O-minimality and Arithmetic.
The Pila-Zannier method

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Some Bibliography

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A general problem scheme

Setting

$\mathcal{C}$ = an underlying family of sets

$\mathcal{S} \subseteq \mathcal{C}$ is a collection of so-called "special" $\mathcal{C}$-sets

$\mathcal{S}_0$ = a set of so-called "special" points, often these are the $\mathcal{S}$-sets of dimension zero.

The problem scheme

Start with an $\mathcal{S}$-set V and consider an arbitrary $\mathcal{C}$-set $X \subseteq V$. Assume that X has "many" special points. Then X contains a special set of positive dimension. Under additional assumptions, X itself is a special set.

The Pila-Wilkie results (viewed in this scheme)

Fix $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, \ldots\ldots \rangle$ an o-minimal expansion of the real field.

$\mathcal{C}$ = the family of all definable sets in $\mathcal{M}$.

$\mathcal{S}$ = The family of semi-algebraic sets (defined over $\mathbb{Q}$).

$\mathcal{S}_0$ = points in $(\mathbb{Q}^{alg})^n \cap \mathbb{R}^n$.

The Pila-Wilkie theorem(s)

Assume that $X \subseteq \mathbb{R}^n$ is definable in $\mathcal{M}$. If $X \cap (\mathbb{Q}^{alg})^n$ is *large** then X contains a connected infinite semi-algebraic set defined over $\mathbb{Q}$. More precisely, if one removes **all** infinite connected semi-algebraic subsets of X then a *small** number of $\mathbb{Q}^{alg}$-points remains.

$X \cap (\mathbb{Q}^{alg})^n$ is **large*** if exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that

$$limsup_T \frac{|\{\bar{q} \in X \cap (\mathbb{Q}_k^{alg})^n \ : \ height_k(\bar{q}) \leqslant T\}|}{T^\epsilon} = \infty.$$

From now on-the algebraic general problem scheme

The algebraic presentation

$\mathcal{C} =$ complex algebraic (irreducible) varieties, (quasi) affine or projective.

$\mathcal{S} =$ a specified subfamily of “special” varieties.

$\mathcal{S}_0 =$ 0-dimensional $\mathcal{S}$-sets: special points.

$V =$ an irreducible $\mathcal{S}$-variety.

$X \subseteq V$ = an irreducible complex algebraic subvariety

Assumption

The set $X \cap \mathcal{S}_0$ is Zariski dense in X.

Goal

The variety X is itself in $\mathcal{S}$.

A test case-the exponential example (algebraic torus)

The algebraic side

Let $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$ (so here V admits the structure of an algebraic group, which is also a complex Lie group).

$\mathcal{C} = \{X \subseteq (\mathbb{G}_m)^n : X$ an irreducible algebraic variety$\}$.

$\mathcal{S} = \{A + p :\ A$ a conn. algebraic subgrp of $\mathbb{G}_m^n$ & p a torsion point$\}$.

$\mathcal{S}_0 =$ Torsion points in $(\mathbb{G}_m)^n$

Theorem (Laurent)

If $X \subseteq (\mathbb{G}_m)^n$ an irreducible algebraic variety and $X \cap Tor(\mathbb{G}_m)^n$ is Zariski dense in X then $X = A + p$ for some $A \leqslant (\mathbb{G}_m)^n$ and $p \in Tor(\mathbb{G}_m)^n$.
Namely,
If $X \in \mathcal{C}$ and $X \cap \mathcal{S}_0$ is Zariski dense in X then $X \in \mathcal{S}$.

Back to the general problem-the analytic presentation

We work with affine (or projective) algebraic variety V and an algebraic subvariety $X \subseteq V$.

An analytic covering map

We have $\widetilde{V}$ = a (semi-algebraic) open subset of $\mathbb{C}^n$ (with $n = \dim V$).
And $\Theta : \widetilde{V} \to V$ a holomorphic, **transcendental,** surjection.

General strategy

Replace V and its algebraic variety $X \subseteq V$ by $\widetilde{V}$ and a complex analytic subvariety $\Theta^{-1}(X) \subseteq \widetilde{V}$.

Caution

In general, Θ and $\Theta^{-1}(X)$ are not definable in any “tame” structure.
We will need to “truncate” them.

The analytic presentation: additional data

An underlying group action

We have $G =$ a real algebraic group acting semi-algebraically and transitively on $\widetilde{V}$. In some cases $\widetilde{V} = G$.

Γ= an infinite discrete subgroup of G (not necessarily normal).

