AROUND PILA-ZANNIER: THE SEMI-ABELIAN CASE

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This note came out as a result of reading of the paper of Pila and Zannier, [3]. Below we suggest a small short-cut for the proof of Manin-Mumford in the case of abelian varieties, and extend their methods in order to prove the semi-abelian case.

1. Definable curves in periodic sets

1.1. **The compact case.** We fix an N-dimensional lattice Λ in $\mathbb{R}^N$. For $0 \neq \vec{v} \in \mathbb{R}^N$, denote by $H(\vec{v})$ the smallest $\mathbb{R}$-linear subspace of $\mathbb{R}^N$ containing $\vec{v}$, such that $H(\vec{v})$ has a basis in Λ.

Lemma 1.1. *Let $\mathcal{M}$ be an o-minimal structure expanding the real field. Assume that $X \subseteq \mathbb{R}^N$ is an arbitrary closed set which is Λ-invariant (namely, $\Lambda + X = X$).*

Let $\gamma : (0, \infty) \to X$ be an $\mathcal{M}$-definable curve whose image $\mathcal{C}$ is unbounded and let $\vec{v}_0 = \lim_{t\to\infty} \frac{\dot{\gamma}(t)}{\|\gamma(t)\|}$ be the limit of the unit tangent vector to γ. Then, there is a translate H of $H(\vec{v}_0)$ such that $H \subseteq X$.

Proof. Let $\ell \subseteq \mathbb{R}^N$ be the 1-dimensional linear subspace generated by $\vec{v}_0$. Because X is Λ-invariant and closed, it is sufficient to show that a translate of ℓ is contained in X.

Let $F \subseteq \mathbb{R}^N$ be a fundamental N-dimensional parallelepiped for Λ (hence $cl(F)$ is a compact), and let $p_n = \gamma(t_n)$ be a sequence of points on $\mathcal{C}$, with $t_n \to \infty$. For each n, there is $\lambda_n \in \Lambda$ with $\lambda_n + p_n \in F$, and because $cl(F)$ is compact, we may assume that the sequence $\lambda_n + p_n$ converges to some $a_0 \in cl(F)$. We will show that $a_0 + \ell \subseteq X$.

Fix $r \in \mathbb{R}^+$. For each n, let $s_n > t_n$ be minimal such that $|\gamma(s_n) - p_n| = r$ (because the r-sphere around p_n is closed and $\mathcal{C}$ is unbounded there is such a point) and let $q_n = \gamma(s_n) \in X$. The vector $q_n - p_n$ tends to $r\vec{v}_0$ as n tends to ∞.

It follows that $\lambda_n + q_n$ tends to $a_0 + r\vec{v}_0$ and because X is Λ-invariant and closed, we have $a_0 + r\vec{v}_0$ in X.

By choosing $s_n < t_n$ to be maximal with $|\gamma(s_n) - p_n| = r$, we will get $q_n - p_n$ tending to $-r\vec{v}_0$ and hence $a_0 - r\vec{v}_0$ is also in X. It follows that $a_0 + \ell \subseteq X$. □

Note that it was sufficient to assume in the last Lemma, instead of the definability of $\mathcal{C}$, that $\mathcal{C}$ is an unbounded smooth curve whose unit tangent vector tends has a limit at infinity.

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1.2. **The noncompact case.** We assume here that $\mathcal{M}$ is an o-minimal expansion of the real field.

Let G be a definable, connected N-dimensional abelian group, and let $(\mathbb{R}^N,+)$ be its universal cover with $\pi:\mathbb{R}^N\to G$ the associated projection. Let $\Lambda_k=ker(\pi)$, with Λ_k the lattice generated by $\mathbb{R}$-independent $\omega_1,\dots,\omega_k\subseteq\mathbb{R}^N$. If G_1 is the maximal definable torsion-free subgroup of G then $\dim(G_1)=N-k$. We let $H_1\subseteq\mathbb{R}^N$ be an $(N-k)$-dimensional linear subspace with $\pi(H_1)=G_1$, and let H_2 be the k-dimensional subspace generated by Λ_k. Then $\mathbb{R}^N=H_1\oplus H_2$. For $\vec{v}\in\mathbb{R}^N$, we let $\pi_1(\vec{v})=\vec{v}_1,\pi_2(\vec{v})=\vec{v}_2$ be the projections of $\vec{v}$ on H_1,H_2, respectively. Given $\gamma:\mathbb{R}\to\mathbb{R}^N$, we let $\gamma_2:\mathbb{R}\to H_2$ be the map $\pi_2\circ\gamma$.

