AROUND PILA-ZANNIER: THE SEMI-ABELIAN CASE

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This note came out as a result of reading of the paper of Pila and Zannier, [3]. Below we suggest a small short-cut for the proof of Manin-Mumford in the case of abelian varieties, and extend their methods in order to prove the semi-abelian case.

1. Definable curves in periodic sets

1.1. The compact case. We fix an $N$-dimensional lattice $\Lambda$ in $\mathbb{R}^N$. For $0 \neq \vec{v} \in \mathbb{R}^N$, denote by $H(\vec{v})$ the smallest $\mathbb{R}$-linear subspace of $\mathbb{R}^N$ containing $\vec{v}$, such that $H(\vec{v})$ has a basis in $\Lambda$.

Lemma 1.1. Let $\mathcal{M}$ be an o-minimal structure expanding the real field. Assume that $X \subseteq \mathbb{R}^N$ is an arbitrary closed set which is $\Lambda$-invariant (namely, $\Lambda + X = X$).

Let $\gamma : (0, \infty) \to X$ be an $\mathcal{M}$-definable curve whose image $C$ is unbounded and let $\vec{v}_0 = \lim_{t \to \infty} \frac{\gamma(t)}{\|\gamma(t)\|}$ be the limit of the unit tangent vector to $\gamma$. Then, there is a translate $H$ of $H(\vec{v}_0)$ such that $H \subseteq X$.

Proof. Let $\ell \subseteq \mathbb{R}^N$ be the 1-dimensional linear subspace generated by $\vec{v}_0$. Because $X$ is $\Lambda$-invariant and closed, it is sufficient to show that a translate of $\ell$ is contained in $X$.

Let $F \subseteq \mathbb{R}^N$ be a fundamental $N$-dimensional parallelepiped for $\Lambda$ (hence $\text{cl}(F)$ is a compact), and let $p_n = \gamma(t_n)$ be a sequence of points on $C$, with $t_n \to \infty$. For each $n$, there is $\lambda_n \in \Lambda$ with $\lambda_n + p_n \in F$, and because $\text{cl}(F)$ is compact, we may assume that the sequence $\lambda_n + p_n$ converges to some $a_0 \in \text{cl}(F)$. We will show that $a_0 + \ell \subseteq X$.

Fix $r \in \mathbb{R}^+$. For each $n$, let $s_n > t_n$ be minimal such that $|\gamma(s_n) - p_n| = r$ (because the $r$-sphere around $p_n$ is closed and $C$ is unbounded there is such a point) and let $q_n = \gamma(s_n) \in X$. The vector $q_n - p_n$ tends to $r\vec{v}_0$ as $n$ tends to $\infty$.

It follows that $\lambda_n + q_n$ tends to $a_0 + r\vec{v}_0$ and because $X$ is $\Lambda$-invariant and closed, we have $a_0 + r\vec{v}_0 \in X$.

By choosing $s_n < t_n$ to be maximal with $|\gamma(s_n) - p_n| = r$, we will get $q_n - p_n$ tending to $-r\vec{v}_0$ and hence $a_0 - r\vec{v}_0$ is also in $X$. It follows that $a_0 + \ell \subseteq X$. \qed

Note that it was sufficient to assume in the last Lemma, instead of the definability of $C$, that $C$ is an unbounded smooth curve whose unit tangent vector tends has a limit at infinity.

Date: February 2009.
1.2. The noncompact case. We assume here that $\mathcal{M}$ is an o-minimal expansion of the real field.

Let $G$ be a definable, connected $N$-dimensional abelian group, and let $(\mathbb{R}^N,+)$ be its universal cover with $\pi: \mathbb{R}^N \to G$ the associated projection. Let $\Lambda_k = ker(\pi)$, with $\Lambda_k$ the lattice generated by $\mathbb{R}$-independent $\omega_1, \ldots, \omega_k \subseteq \mathbb{R}^N$. If $G_1$ is the maximal definable torsion-free subgroup of $G$ then $\dim(G_1) = N - k$. We let $H_1 \subseteq \mathbb{R}^N$ be an $(N-k)$-dimensional linear subspace with $\pi(H_1) = G_1$, and let $H_2$ be the $k$-dimensional subspace generated by $\Lambda_k$. Then $\mathbb{R}^N = H_1 \oplus H_2$. For $\vec{v} \in \mathbb{R}^N$, we let $\pi_1(\vec{v}) = \vec{v}_1, \pi_2(\vec{v}) = \vec{v}_2$ be the projections of $\vec{v}$ on $H_1, H_2$, respectively. Given $\gamma: \mathbb{R} \to \mathbb{R}^N$, we let $\gamma_2: \mathbb{R} \to H_2$ be the map $\pi_2 \circ \gamma$.