The map $\Theta : \widetilde{V} \to V$ is Γ-invariant. Namely, $\Theta(x) = \Theta(y)$ if and only if $\Gamma x = \Gamma y$.

So, V can be identified with $\Gamma \backslash \widetilde{V}$.

If $X \subseteq V$ is a complex algebraic subvariety then $\Theta^{-1}(X) = \widetilde{X}$ is a Γ-invariant analytic subvariety of $\widetilde{V}$.

In general, $\widetilde{X}$ might have infinitely many connected components.

The analytic presentation-special varieties

$\widetilde{special}$-varieties and points

An irreducible analytic subvariety $Y \subseteq \widetilde{V}$ **is called a $\widetilde{\textbf{special}}$ variety** if $\Theta(Y)$ is a spacial subvariety of V. In particular, $\Theta(Y)$ is algebraic (**!**).

A point $z \in \widetilde{V}$ is $\widetilde{\textbf{special}}$ if $\Theta(z)$ is a special point. Namely $\Theta(z) \in \mathcal{S}_0$.

Fact (an alternative definition): $\widetilde{\text{special}}$ varieties as orbits

An irreducible complex analytic variety $\widetilde{X} \subseteq \widetilde{V}$ is $\widetilde{\text{special}}$ iff
(i) $\Theta(\widetilde{X})$ is an algebraic subvariety of V.
(ii) There exists a real algebraic subgroup $H \subseteq G$ such that $\widetilde{X}$ is an orbit of H. In case $\widetilde{V} = G$ it means that $\widetilde{X}$ is a coset. (Note: it follows in either case that $\widetilde{X}$ is real algebraic).
(iii) $\widetilde{X} \cap \widetilde{S}_0 \neq \varnothing$.

- If only (i) and (ii) hold then $\widetilde{X}$ is called **weakly** $\widetilde{\textbf{special}}$.

The exponential example: the analytic presentation

Recall $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$. Take $\widetilde{V} = \mathbb{C}^n$ and $\Theta := \exp : \mathbb{C}^n \to (\mathbb{C}^*)^n$ defined by $exp(z_1, \dots, z_n) = (e^{z_1}, \dots, e^{z_n})$.

So $\Theta : (\mathbb{C}^n, +) \to ((\mathbb{G}_m)^n, *)$ is a holomorphic group homomorphism.

$\Gamma := Ker(\Theta) = (2\pi i\mathbb{Z})^n$. Clearly, Θ is Γ-invariant.

$\widetilde{\text{special}}$ points

Because Θ is a homomorphism, $\Theta(z)$ is a torsion point of order k iff $kz \in \Gamma$. So, $\widetilde{\mathbb{S}}_0 = \{\bar{z} \in \mathbb{C}^n : \exists k \; k\bar{z} \in (2\pi i\mathbb{Z})^n\} = (2\pi i\mathbb{Q})^n$.

$\widetilde{\text{special}}$ varieties

An irreducible analytic $Y \subseteq \mathbb{C}^n$ is $\widetilde{\text{special}}$ if $\Theta(Y) = p + A$, where A is an algebraic subgroup of $(\mathbb{G}_m)^n$ and $p \in Tor(\mathbb{G}_m)^n$. So, $Y = \bar{q} + H$, where H is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ **defined over** $\mathbb{Q}$, and $\bar{q} \in (2\pi i\mathbb{Q})^n$. (Note: these are exactly the cosets of real subgroups of $\mathbb{C}^n$ which project onto algebraic varieties).

The general case: The (non) definability of Θ

We have $\Theta : \widetilde{V} \to V \quad \sim \Gamma\backslash\widetilde{V}$

The **non-definability of Θ**

Since Γ is infinite discrete, $\Theta^{-1}(p)$ is an infinite discrete set (for every $p \in V$). Hence, the map Θ is never definable in an o-minimal structure.

Instead we aim for a small subset of $\widetilde{V}$ on which Θ is definable in **some** o-minimal structure.

Fundamental sets

A closed semi-algebraic set $\mathfrak{F} \subseteq \widetilde{V}$ is **a fundamental set for Θ** if:
(i) $\Theta(\mathfrak{F}) = V$ (i.e. $\Gamma \cdot \mathfrak{F} = \widetilde{V}$)
(ii) There are only finitely many $\gamma \in \Gamma$ such that $\gamma \cdot \mathfrak{F} \cap \mathfrak{F} \neq \varnothing$.

By (i), the quotient $\Gamma\backslash\mathfrak{F}$ can be identified with V. By (ii), the relation $\Theta(z) = \Theta(w)$ is semi-algebraic on $\mathfrak{F}$. So, the quotient $\Gamma\backslash\mathfrak{F}$ is semi-algebraic.

