trivial?

Namely, is every group with no normal torsion-free subgroup isomorphic to a group over $\mathbb{R}$, so we are always in the "simple case" like in the example of $SL(2, \mathbb{R})$?

Answer: No!

Fact

There exists a semi-algebraic group G over a real closed R, with no definable normal torsion-free subgroup,such that G is not definably isomorphic (in any structure) to a group defined over $\mathbb{R}$.

Proof:

This is a (small) variation of an example of Conversano-Pillay:

1. Start from the universal cover of $SL(2,\mathbb{R})$:

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } \widetilde{S}L(2,\mathbb{R}) \xrightarrow{\ \pi\ } SL(2,\mathbb{R}) \longrightarrow 1 \ .$$

Here $\widetilde{S}L(2,\mathbb{R})$ is locally definable, on the set $\mathbb{Z}\times SL(2,\mathbb{R})$, via a definable cocycle $h : SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ into $\mathbb{Z}$.

2. Pick an infinitesimal element $g \in SO(2,R)^{00}$ and identify $\mathbb{Z}$ with $\langle g\rangle$.

3. Using the cocycle h obtain **a definable** extension G living on $SO(2,R)\times SL(2,R)$,

$$1 \longrightarrow SO(2,R) \xrightarrow{\ i\ } G \xrightarrow{\ \pi\ } SL(2,\mathbb{R}) \longrightarrow 1 \ .$$

4. G has no definable normal torsion-free subgroup.

And since $\langle g\rangle = [G,G]\cap SO(2,R) \subseteq Z(G)^{00}$, the group G is not definably isomorphic to any group over $\mathbb{R}$.

So what is the real shadow of the above group?

(Probably) $SO(2, \mathbb{R}) \times SL(2, \mathbb{R})$.