We let $F\subseteq H_2$ be a compact fundamental domain for H_2/Λ_k and hence $D=H_1+F$ is a fundamental (not compact!) domain for $\mathbb{R}^N/\Lambda_k$. Namely, every element in $Int(D)$ is in a single Λ_k-coset and every coset is represented in D.

Theorem 1.2. *Let $X\subseteq\mathbb{R}^N$ be a closed Λ_k-invariant set such that $X\cap D$ is definable in $\mathcal{M}$. Let $\gamma:(0,\infty)\to\mathbb{R}^N$ be an $\mathcal{M}$-definable curve whose image $\mathcal{C}$ is contained in X and such that γ_2 is unbounded in H_2.*

Let $\vec{v}_2\in H_2$ be the limit of $\frac{\dot\gamma_2(t)}{\|\dot\gamma_2(t)\|}$ as t tends to ∞. If $\lambda\in\Lambda_k$ is sufficiently close to the space $\mathbb{R}\vec{v}_2$ and ℓ is the linear $\mathbb{R}$-subspace generated by λ then a translate of some infinite segment of ℓ is contained in X.

Proof. We first make several observations:
(1) If $B\subseteq H_2$ is a bounded definable set then $X\cap(H_1+B)$ is definable (but not uniformly in B).

Indeed, his is immediate from the fact that finitely many Λ_k-translates of D cover H_1+B, together with the fact that X is Λ-invariant.
(2) Let $U\subseteq\mathbb{R}^N$ be a definable set such that $\pi_2(U)$ is bounded, and let $\{A_s:s\in S\}$ be a definable family of subsets of $\mathbb{R}^N$. Then there is a number K such that for all $\tau\in\mathbb{R}^N$ and $s\in S$, the set $A_s\cap(U+\tau)\cap X$ has at most K connected components.

Proof. Note first that the family $A_s\cap(U+\tau)\cap X$ is not uniformly definable in s,τ so the straightforward o-minimal argument cannot work here. Let $V=U+D$. By (1), $V\cap X$ is definable. Let $\tau\in\mathbb{R}^N$. By the choice of F we can find $\lambda\in\Lambda$ such that $\tau-\lambda\in D$. Since X is Λ-invariant, we have that for any $s\in S$ the set $A_s\cap(U+\tau)\cap X$ is homeomorphic to $(A_s-\lambda)\cap(U+\tau-\lambda)\cap X$. Thus each set $A_s\cap(U+\tau)\cap X$ is homeomorphic to a set in the definable family

$$\{(A_S+a)\cap(U+b)\cap(V\cap X):s\in S,a,b\in\mathbb{R}^N\}.$$

By o-minimality, there is a uniform bound on the number of connected components of this family. *End of (2)*

We now return to the setting of the theorem, with $\vec{v}_2\in H_2$ the limit of $\frac{\dot\gamma_2(t)}{\|\dot\gamma_2(t)\|}$. We can re-coordinate $\mathbb{R}^N$ such that $H_2=\{0_{N-k}\}\times\mathbb{R}^k$, and because $\gamma_2(t)$ is unbounded we may assume that the last coordinate of v_2, call it α, is non zero (and in particular that the last coordinate of $\gamma_2(t)$, call $h_N(t)$, is unbounded). Using the last coordinate, we can re-parameterize γ so that

$\lim_{t\to\infty}\dot{\gamma_2}(t) = \vec{v}_2$ (we do that by replacing $\gamma(t)$ by $\gamma(h_N^{-1}(\alpha t))$) Note that in this case, for every $r > 0$, we have $\lim_{t\to\infty}(\gamma_2(t+r) - \gamma_2(t)) = r\vec{v}_2$.

Fixing K:

For $T \in \mathbb{R}^+$ and $\vec{u} \in H_2$, let $A_{T,\vec{u}} = \{\gamma(T+t) - t\vec{u} : t \in \mathbb{R}\}$, and let $U = H_1 + B$, for $B \subseteq \mathbb{R}^k$ a ball of radius 1 around 0. Choose K as in (2) above with respect to the family $\{A_{T,\vec{u}} : T \in \mathbb{R}^+, \vec{u} \in H_2\}$ and the set U.