The ingredients for the Pila-Zannier method

We have $\Theta : \widetilde{V} \to V \sim \Gamma\backslash\widetilde{V}$. $\mathcal{S}_0 \subseteq V$ the set of special points.

I. Definability requirements (from algebraic to o-minimal)

One needs to establish the existence of a semialgebraic fundamental set $\mathfrak{F} \subseteq \widetilde{V}$ and the definability of $\Theta \restriction \mathfrak{F}$ in some o-minimal structure $\mathcal{M}$. In all examples, $\mathcal{M}$ is $\mathbb{R}_{an,exp}$.

For $X \subseteq V$ algebraic, let $\widetilde{X} \subseteq \widetilde{V}$ be an irreducible analytic component of $\Theta^{-1}(X)$. Note that $\widetilde{X} \cap \mathfrak{F} = (\Theta \restriction \mathfrak{F})^{-1}(X)$ is definable in $\mathcal{M}$.

II. Number theory goal

• The set $\widetilde{\mathcal{S}}_0 = \Theta^{-1}(\mathcal{S}_0)$ is contained in $\mathbb{Q}_k^{alg}$ for some k(up to definable bijection).

►► If $X \cap \mathcal{S}_0$ (on the algebraic side) is Zariski dense in X then $\widetilde{\mathcal{S}}_0 \cap (\widetilde{X} \cap \mathfrak{F})$ (on the analytic side) is large* (in the sense of Pila-Wilkie). This is "the lower bound".

The ingredients of the Pila-Zannier method (cont)

The Pila-Wilkie input

- Assume that we established that $\widetilde{\mathcal{S}}_0 \cap (\widetilde{X} \cap \mathfrak{F})$ is large*.
- By PW, There exists a connected semi-algebraic nontrivial curve $C \subseteq \widetilde{X} \cap \mathfrak{F}$.
- Let $\overline{C} \subseteq \mathbb{C}^n$ be the Zariski closure of C. It is a complex algebraic curve, and by dimension considerations $(\overline{C} \cap \widetilde{V}) \subseteq \widetilde{X}$.
- So $\widetilde{X}$ **contains a complex algebraic curve** (relative to the open semialgebraic $\widetilde{V}$).

The Pila-Zannier method: The punch-line!

We have $\Theta : \widetilde{V} \to V \sim \Gamma\backslash\widetilde{V}$. $\widetilde{X} \subseteq \widetilde{V}$ a component of $\Theta^{-1}(X)$.

The general idea

Take $\widetilde{A}$ a maximal algebraic subset of $\Theta^{-1}(X)$.
The Γ-periodicity of $\Theta^{-1}(X)$ together with the algebraicity of $\widetilde{A}$ is "unlikely" and should imply that the stabilizer of $\widetilde{A}$ in $G(\mathbb{R})$ is nontrivial.
In fact, it should imply that $\widetilde{A}$ is "special".

More precisely,

Ingredient III, the "Ax-Lindemann" goal

Assume that $\widetilde{A}$ is a maximal irreducible algebraic (relative to $\widetilde{V}$) subset of $\widetilde{X}$.
Then $\widetilde{A}$ is a weakly special variety. Namely,

(i) $\widetilde{A}$ is an orbit of a real algebraic subgroup of G (defined over $\mathbb{Q}$).

(ii) $\Theta(\widetilde{A})$ is an algebraic subvariety of V.

Summary of the Pila-Zannier method

We have $X \subseteq V$, $\Theta : \widetilde{V} \to V$ and $X \cap \mathcal{S}_0$ Zariski dense in X.

I. Definability

$\Theta \restriction \mathfrak{F}$ is definable in an o-minimal structure.

II. Number Theory

The set $\widetilde{S_0} \cap (\Theta^{-1}(X) \cap \mathfrak{F})$ is large*.

Application of the Pila-Wilkie Theorem.

III. Ax-Lindemann

If $\widetilde{A} \subseteq \Theta^{-1}(X)$ is maximal irreducible algebraic then it is weakly special. (So, f in addition $\widetilde{A} \cap \widetilde{\mathcal{S}_0} \neq \varnothing$ then $\widetilde{A}$ is special).

We conclude: X contains a special variety $\Theta(\widetilde{A})$.

Applying Pila-Zannier to the exponential case

We have $V = (\mathbb{C}^*)^n$

$\mathcal{S}_0 = Tor(\mathbb{C}^*)^n$

$\mathcal{S} = \{A + p : A \leqslant (\mathbb{C}^*)^n \ , \ p \in \mathcal{S}_0\}$.