We let $F \subseteq H_2$ be a compact fundamental domain for $H_2/\Lambda_k$ and hence $D = H_1 + F$ is a fundamental (not compact!) domain for $\mathbb{R}^N/\Lambda_k$. Namely, every element in $Int(D)$ is in a single $\Lambda_k$-coset and every coset is represented in $D$.

**Theorem 1.2.** Let $X \subseteq \mathbb{R}^N$ be a closed $\Lambda_k$-invariant set such that $X \cap D$ is definable in $\mathcal{M}$. Let $\gamma: (0,\infty) \to \mathbb{R}^N$ be an $\mathcal{M}$-definable curve whose image $\mathcal{C}$ is contained in $X$ and such that $\gamma_2$ is unbounded in $H_2$.

Let $\vec{v}_2 \in H_2$ be the limit of $\gamma_2(t)\|\gamma_2(t)\|$ as $t$ tends to $\infty$. If $\lambda \in \Lambda_k$ is sufficiently close to the space $\mathbb{R} \vec{v}_2$ and $\ell$ is the linear $\mathbb{R}$-subspace generated by $\lambda$ then a translate of some infinite segment of $\ell$ is contained in $X$.

**Proof.** We first make several observations:

1. If $B \subseteq H_2$ is a bounded definable set then $X \cap (H_1 + B)$ is definable (but not uniformly in $B$).

   Indeed, his is immediate from the fact that finitely many $\Lambda_k$-translates of $D$ cover $H_1 + B$, together with the fact that $X$ is $\Lambda$-invariant.

2. Let $U \subseteq \mathbb{R}^N$ be a definable set such that $\pi_2(U)$ is bounded, and let $\{A_s : s \in S\}$ be a definable family of subsets of $\mathbb{R}^N$. Then there is a number $K$ such that for all $\tau \in \mathbb{R}^N$ and $s \in S$, the set $A_s \cap (U + \tau) \cap X$ has at most $K$ connected components.

**Proof.** Note first that the family $A_s \cap (U + \tau) \cap X$ is not uniformly definable in $s, \tau$, so the straightforward o-minimal argument cannot work here. Let $V = U + D$. By (1), $V \cap X$ is definable. Let $\tau \in \mathbb{R}^N$. By the choice of $F$ we can find $\lambda \in \Lambda$ such that $\tau - \lambda \in D$. Since $X$ is $\Lambda$-invariant, we have that for any $s \in S$ the set $A_s \cap (U + \tau) \cap X$ is homeomorphic to $(A_s - \lambda) \cap (U + \tau - \lambda) \cap X$. Thus each set $A_s \cap (U + \tau) \cap X$ is homeomorphic to a set in the definable family

$$\{ (A_S + a) \cap (U + b) \cap (V \cap X) : s \in S, a, b \in \mathbb{R}^N \}.$$

By o-minimality, there is a uniform bound on the number of connected components of this family. **End of (2)**

We now return to the setting of the theorem, with $\vec{v}_2 \in H_2$ the limit of $\gamma_2(t)\|\gamma_2(t)\|$. We can re-coordinate $\mathbb{R}^N$ such that $H_2 = \{0_{N-k}\} \times \mathbb{R}^k$, and because $\gamma_2(t)$ is unbounded we may assume that the last coordinate of $v_2$, call it $\alpha$, is non zero (and in particular that the last coordinate of $\gamma_2(t)$, call $h_N(t)$, is unbounded). Using the last coordinate, we can re-parameterize $\gamma$ so that
lim_{t \to \infty} \gamma_2(t) = \hat{v}_2 \text{ (we do that by replacing } \gamma(t) \text{ by } \gamma(h_N^{-1}(\omega t))) \text{ Note that in this case, for every } r > 0, \text{ we have } lim_{t \to \infty} (\gamma_2(t + r) - \gamma_2(t)) = r \hat{v}_2.