Fixing T_0:

Claim 1.3. *Assume $\lambda \in \Lambda_k$ satisfies $|\lambda - r\vec{v}_2| \leq \frac{1}{2}(K+1)$ for some $r \in \mathbb{R}$. Then there exists $T_0 \in \mathbb{R}^+$ such that for all $T > T_0$ there is an open interval $I(T) \subseteq [0, K+1]$ such that the set $\{\gamma(T+t) - t\lambda : t \in I(T)\}$ is contained in X.*

Proof. Because $\gamma_2(T+r) - \gamma_2(T)$ tends to $r\vec{v}_2$ as T goes to ∞, there exists T_0 such that for every $T > T_0$,

$$|\gamma_2(T+r) - \gamma_2(T) - \lambda| \leq |\gamma_2(T+r) - \gamma_2(T) - r\vec{v}_2| + |r\vec{v}_2 - \lambda| \leq 1/(K+1).$$

By a simple calculation, it follows that for every $n = 1, \dots, K+1$, we have $|\gamma_2(T+nr) - \gamma_2(T) - n\lambda| \leq 1$. Said differently, $\gamma(T+nr) - n\lambda$ is in $U + \gamma(T)$, for all $n = 1, \dots, K+1$ and $T > T_0$.

Because $\mathcal{C} \subseteq X$ and X is Λ_k-invariant, each $\gamma(T+nr) - n\lambda$ belongs to $A_{T,\lambda} \cap (U + \gamma(T)) \cap X$, for $n = 1, \dots, K+1$. However, $A_{T,\lambda} \cap (U + \gamma(T)) \cap X$ has at most K components, and therefore, for some interval $I(T) \subseteq [0, K+1]$, the whole set $\{\gamma(T+t) - t\lambda : t \in I(T)\}$ is contained in $A_{T,\lambda} \cap X \cap (U + \gamma(T))$. *End of Claim.*

We now choose a bounded interval J all of whose elements are greater than T_0, and observe that the set

$$V = \bigcup_{T \in J}(U + \gamma(T))$$

projects onto a bounded subset of H_2 and hence $X \cap V$ is definable. Consider the definable set

$$Y = \{(T,t) : T \in J \,\&\, t \in [0, K+1] \,\&\, \gamma(T+t) - t\lambda \in X \cap V\}.$$

As we showed above, for every $T \in J$ there exists an infinite subset of $t \in [0, K+1]$ such that $(T,t) \in Y$, and therefore $\dim(Y) = 2$. It follows that for some fixed $T' \in \mathbb{R}$ there are infinitely many $(T,t) \in Y$ such that $T + t = T'$. In particular, for a whole interval of t's we have $\gamma(T') - t\lambda \in X$. We thus found an interval in $\ell = \mathbb{R}\lambda$ whose translate (by $\gamma(T')$) is contained in X. □

2. A lower bound for the number of torsion points

For a definable abelian G, let $Tor_m(G)$ be the subgroup of all elements $x \in G$ such that $mx = 0$. For a definable subset D of G and $m \in \mathbb{N}$ let $Tor_m(D)$ denote the intersection of D with $Tor_m(G)$. Let $Tor(D)$ denote all torsion elements in D.

Roots of unity Let us consider the torsion points of $\mathbb{C}^*$. It is known that (i) all primitive roots of unity of a fixed order m are roots of the same irreducible polynomial, hence conjugates to each other. (ii) there are $\varphi(m)$ many such elements, with $\varphi(m)$ the Euler totient function. Furthermore, it is known that for every $0 < \epsilon < 1$, for all sufficiently large m, $\varphi(m) > m^\epsilon$, and hence $\varphi(m)/m^\epsilon \to \infty$ as m tends to ∞.

Theorem 2.1. *Let S be a semiabelian variety, defined over a number field F, given by an exact sequence*

$$0 \longrightarrow (\mathbb{C}^*)^n \xrightarrow{\theta} S \xrightarrow{\sigma} A \longrightarrow 0 \, ,$$

with A an abelian variety. Let $X \subseteq S$ be an algebraic variety defined over F and assume that $Tor(X)$ is infinite. Then there exists an $\epsilon > 0$ such that

$$\limsup_{m \in \mathbb{N}} \frac{|Tor_m(X)|}{m^\epsilon} = \infty.$$

Proof. Given $Q \in Tor(S)$ of order m, let $a(Q)$ be the order of $\sigma(Q)$ in A. Then $m = a(Q)c(Q)$, where $c(Q)$ is the order of the element $a(Q) \cdot Q$ in $\theta((\mathbb{C}^*)^n)$. It follows that either $a(Q) \geq \sqrt{m}$ or $c(Q) \geq \sqrt{m}$.