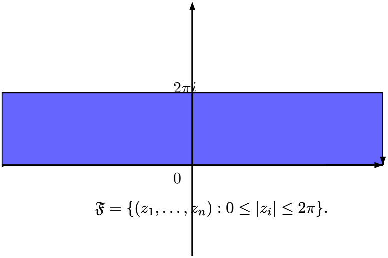
I. Fundamental set and the definability of Θ

We have $\Theta : \mathbb{C}^n \to (\mathbb{C}^*)^n$ given by $\Theta(z_1, \dots, z_n) = (e^{z_1}, \dots, e^{z_n})$.

- A fundamental set for Θ is:

$$\mathfrak{F} = \{\bar{z} = (z_1, \dots, z_n) \in \mathbb{C}^n : 0 \leqslant |Im(z_i)| \leqslant \pi\}.$$

The exponential case: the definability of $\Theta \restriction \mathfrak{F}$



$\Theta \restriction \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$:

We have $e^z = e^{x+iy} = e^x(cosy + i\,siny)$.

The map e^x is definable in $\mathbb{R}_{exp}$; the maps $cos, sin \restriction [0, 2\pi]$ are definable in $\mathbb{R}_{an}$, hence:

The map $e^z \restriction \{0 \leqslant Im(z) \leqslant 2\pi\}$ is definable in $\mathbb{R}_{an,exp}$.

It follows that $\Theta \restriction \mathfrak{F} = exp \restriction \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

The exponential case: II. Basic Number Theory

• Assume that $X \subseteq (\mathbb{C}^*)^n$ is an irreducible algebraic variety and that $X \cap Tor(\mathbb{C}^*)^n$ is Zariski dense in X.

We want to show that $(2\pi i\mathbb{Q})^n \cap (\Theta^{-1}(X) \cap \mathfrak{F})$ is large*.

• X is defined over a number field k. For simplicity, $k = \mathbb{Q}$.

• Since $X \cap Tor(\mathbb{C}^*)^n$ is infinite there are natural numbers $m_1 < m_2 < \ldots$ and elements $g_i \in X$, with $ord(g_i) = m_i$.

• If $g \in (\mathbb{C}^*)^n$, and $ord(g) = m$ then g has at least $\phi(m)$ conjugates over $\mathbb{Q}$, where $\phi(m) = \#\{i \leqslant m : (i, m) = 1\}$ is the Euler function.

Fact For every $0 < \epsilon < 1$, $\lim \phi(m)/m^\epsilon = \infty$.

Hence, $\lim_i \frac{|\{g \in X : ord(g) = m_i\}|}{m_i^{1/2}} = \infty$.

Corollary

The following set is large*

$$\{(q_1, \ldots, q_n) \in \mathbb{Q}^n : \Sigma_j 2\pi i q_j \in \Theta^{-1}(X) \cap \mathfrak{F}\}$$

The exponential case: the Ax Lindemann statement

The Pila-Wilkie input

The analytic set $\Theta^{-1}(X) \subseteq \mathbb{C}^n$ contains a nontrivial algebraic set. Take a maximal such irreducible algebraic set A.

Goal: A **is weakly special** $=$ a coset of a linear s.space of $\mathbb{C}^n$ over $\mathbb{Q}$.

A proof using the classical Ax's theorem (corrected)

Ax's Theorem If $\xi_1, \ldots, \xi_n \in \mathbb{C}(A)$ and $lin.dim_{\mathbb{Q}}(\bar{\xi}/\mathbb{C}) = m$ then $tr.deg(\mathbb{C}(exp(\xi_1), \ldots, exp(\xi_n))/\mathbb{C}) = m$.

- Take $H \subseteq \mathbb{C}^n$ a minimal subspace $/\mathbb{Q}$ with $A \subseteq H + p$ for $p \in \mathbb{C}^n$. Let $m = \dim H$.
- We have $\Theta(A) \subseteq \Theta(H) + \Theta(p)$, and $\Theta(H) \leqslant (\mathbb{C}^*)^n$ algebraic.
- Take $\xi_1, \ldots, \xi_n \in \mathbb{C}(A)$ coordinate functions in the function field of A. Then $lin.dim_{\mathbb{Q}}(\bar{\xi}/\mathbb{C}) = m$, so by Ax's theorem $tr.deg(\mathbb{C}(\Theta(\bar{\xi}))/\mathbb{C}) = m$ $= \dim(\Theta(H) + \Theta(p))$.

Ax-Lindemann (cont)

It follows that $\Theta(A)$ is Zariski dense in $\Theta(H) + \Theta(p)$, so $\Theta(H) + \Theta(p) \subseteq X$.