**Fixing } K:\)

For } T \in \mathbb{R}^+ \text{ and } \bar{u} \in H_2, \text{ let } A_{T,\bar{u}} = \{ \gamma(T + t) - t\bar{u} : t \in \mathbb{R} \}, \text{ and let } U = H_1 + B, \text{ for } B \subseteq \mathbb{R}^k \text{ a ball of radius 1 around 0. Choose } K \text{ as in (2) above with respect to the family } \{ A_{T,\bar{u}} : T \in \mathbb{R}^+, \bar{u} \in H_2 \} \text{ and the set } U.

**Fixing } T_0:\)

**Claim 1.3.** Assume } \lambda \in \Lambda_k \text{ satisfies } |\lambda - r\hat{v}_2| \leq \frac{1}{2}(K + 1) \text{ for some } r \in \mathbb{R}. \text{ Then there exists } T_0 \in \mathbb{R}^+ \text{ such that for all } T > T_0 \text{ there is an open interval } I(T) \subseteq [0, K + 1] \text{ such that the set } \{ \gamma(T + t) - t\lambda : t \in I(T) \} \text{ is contained in } X.

**Proof.** Because } \gamma_2(T + r) - \gamma_2(T) \text{ tends to } r\hat{v}_2 \text{ as } T \text{ goes to } \infty, \text{ there exists } T_0 \text{ such that for every } T > T_0,

\begin{equation}
|\gamma_2(T + r) - \gamma_2(T) - \lambda| \leq |\gamma_2(T + r) - \gamma_2(T) - r\hat{v}_2| + |r\hat{v}_2 - \lambda| \leq 1/(K + 1).
\end{equation}

By a simple calculation, it follows that for every } n = 1, \ldots, K + 1, \text{ we have }

\begin{equation}
|\gamma_2(T + nr) - \gamma_2(T) - n\lambda| \leq 1. \text{ Said differently, } \gamma(T + nr) - n\lambda \text{ is in } U + \gamma(T), \text{ for all } n = 1, \ldots, K + 1 \text{ and } T > T_0.
\end{equation}

Because } C \subseteq X \text{ and } X \text{ is } \Lambda_k\text{-invariant, each } \gamma(T + nr) - n\lambda \text{ belongs to } A_{T,\lambda} \cap (U + \gamma(T)) \cap X, \text{ for } n = 1, \ldots, K+1. \text{ However, } A_{T,\lambda} \cap (U + \gamma(T)) \cap X \text{ has at most } K \text{ components, and therefore, for some interval } I(T) \subseteq [0, K + 1], \text{ the whole set } \{ \gamma(T + t) - t\lambda : t \in I(T) \} \text{ is contained in } A_{T,\lambda} \cap X \cap (U + \gamma(T)).

**End of Claim.**

We now choose a bounded interval } J \text{ all of whose elements are greater than } T_0, \text{ and observe that the set

\begin{equation}
V = \bigcup_{T \in J} (U + \gamma(T))
\end{equation}

projects onto a bounded subset of } H_2 \text{ and hence } X \cap V \text{ is definable. Consider the definable set

\begin{equation}
Y = \{ (T, t) : T \in J \& t \in [0, K + 1] \& \gamma(T + t) - t\lambda \in X \cap V \}.
\end{equation}

As we showed above, for every } T \in J \text{ there exists an infinite subset of } t \in [0, K + 1] \text{ such that } (T, t) \in Y, \text{ and therefore } dim(Y) = 2. \text{ It follows that for some fixed } T' \in \mathbb{R} \text{ there are infinitely many } (T, t) \in Y \text{ such that } T + t = T'. \text{ In particular, for a whole interval of } t's \text{ we have } \gamma(T') - t\lambda \in X. \text{ We thus found an interval in } \ell = \mathbb{R}\lambda \text{ whose translate (by } \gamma(T')) \text{ is contained in } X.

\[ \square \]