Case 1 Assume that there are infinitely many $Q \in Tor(X)$ such that $a(Q) \geq \sqrt{m}$.

By the result of Masser, [1], there exists an $\rho > 0$ and a constant $c_2(A)$, such that for every $P \in A$, if $ord(P) = a$ then $[F(P) : \mathbb{Q}] \geq c_2(A)a^\rho$. Because $\sigma(X)$ is defined over F, it is invariant under F-automorphisms and therefore if $ord(P) = a$ then every F-conjugate of P is also in $\sigma(X)$ (and has the same order a). It is now claimed that P has at least $c_3(A)a^\rho$ such conjugates, for some constant c_3. (Indeed, this is stated in Pila-Zannier, but needs some argument because P is represented in projective coordinates as a tuple of complex algebraic numbers $(p_1, \ldots, p_k)$ with

$$[F(p_1, \ldots, p_k) : \mathbb{Q}] \geq c_2(A)a^\rho.$$

However, every such algebraic extension has at least as many automorphisms as its degree, as can be seen by taking a primitive element and considering its conjugates. Finally, any two distinct automorphisms must send P to a different tuples).

If $Q \in Tor(X)$ of order m such that $a = a(Q) \geq \sqrt{m}$ and $P = \sigma(Q)$ then every F-conjugate of P is the image of some F-conjugate of Q. Hence, Q has also at least $c_3(A)a^\rho \geq c_3(A)m^{\frac{1}{2}\rho}$-many conjugates, all of order m. If we take $0 < \epsilon < 1/2\rho$ then, since there are elements Q of unbounded order with this property, we have

$$\limsup_{m \in \mathbb{N}} \frac{|Tor_m(X)|}{m^\epsilon} = \infty.$$

Case 2 Assume that there are infinitely many $Q \in Tor(X)$ of order m, such that $c(Q) \geq \sqrt{m}$.

In this case, if we let $c = c(Q)$ and $a = a(Q)$, then the element $aQ \in \theta((\mathbb{C}^*)^n)$ has order c. By our above observations, aQ has $\varphi(c)$-many conjugates over $\mathbb{Q}$ and hence $\varphi(c)/d$-many conjugates over F (with $d = [F : \mathbb{Q}]$). Every such conjugate is of the form aQ' for some F-conjugate Q' of Q. It follows that Q has at least $\varphi(c)/d$-many conjugates over F, all of order m and all in X. Again, by the above facts about φ, if we pick $\rho < 1$ then there is some constant B such that Q has at least $Bc(Q)^\rho$ F-conjugates. If $c(Q) > \sqrt{m}$ then we have at least $Bm^{1/2\rho}$-many F-conjugates. Since this is true for every $\rho < 1$ and there are infinitely many such Q's, we have

$$\limsup_m \frac{|Tor_m(X)|}{m^\epsilon} = \infty,$$

for $\epsilon < 1/2$. □

3. Manin-Mumford for semi-abelian varieties

In this section we identify varieties defined over $\mathbb{C}$ with their $\mathbb{C}$-points. We also consider $\mathbb{R}$ as a subfield of $\mathbb{C}$, hence for each $l \in \mathbb{N}$ we have $\mathbb{R}^l \subset \mathbb{C}^l$.

We prove here Manin-Mumford for semi-abelian varieties:

Theorem 3.1. *Let S be a semi-abelian variety over a number field K and let $Y \subseteq S$ be an algebraic variety over K. If $Tor(S) \cap Y$ is infinite then Y contains an algebraic subgroup of S.*

Proof. We have an exact sequence

$$0 \longrightarrow (\mathbb{C}^*)^n \xrightarrow{\theta} S \xrightarrow{\sigma} A \longrightarrow 0$$

with A an abelian variety over K, and $\theta\colon (\mathbb{C}^*)^n \to G$ a morphism defined over F. We assume that S has complex dimension m.

Let L be the tangent space to S at the identity. It is a $\mathbb{C}$-vector space of complex dimension m.

Since S is a connected abelian complex Lie group, there is a complex analytic surjective group homomorphism $Exp_S\colon L \to S$, such that for any complex Lie subgroup S_1 of S the tangent space to S_1 at the identity is $\{\vec{v} \in L\colon Exp_S(\vec{v}) \in S_1\}$. Let $\Lambda = ker(Exp_S)$.