Hence, $A \subseteq H + p \subseteq \Theta^{-1}(X)$.

By maximality, $A = H + p$, so A is weakly special. □

Summary of proof in the exponential case

• We started with $X \subseteq (\mathbb{G}^m)^n$ such that $Tor(\mathbb{G}_m)^n \cap X$ is Zariski dense in X.

• Using Pila-Wilkie, we concluded that $\Theta^{-1}(X)$ contained a nontrivial complex algebraic set A. Furthermore we can choose it so $A \cap \widetilde{\mathbb{S}}_0$ is nonempty. Take such A maximal.

• By Ax, A is weakly $\widetilde{\text{special}}$, hence $\widetilde{\text{special}}$ ($A \cap \widetilde{\mathbb{S}}_0 \neq \varnothing$).

• It follows that X contains a nontrivial special set $\Theta(A)$..

• By using the full strength of Pila-Wilkie we could show that X **is** actually special.

Another example: The Manin-Mumford conjecture

The setting

$V =$ an abelian variety in $\mathbb{P}^n(\mathbb{C})$.

So, V is a projective algebraic variety which admits an algebraic group structure, abelian. It is also a compact, complex Lie group.

$\mathcal{C} =$ all irreducible algebraic subvarieties of V.

$\mathcal{S} =$ all cosets of the form $A + p$, where $p \in Tor(V)$ and A a connected algebraic subgroup (i.e. abelian subvariety) of V.

$\mathcal{S}_0 = Tor(V)$ the torsion elements.

The Manin-Mumford conjecture (Raynaud's Theorem, 1983)

Assume that V is a complex abelian variety defined over a number field, and $X \subseteq V$ an irreducible algebraic subvariety. If $X \cap Tor(V)$ is Zariski dense in V then $X = A + p$ as above.

The analytic presentation

- There exists a holomorphic group homomorphism $\Theta : (\mathbb{C}^n, +) \to V$.
- $\Gamma := Ker(\Theta)$ is a $2n$-lattice. I.e., $\Gamma = \Sigma_{i=1}^{2n} \mathbb{Z}\omega_i$, where $\omega_1, \ldots, \omega_{2n}$ are linearly independent over $\mathbb{R}$.
(Note: While every $2n$-lattice gives rise to a complex torus, it might not give rise, if $n > 1$, to an **projective** complex torus, i.e. abelian variety.)
- **special points** $= \Theta^{-1}(Tor(V)) = \mathbb{Q}\Gamma = \Sigma_{i=1}^{2n} \mathbb{Q}\omega_i$.
- **special varieties** are cosets of the form $\bar{z} + H$, where H a complex linear subspace defined over $\mathbb{Q}$ and $\bar{z} \in \mathbb{Q}\Gamma$.
- **Weakly special varieties** are arbitrary cosets of such H.

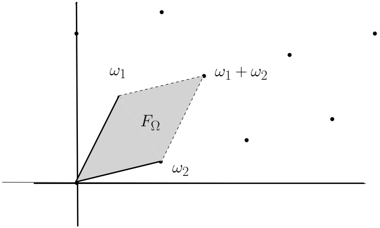
(weakly) special varieties as orbits

Note that the weakly special varieties are exactly those orbits (i.e., cosets) of real subgroups of $(\mathbb{C}^n, +)$ which project onto algebraic subvarieties of V.

The Pila-Zannier method for Manin-Mumford

I. The fundamental set and definability of $\Theta \restriction \mathfrak{F}$

Consider the compact semilinear parallelogram
$\mathfrak{F} = \{\sum_{i=1}^{2n} t_i\omega_i : 0 \leqslant t_i \leqslant 1\}$. Then:
(i) $\Gamma + \mathfrak{F} = \mathbb{C}^n$.
(ii) The set $\{\gamma \in \Gamma : (\gamma + \mathfrak{F}) \cap \mathfrak{F} \neq \varnothing\}$ is finite.
$\mathfrak{F}$ is a fundamental set for Θ.



Since Θ is analytic on $\mathbb{C}^n$ and $\mathfrak{F}$ compact, $\Theta \restriction \mathfrak{F}$ is definable in the o-minimal $\mathbb{R}_{an}$ (by considering the real and imaginary parts of Θ).

Pila-Zannier for Manin-Mumford (cont)

II. Number Theory (on the algebraic side)

- V is an abelian variety defined over a number field F.
- $X \subseteq V$ is irreducible algebraic, with $X \cap Tor(V)$ Zariski dense in X.
- So, X is also defined over a number field $k \supseteq F$.