2. A LOWER BOUND FOR THE NUMBER OF TORSION POINTS

For a definable abelian } G, \text{ let } Tor_m(G) \text{ be the subgroup of all elements } x \in G \text{ such that } mx = 0. \text{ For a definable subset } D \text{ of } G \text{ and } m \in \mathbb{N} \text{ let } Tor_m(D) \text{ denote the intersection of } D \text{ with } Tor_m(G). \text{ Let } Tor(D) \text{ denote all torsion elements in } D.
**Roots of unity** Let us consider the torsion points of $\mathbb{C}^*$. It is known that (i) all primitive roots of unity of a fixed order $m$ are roots of the same irreducible polynomial, hence conjugates to each other. (ii) there are $\varphi(m)$ many such elements, with $\varphi(m)$ the Euler totient function. Furthermore, it is known that for every $0 < \epsilon < 1$, for all sufficiently large $m$, $\varphi(m) > m^\epsilon$, and hence $\varphi(m)/m^\epsilon \to \infty$ as $m$ tends to $\infty$.

**Theorem 2.1.** Let $S$ be a semiabelian variety, defined over a number field $F$, given by an exact sequence

$$
0 \longrightarrow (\mathbb{C}^*)^n \overset{\theta}{\longrightarrow} S \overset{\sigma}{\longrightarrow} A \longrightarrow 0,
$$

with $A$ an abelian variety. Let $X \subseteq S$ be an algebraic variety defined over $F'$ and assume that $\text{Tor}(X)$ is infinite. Then there exists an $\epsilon > 0$ such that

$$
\limsup_{m \in \mathbb{N}} \frac{\text{Tor}_m(X)}{m^\epsilon} = \infty.
$$

**Proof.** Given $Q \in \text{Tor}(S)$ of order $m$, let $a(Q)$ be the order of $\sigma(Q)$ in $A$. Then $m = a(Q)c(Q)$, where $c(Q)$ is the order of the element $a(Q) \cdot Q$ in $\theta((\mathbb{C}^*)^n)$. It follows that either $a(Q) \geq \sqrt{m}$ or $c(Q) \geq \sqrt{m}$.

**Case 1** Assume that there are infinitely many $Q \in \text{Tor}(X)$ such that $a(Q) \geq \sqrt{m}$.

By the result of Masser, [1], there exists an $\rho > 0$ and a constant $c_2(A)$, such that for every $P \in A$, if $\text{ord}(P) = a$ then $[F(P) : \mathbb{Q}] \geq c_2(A)a^\rho$. Because $\sigma(X)$ is defined over $F$, it is invariant under $F$-automorphisms and therefore if $\text{ord}(P) = a$ then every $F$-conjugate of $P$ is also in $\sigma(X)$ (and has the same order $a$). It is now claimed that $P$ has at least $c_3(A)a^\rho$ such conjugates, for some constant $c_3$. (Indeed, this is stated in Pila-Zannier, but needs some argument because $P$ is represented in projective coordinates as a tuple of complex algebraic numbers $(p_1, \ldots, p_k)$ with

$$
[F(p_1, \ldots, p_k) : \mathbb{Q}] \geq c_2(A)a^\rho.
$$

However, every such algebraic extension has at least as many automorphisms as its degree, as can be seen by taking a primitive element and considering its conjugates. Finally, any two distinct automorphisms must send $P$ to a different tuples).

If $Q \in \text{Tor}(X)$ of order $m$ such that $a = a(Q) \geq \sqrt{m}$ and $P = \sigma(Q)$ then every $F$-conjugate of $P$ is the image of some $F$-conjugate of $Q$. Hence, $Q$ has also at least $c_3(A)a^\rho \geq c_3(A)m^{1/\rho}$-many conjugates, all of order $m$. If we take $0 < \epsilon < 1/2\rho$ then, since there are elements $Q$ of unbounded order with this property, we have