Claim 3.2. *There exists a real n-dimensional space $H_1 \subseteq L$ such that, as a real space, $L = H_1 \oplus H_2$, with H_2 the real span of Λ, and such that:*
(i) The restriction of Exp_S to H_1 is definable in $\mathbb{R}_{exp}$.
(ii) If $F \subseteq H_2$ is a fundamental parallelogram for Λ then the restriction of Exp_S to F is definable in $\mathbb{R}_{an}$. In particular, the restriction of Exp_S to $H_1 + F$ is surjective on S and definable in $\mathbb{R}_{an,exp}$.
(iii) If $X \subseteq L$ is an algebraic variety of positive dimension then its image under the projection to H_2 is unbounded.

Proof. We can write $(\mathbb{C}^*)^n$ as $\mathbb{C}^* \oplus \ldots \oplus \mathbb{C}^*$ and θ as $(\theta_1, \ldots, \theta_n)$, where each $\theta_j\colon \mathbb{C}^* \to S$ is an embedding. Let $T_j = \theta_j(\mathbb{C}^*)$.

We choose a $\mathbb{C}$-basis $\vec{v}_1, \ldots, \vec{v}_m$ for L so that for $j = 1, \ldots, n$, the vector $\vec{v}_j$ is in the tangent space to T_j at the identity. It follows then that $Exp_S(z\vec{v}_j) \in T_j$ for any $z \in \mathbb{C}$.

For $j = 1, \dots, n$ let $\alpha_j \colon \mathbb{C} \to \mathbb{C}^*$ be the map

$$\alpha_j \colon z \mapsto \theta_j^{-1}(Exp_S(z\vec{v}_j)).$$

Each α_j is a surjective group homomorphism that is complex analytic. Replacing $\vec{v}_j$ by $1/a_j\vec{v}_j$, where a_j is the complex derivative of α_j at zero, we may assume that the differential of α_j at zero equals 1. It implies that each α_j is the complex exponentiation $z \mapsto e^z$. In particular, $\alpha(1) = e \in \mathbb{R}$ and hence for every $r \in \mathbb{R}$, $Exp_S(r\vec{v}_j) \in \theta_j(\mathbb{R}^*)$.

We now consider L as a real vector space of real dimension $2m$.

Let H_1 be the $\mathbb{R}$-span of $\vec{v}_1, \dots, \vec{v}_n$. Using $\vec{v}_1, \dots, \vec{v}_n$ we can identify H_1 with $\mathbb{R}^n$. Under this identification, for $(r_1, \dots, r_n) \in \mathbb{R}^n$, we have $\theta^{-1}(Exp_S(r_1, \dots, r_n)) = (e^{r_1}, \dots, e^{r_n})$. In particular, the restriction of Exp_S to H_1 is definable in $\mathbb{R}_{exp}$, since θ is a semi-algebraic map, and $Exp_S(H_1)$ is a definable (maximal) torsion free subgroup of S.

The lattice Λ is a discrete subgroup of L of rank $2m - n$. We can extend $\vec{v}_1, \dots, \vec{v}_n$ to an $\mathbb{R}$-basis $\{\vec{w}_1, \dots, \vec{w}_{2m}\}$ of L so that $\vec{w}_j = \vec{v}_j$ for $j = 1, \dots, n$, $\{\vec{w}_{n+1}, \dots, \vec{w}_{2m}\}$ form a $\mathbb{Z}$-basis for Λ and furthermore $\vec{w}_{n+1}, \dots, \vec{w}_{2n}$ belong to the tangent spaces of $T_1, \dots, T_n$, respectively. If H_2 is the $\mathbb{R}$-span of Λ then $L = H_1 \oplus H_2$. If we choose $F \subseteq H_2$ to be a fundamental parallelogram for $\theta|H_2$ then, since θ is analytic, its restriction to the compact $cl(F)$ is definable in $\mathbb{R}_{an}$.

For (iii), see the appendix. □

We now proceed exactly as in Pila-Zannier. For simplicity, we let $\pi : L \to S$ be the holomorphic map Exp_S. As a real space, we can write $L = H_1 \oplus H_2$, as above. Let $\lambda_1, \dots, \lambda_{2m-n}$ generate Λ and fix a fundamental parallelogram

$$F = \{\Sigma_{i=1}^{2m-n} t_i\lambda_i : t_i \in [0, 1)\} \subseteq H_2$$

and $D = H_1 + F$, with $\pi|D$ definable in $\mathbb{R}_{an,exp}$.