Number theoretic input (Masser)

There exists $\rho = \rho(V) > 0$ and a constant c, such that for every $P \in V$, if $ord(P) = T$ then $[F(P) : \mathbb{Q}] \geqslant cT^{\rho}$.

By conjugating $X \cap Tor(V)$ over k we conclude: if $\epsilon < \rho(V)$ then

$$limsup_T \frac{|\{P \in X : ord(P) \leqslant T\}|}{T^{\epsilon}} = \infty.$$

Conclusion: on the analytic side

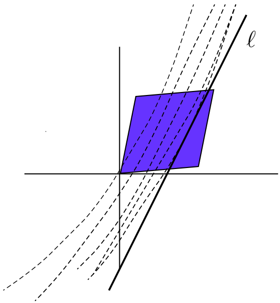
The set $\{(q_1, \ldots, q_{2n}) \in \mathbb{Q}^{2n} : \Sigma_{j=1}^{2n} q_j \omega_i \in \Theta^{-1}(X) \cap \mathfrak{F}\}$ is **large***.

III. Ax-Lindemann: an o-minimal argument

The Pila-Wilkie input

The analytic variety $\Theta^{-1}(X)$ contains an unbounded semialgebraic curve σ.

By the o-minimality of σ, when we translate it into $\mathfrak{F}$ by elements of Γ we get (inside $\widetilde{X}$) curves which are more and more "linear". Since $\widetilde{X} \cap \mathfrak{F}$ is compact, at the limit we get an affine line $\ell \subseteq \widetilde{X}$.



Finishing the proof of MM

On the analytic side

We saw that $\Theta^{-1}(X)$ contains an affine line $\ell \subseteq \mathbb{C}^n$.

Back to the algebraic side

The variety $X \subseteq V$ contains $\Theta(\ell)$, a coset of a subgroup.

The Zariski closure of $\Theta(\ell)$ is a coset of an algebraic subgroup of V, which is contained in X.

Hence, X contains a (weakly) special variety $z + A$, for $A \leqslant X$.

By using the full strength of Pila-Wilkie, together with the ability to write V as a an almost direct product $A \oplus B$, we can show that X itself is a special variety.
END of the proof of Manin-Mumford.

Andre-Oort setting

The general analytic setting for Shimura varieties (simplified)

• $G(\mathbb{R})$ is the $\mathbb{R}$-points of an algebraic semisimple group G over $\mathbb{R}$.

• $K \leqslant G(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$.

• (with additional assumptions) the quotient space $G(\mathbb{R})/K$ admits the structure of an open semi-algebraic subset of $\mathbb{C}^n$. This set is our $\widetilde{V}$.
• $G(\mathbb{R})$ acts on $\widetilde{V}$. Actually, for every $g \in G(\mathbb{R})$, $g : \widetilde{V} \to \widetilde{V}$ is a biholomorphism.

• Let $\Gamma = G(\mathbb{Z})$ (more generally, an arithmetic subgroup), and consider the quotient $V = \Gamma \backslash \widetilde{V}$.

The Baily-Borel Theorem (1966)

There exists a holomorphic embedding $\Theta : \Gamma \backslash \widetilde{V} \to \mathbb{P}^m(\mathbb{C})$ whose image is a quasi-projective variety.

$Im(\Theta) = V$ is ***a Shimura variety*** (a non-specialist viewpoint).

The Shimura variety $\mathbb{C}^n$: Preliminaries

We start with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$.

The group $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ (transitively) as follows:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\tau \in \mathbb{H}$ then $A \cdot \tau = \frac{a\tau+b}{c\tau+d}$.

Connection to elliptic curves

$\mathbb{H}$ is a parameter space for elliptic curves, namely, every τ represents the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ where Λ_τ the lattice $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$.

$E_{\tau_1} \cong E_{\tau_2} \Leftrightarrow \tau_1, \tau_2$ are in the same $SL(2, \mathbb{Z})$-orbit. So, $SL(2, \mathbb{Z})\backslash\mathbb{H}$ is the moduli space of elliptic curves.

The J-invariant

There exists a holomorphic, transcendental surjection $J : \mathbb{H} \to \mathbb{C}$ such that $J(\tau_1) = J(\tau_2) \Leftrightarrow SL(2, \mathbb{Z})\tau_1 = SL(2, \mathbb{Z})\tau_2$.