$$
\limsup_{m \in \mathbb{N}} \frac{\text{Tor}_m(X)}{m^\epsilon} = \infty.
$$

**Case 2** Assume that there are infinitely many $Q \in \text{Tor}(X)$ of order $m$, such that $c(Q) \geq \sqrt{m}$.
In this case, if we let \( c = c(Q) \) and \( a = a(Q) \), then the element \( aQ \in \theta((\mathbb{C}^*)^n) \) has order \( c \). By our above observations, \( aQ \) has \( \varphi(c) \)-many conjugates over \( Q \) and hence \( \varphi(c)/d \)-many conjugates over \( F \) (with \( d = [F : Q] \)). Every such conjugate is of the form \( aQ' \) for some \( F \)-conjugate \( Q' \) of \( Q \). It follows that \( Q \) has at least \( \varphi(c)/d \)-many conjugates over \( F \), all of order \( m \) and all in \( X \). Again, by the above facts about \( \varphi \), if we pick \( \rho < 1 \) then there is some constant \( B \) such that \( Q \) has at least \( Bc(Q)^\rho \) \( F \)-conjugates. If \( c(Q) > \sqrt{m} \) then we have at least \( Bm^{1/2\rho} \)-many \( F \)-conjugates. Since this is true for every \( \rho < 1 \) and there are infinitely many such \( Q \)'s, we have \[
\limsup_m \frac{|\text{Tor}_m(X)|}{m^\epsilon} = \infty,\]
for \( \epsilon < 1/2 \). \( \square \)

3. MANIN-MUMFORD FOR SEMI-ABELIAN VARIETIES

In this section we identify varieties defined over \( \mathbb{C} \) with their \( \mathbb{C} \)-points. We also consider \( \mathbb{R} \) as a subfield of \( \mathbb{C} \), hence for each \( l \in \mathbb{N} \) we have \( \mathbb{R}^l \subset \mathbb{C}^l \).

We prove here Manin-Mumford for semi-abelian varieties:

**Theorem 3.1.** Let \( S \) be a semi-abelian variety over a number field \( K \) and let \( Y \subseteq S \) be an algebraic variety over \( K \). If \( \text{Tor}(S) \cap Y \) is infinite then \( Y \) contains an algebraic subgroup of \( S \).

**Proof.** We have an exact sequence

\[
0 \longrightarrow (\mathbb{C}^*)^n \xrightarrow{\theta} S \xrightarrow{\sigma} A \longrightarrow 0
\]

with \( A \) an abelian variety over \( K \), and \( \theta: (\mathbb{C}^*)^n \to G \) a morphism defined over \( F \). We assume that \( S \) has complex dimension \( m \).

Let \( L \) be the tangent space to \( S \) at the identity. It is a \( \mathbb{C} \)-vector space of complex dimension \( m \).

Since \( S \) is a connected abelian complex Lie group, there is a complex analytic surjective group homomorphism \( \text{Exp}_S: L \to S \), such that for any complex Lie subgroup \( S_1 \) of \( S \) the tangent space to \( S_1 \) at the identity is \( \{ \vec{v} \in L : \text{Exp}_S(\vec{v}) \in S_1 \} \). Let \( \Lambda = \ker(\text{Exp}_S) \).

**Claim 3.2.** There exists a real \( n \)-dimensional space \( H_1 \subseteq L \) such that, as a real space, \( L = H_1 \oplus H_2 \), with \( H_2 \) the real span of \( \Lambda \), and such that:

(i) The restriction of \( \text{Exp}_S \) to \( H_1 \) is definable in \( \mathbb{R}_{\exp} \).

(ii) If \( F \subseteq H_2 \) is a fundamental parallelogram for \( \Lambda \) then the restriction of \( \text{Exp}_S \) to \( F \) is definable in \( \mathbb{R}_{\text{an}} \). In particular, the restriction of \( \text{Exp}_S \) to \( H_1 + F \) is surjective on \( S \) and definable in \( \mathbb{R}_{\text{an,exp}} \).

(iii) If \( X \subseteq L \) is an algebraic variety of positive dimension then its image under the projection to \( H_2 \) is unbounded.

**Proof.** We can write \( (\mathbb{C}^*)^n \) as \( \mathbb{C}^* \oplus \ldots \oplus \mathbb{C}^* \) and \( \theta \) as \( (\theta_1, \ldots, \theta_n) \), where each \( \theta_j: \mathbb{C}^* \to S \) is an embedding. Let \( T_j = \theta_j(\mathbb{C}^*) \).