Assume now that $Y \subseteq S$ is an algebraic variety such that $Y \cap Tor(S)$ is infinite. By Theorem 2.1, there exists an $\epsilon > 0$ such that

$$\limsup_{m \in \mathbb{N}} \frac{|Tor_m(Y)|}{m^\epsilon} = \infty.$$

If $X = \pi^{-1}(Y)$ then it is a complex analytic, Λ-invariant set, such that $X_1 = X \cap D$ is definable (as the pre-image of an algebraic set under a definable map). We now translate the data into L. We let $\mathbb{Q}\Lambda$ be the rational span of Λ and for $h > 0$ we let $(\mathbb{Q}\Lambda)_h$ be all elements $\Sigma_{i=1}^{2m-n} q_i\lambda_i \in \mathbb{Q}\Lambda$ such that each $q_i \in \mathbb{Q}$ can be written as m_i/n_i with $|m_i|, |n_i| \le h$.

We now have:

$$\limsup_{m \in \mathbb{N}} \frac{|X_1 \cap (\mathbb{Q}\Lambda)_h|}{h^\epsilon} = \infty.$$

We now apply Pila-Wilkie, [2], and conclude that X_1 contains some piece of an irreducible real algebraic curve, call it C. If we let $\mathcal{C} \subseteq L$ be the (complex algebraic) Zariski closure of C then $\mathcal{C}$ is a complex algebraic one-dimensional variety which is still contained in X. By Claim 3.2, the projection of $\mathcal{C}$ on H_2 must be unbounded. It follows that $\mathcal{C}$ contains a semi-algebraic curve $\gamma : (0, \infty) \to \mathcal{C}$ whose projection onto H_2 is unbounded.

Let $\vec{v}_2 \in H_2$ be the limit of the unit tangent vector to γ_2 (the projection of γ onto H_2). Then (see Pila-Zannier) there are elements of Λ as close as we wish to $\mathbb{R}\vec{v}_2$. We can now apply Theorem 1.2 (or, in the abelian case Lemma 1.1) and conclude that X contains an infinite interval I in some translate of a real one-dimensional subspace of L. The Zariski closure H of I, which must still be contained in X, is then a translate of a complex one-dimensional subspace of L. It follows that Y contains $\pi(H)$, a coset of a subgroup of S, and by taking the Zariski closure of $\pi(H)$ we see that Y contains a coset of a complex algebraic group. □

4. APPENDIX: Bounded projections of algebraic sets

Lemma 4.1. *Let L be complex linear space. Assume that, as a real space $L = H_1 \oplus H_2$, for H_1, H_2 real subspaces and that $\pi_2 : L \to H_2$ is the projection with respect to this decomposition. If $X \subseteq L$ is an irreducible complex algebraic variety and $\pi_2(X)$ is bounded then X is contained in a translate of H_1*

Proof. Note that for any real subspace H' containing H_1, the projection of X into the real space L/H' is bounded (independently of the coordinate system we choose for the quotient).
Special case Assume that $H_2 = iH_1$.

If m is the real dimension of H_1 then L is isomorphic as a complex space to $\mathbb{C}^m$, with H_1 going to $\mathbb{R}^n$ and H_2 to $i\mathbb{R}^n$. Under this identification all elements of X are of the form $\vec{z} = (z_1, \ldots, z_m)$ with the imaginary part of the z_i's all contained in some bounded set, as $\vec{z}$ varies in X. It easily follows that X is a singleton.

We use induction on the complex dimension of L.

If H_1 contains an complex subspace K of positive dimension then we replace L by $L/K = H_1/K \oplus H_2/K$ (with $H_2/K \simeq H_2$). By induction, the image of X in L/K is contained in a translate of H_1/K, which implies that X is contained in a a translate of H_1. We may therefore assume that H_1 contains no nonzero complex subspace. It follows that $iH_1 \cap H_1 = \{0\}$.

Let $\tilde{H}_1$ be the complex space $H_1 \oplus iH_1$ and write L as $\tilde{H}_1 \oplus \tilde{H}_2$ for some complex subspace $\tilde{H}_2$. The projection of X into $\tilde{H}_2$ is an irreducible complex constructible set and, as was observed above, is still bounded. This image must then be a singleton and hence X is contained in a translate of $\tilde{H}$.

We may then assume that $L = \tilde{H} = H_1 \oplus iH_1$ and that the projection of X into iH_1 is bounded. By the special case, we are done. □

References

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