André-Oort for $\mathbb{C}^n$

We now begin on the analytic side

- $\widetilde{V} = \mathbb{H}^n$.
- $G(\mathbb{R}) = SL(2,\mathbb{R})^n$ acts on $\mathbb{H}^n$ in coordinates.
- The action is transitive so $\mathbb{H}^n = G(\mathbb{R})/stab_G(\bar{z})$ for any $\bar{z} \in \mathbb{H}^n$.
- Since $stab(i,\ldots,i) = O(2,\mathbb{R})^n$, we have $\mathbb{H}^n = SL(2,\mathbb{R})^n/O(2,\mathbb{R})^n$ (namely, $K = O(2,\mathbb{R})^n$).

Note: $\widetilde{V}$ is not a group anymore. It is a semialgebraic homogenous space.

- Let $\Gamma = SL(2,\mathbb{Z})^n$ and $\Theta := (J,\ldots,J) : \mathbb{H}^n \to \mathbb{C}^n$. Θ is a Γ-invariant surjection.

On the algebraic side

We define $V := \mathbb{C}^n \sim \Gamma\backslash\mathbb{H}^n$, via Θ.

Special varieties and points

Again, the definition begins on the analytic side.

Definition of special points: The set $\widetilde{\mathbb{S}}_0$

$(\tau_1, \ldots, \tau_n) \in \mathbb{H}^n$ is **special**, if for every i, the elliptic curve E_{τ_i} has complex multiplication ($End(E_\tau) \neq \mathbb{Z}$).
Equivalently, τ_i belongs to an imaginary quadratic extension of $\mathbb{Q}$.
(abstract definition of special points in Shimura varieties-omitted here).

Definition of special varieties

Recall: An irreducible analytic variety $Y \subseteq \mathbb{H}^n$ is **special** if
(i) Y is an orbit of a real algebraic group $H \leqslant SL(2, \mathbb{R})^n$.
(ii) $\Theta(Y) \subseteq \mathbb{C}^n$ is an algebraic variety.
(iii) $Y \cap \widetilde{\mathbb{S}}_0 \neq \varnothing$.

Special varieties and points in $V = \mathbb{C}^n$

The image under Θ of a $\widetilde{\text{special}}$ point is **special** in $\mathbb{C}^n$. $\mathbb{S}_0 := \Theta(\widetilde{\mathbb{S}}_0)$.

The Image under Θ of $\widetilde{\text{special}}$ variety is **special** in $\mathbb{C}^n$.

Examples of special varieties

• $\widetilde{X} = \{\tau\} \times \mathbb{H}^{n-1}$, with $\tau \in \widetilde{\mathbb{S}}_0$; it is an orbit of $H = \{1\} \times SL(2,\mathbb{R})^{n-1}$.
$\Theta(\widetilde{X}) = \{p\} \times \mathbb{C}^{n-1}$ is a special variety.

• $\widetilde{X} = \{(\tau, N\tau) : \tau \in \mathbb{H}\} \times \mathbb{H}^{n-2}$, for some $N \in \mathbb{N}$. It is an orbit of $H_1 \times SL(2,\mathbb{R})^{n-2}$, with $H_1 = \{(g, hgh^{-1}) : g \in SL(2,\mathbb{R})\}$ and

$$h = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$$

$\Theta(\widetilde{X}) = Z(\Phi_N) \times \mathbb{C}^{n-2}$ where Φ_N is the zero set of a modular polynomial.

Moonen's work

Every special variety in $\mathbb{C}^n$ is obtained from the above examples by permutation of variables and cartesian products.

The statement of the theorem

The André-Oort Conjecture for $\mathbb{C}^n$ (a theorem of Pila)

If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety and $X \cap \mathcal{S}_0$ is Zariski dense in X then X is special.

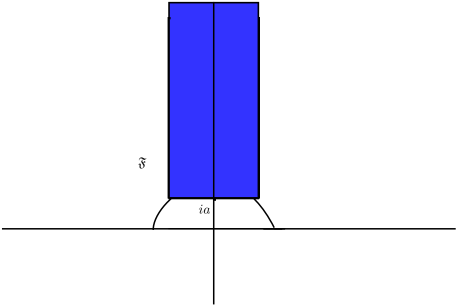
By the nature of the definitions, we immediately have an analytic presentation of the problem:

- We have $\Theta : \mathbb{H}^n \to \mathbb{C}^n$ given by the J function in each coordinate.
- We have notions of $\widetilde{\text{special}}$ points and varieties in $\mathbb{H}^n$.