We choose a \( \mathbb{C} \)-basis \( \vec{v}_1, \ldots, \vec{v}_m \) for \( L \) so that for \( j = 1, \ldots, n \), the vector \( \vec{v}_j \) is in the tangent space to \( T_j \) at the identity. It follows then that \( \text{Exp}_S(z\vec{v}_j) \in T_j \) for any \( z \in \mathbb{C} \).
For $j = 1, \ldots, n$ let $\alpha_j : \mathbb{C} \to \mathbb{C}^*$ be the map
$$\alpha_j : z \mapsto \theta_j^{-1}(ExpS(z\vec{v}_j)).$$
Each $\alpha_j$ is a surjective group homomorphism that is complex analytic. Replacing $\vec{v}_j$ by $1/a_j\vec{v}_j$, where $a_j$ is the complex derivative of $\alpha_j$ at zero, we may assume that the differential of $\alpha_j$ at zero equals 1. It implies that each $\alpha_j$ is the complex exponentiation $z \mapsto e^z$. In particular, $\alpha(1) = e \in \mathbb{R}$ and hence for every $r \in \mathbb{R}$, $ExpS(r\vec{v}_j) \in \theta_j(\mathbb{R}^*)$.

We now consider $L$ as a real vector space of real dimension $2m$.

Let $H_1$ be the $\mathbb{R}$-span of $\vec{v}_1, \ldots, \vec{v}_n$. Using $\vec{v}_1, \ldots, \vec{v}_n$ we can identify $H_1$ with $\mathbb{R}^n$. Under this identification, for $(r_1, \ldots, r_n) \in \mathbb{R}^n$, we have $\theta^{-1}(ExpS(r_1, \ldots, r_n)) = (e^{r_1}, \ldots, e^{r_n})$. In particular, the restriction of $ExpS$ to $H_1$ is definable in $\mathbb{R}_{exp}$, since $\theta$ is a semi-algebraic map, and $ExpS(H_1)$ is a definable (maximal) torsion free subgroup of $S$.

The lattice $\Lambda$ is a discrete subgroup of $L$ of rank $2m - n$. We can extend $\vec{v}_1, \ldots, \vec{v}_n$ to an $\mathbb{R}$-basis $\{\vec{w}_1, \ldots, \vec{w}_{2m}\}$ of $L$ so that $\vec{w}_j = \vec{v}_j$ for $j = 1, \ldots, n$, $\{\vec{w}_{n+1}, \ldots, \vec{w}_{2m}\}$ form a $\mathbb{Z}$-basis for $\Lambda$ and furthermore $\vec{w}_{n+1}, \ldots, \vec{w}_{2m}$ belong to the tangent spaces of $T_1, \ldots, T_n$, respectively. If $H_2$ is the $\mathbb{R}$-span of $\Lambda$ then $L = H_1 \oplus H_2$. If we choose $F \subseteq H_2$ to be a fundamental parallelogram for $\theta|H_2$, then, since $\theta$ is analytic, its restriction to the compact $cl(F)$ is definable in $\mathbb{R}_{an}$.

For (iii), see the appendix. \qed

We now proceed exactly as in Pila-Zannier. For simplicity, we let $\pi : L \to S$ be the holomorphic map $ExpS$. As a real space, we can write $L = H_1 \oplus H_2$, as above. Let $\lambda_1, \ldots, \lambda_{2m-n}$ generate $\Lambda$ and fix a fundamental parallelogram
$$F = \{\Sigma_{i=1}^{2m-n} t_i \lambda_i : t_i \in [0, 1]\} \subseteq H_2$$
and $D = H_1 + F$, with $\pi|D$ definable in $\mathbb{R}_{an,exp}$.

Assume now that $Y \subseteq S$ is an algebraic variety such that $Y \cap Tor(S)$ is infinite. By Theorem 2.1, there exists an $\epsilon > 0$ such that
$$\limsup_{m \in \mathbb{N}} \frac{|Tor_m(Y)|}{m^\epsilon} = \infty.$$ If $X = \pi^{-1}(Y)$ then it is a complex analytic, $\Lambda$-invariant set, such that $X_1 = X \cap D$ is definable (as the pre-image of an algebraic set under a definable map). We now translate the data into $L$. We let $\mathbb{Q}\Lambda$ be the rational span of $\Lambda$ and for $h > 0$ we let $(\mathbb{Q}\Lambda)_h$ be all elements $\Sigma_{i=1}^{2m-n} q_i \lambda_i \in \mathbb{Q}\Lambda$ such that each $q_i \in \mathbb{Q}$ can be written as $m_i/n_i$ with $|m_i|, |n_i| \leq h$.

We now have:
$$\limsup_{m \in \mathbb{N}} \frac{|X_1 \cap (\mathbb{Q}\Lambda)_h|}{h^\epsilon} = \infty.$$ We now apply Pila-Wilkie, [2], and conclude that $X_1$ contains some piece of an irreducible real algebraic curve, call it $C$. If we let $C \subseteq L$ be the (complex algebraic) Zariski closure of $C$ then $C$ is a complex algebraic one-dimensional variety which is still contained in $X$. By Claim 3.2, the projection of $C$ on $H_2$ must be unbounded. It follows that $C$ contains a semi-algebraic curve $\gamma : (0, \infty) \to C$ whose projection onto $H_2$ is unbounded.

Let $\vec{v}_2 \in H_2$ be the limit of the unit tangent vector to $\gamma$ (the projection of $\gamma$ onto $H_2$). Then (see Pila-Zannier) there are elements of $\Lambda$ as close as
we wish to $\mathbb{R}v_2$. We can now apply Theorem 1.2 (or, in the abelian case Lemma 1.1) and conclude that $X$ contains an infinite interval $I$ in some translate of a real one-dimensional subspace of $L$. The Zariski closure $H$ of $I$, which must still be contained in $X$, is then a translate of a complex one-dimensional subspace of $L$. It follows that $Y$ contains $\pi(H)$, a coset of a subgroup of $S$, and by taking the Zariski closure of $\pi(H)$ we see that $Y$ contains a coset of a complex algebraic group.

\[\square\]

4. APPENDIX: Bounded projections of algebraic sets

**Lemma 4.1.** Let $L$ be complex linear space. Assume that, as a real space $L = H_1 \oplus H_2$, for $H_1, H_2$ real subspaces and that $\pi_2 : L \rightarrow H_2$ is the projection with respect to this decomposition. If $X \subseteq L$ is an irreducible complex algebraic variety and $\pi_2(X)$ is bounded then $X$ is contained in a translate of $H_1$.

**Proof.** Note that for any real subspace $H$ containing $H_1$, the projection of $X$ into the real space $L/H$ is bounded (independently of the coordinate system we choose for the quotient).

**Special case** Assume that $H_2 = iH_1$.

If $m$ is the real dimension of $H_1$ then $L$ is isomorphic as a complex space to $\mathbb{C}^m$, with $H_1$ going to $\mathbb{R}^n$ and $H_2$ to $i\mathbb{R}^n$. Under this identification all elements of $X$ are of the form $\vec{z} = (z_1, \ldots, z_m)$ with the imaginary part of the $z_i$'s all contained in some bounded set, as $\vec{z}$ varies in $X$. It easily follows that $X$ is a singleton.

We use induction on the complex dimension of $L$.

If $H_1$ contains an complex subspace $K$ of positive dimension then we replace $L$ by $L/K = H_1/K \oplus H_2/K$ (with $H_2/K \simeq H_2$). By induction, the image of $X$ in $L/K$ is contained in a translate of $H_1/K$, which implies that $X$ is contained in a a translate of $H_1$. We may therefore assume that $H_1$ contains no nonzero complex subspace. It follows that $iH_1 \cap H_1 = \{0\}$.

Let $\bar{H}_1$ be the complex space $H_1 \oplus iH_1$ and write $L$ as $\bar{H}_1 \oplus \bar{H}_2$ for some complex subspace $\bar{H}_2$. The projection of $X$ into $\bar{H}_2$ is an irreducible complex constructible set and, as was observed above, is still bounded. This image must then be a singleton and hence $X$ is contained in a translate of $\bar{H}$.

We may then assume that $L = \bar{H} = H_1 \oplus iH_1$ and that the projection of $X$ into $iH_1$ is bounded. By the special case, we are done.

\[\square\]

**References**