The Pila Zannier method: I. The fundamental set

By the basic theory of elliptic curves, the following is a fundamental set for $SL(2,\mathbb{Z})$ (for every $0 < a < \sqrt{3}/2$):

$$\mathfrak{F} = \{z \in \mathbb{H} : \ -1/2 \leqslant Re(z) \leqslant 1/2 \,\&\, Im(z) > a\}.$$



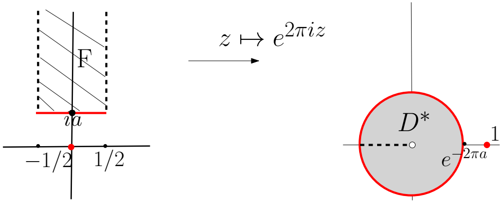
So $\mathfrak{F}^n$ is a fundamental set for $SL(2,\mathbb{Z})^n$.

Pila-zanner method I: Definability of $J \restriction \mathfrak{F}$

Theorem

The restriction of J to $\mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

Proof Consider first the map $z \mapsto e^{2\pi iz}$. It sends $\mathfrak{F}$ onto a punctured disc D^*. The “point” $Im(z) = \infty$ is sent to 0.



• The restriction of $e^{2\pi iz}$ to $\mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.
• As pointed out in an earlier talk, we may write J in the variable $q = e^{2\pi iz}$ and obtain a meromorphic function on D^*. Hence (???) $J(q)$ is definable in $\mathbb{R}_{an}$. It follows that $J(z)$ is definable in $\mathbb{R}_{an,exp}$.

II. Number Theory

We have $\Theta : \mathbb{H}^n \to \mathbb{C}^n$, and $X \subseteq \mathbb{C}^n$ algebraic, with $X \cap \mathcal{S}_0$ Zariski dense in X. We use $\mathfrak{F}$ for the fundamental set for Θ $(= \mathfrak{F}^n)$.

On the analytic side

Let $\widetilde{X} \subseteq \mathbb{H}^n$ be an irreducible **analytic** $\underbrace{\text{component of}}$ $\Theta^{-1}(X)$.

We already saw that if $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{H}^n$ is special then each τ_i is imaginary quadratic, so $\widetilde{\mathcal{S}}_0 \subseteq (\mathbb{Q}_2^{alg})^n$..

Using a theorem of Siegel on imaginary quadratic fields, Pila proves:

Largness of special points

The set $\widetilde{\mathcal{S}}_0 \cap \widetilde{X} \cap \mathfrak{F}$ is large*.

III. The Ax-Lindemann statement

The Pila-Wilkie input

$\widetilde{X}$ contains an algebraic set of positive dimension (relative to $\mathbb{H}^n$). Let A be maximal irreducible such set.

Goal

A is weakly special. Namely
(i) it is the orbit of a real algebraic subgroup of $SL(2, \mathbb{R})^n$, and
(ii) $\Theta(A)$ is algebraic.

Ax-Lindemann for $\mathbb{H}^n$ (third type of proof)

• We have $A \subseteq \Theta^{-1}(X)$ a maximal, irreducible relatively algebraic subset, of positive dimension. Namely, there exists an algebraic $\bar{A} \subseteq \mathbb{C}^n$ such that $A = \bar{A} \cap \mathbb{H}^n$.

Write $G := SL(2, \mathbb{R})^n$, and $\Gamma = SL(2, \mathbb{Z})^n$.

• Without loss of generality $\dim(A \cap \mathfrak{F}) = \dim A$ (if not, replace $\widetilde{X}$ and A by $\gamma\widetilde{X}$ and γA, for some $\gamma \in \Gamma$).

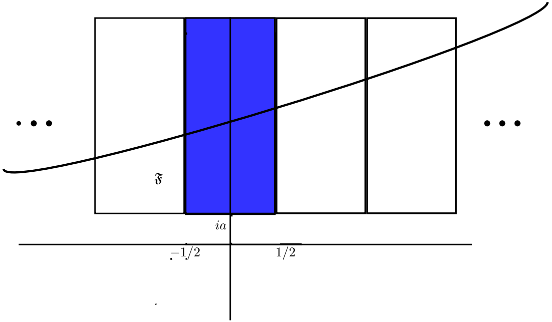
Fact A is not contained in finitely many Γ-translates of $\mathfrak{F}$.

WHY?

Otherwise $A \subseteq \bigcup_{i=1}^{k} \gamma_i \mathfrak{F}$. Because the real part of $\mathfrak{F}$ is bounded, it follows that $Re(z)$ is bounded for $z \in \bar{A} \cap \mathbb{H}^n$. This would imply (?) that A must be compact. But a compact complex analytic subset of $\mathbb{H}^n$ is finite. Contradiction.

Proof of A-L (cont)

We showed that A is not contained in finitely many Γ-translates of $\mathfrak{F}$



Hence, there are infinitely many $\gamma \in \Gamma$ such that $\dim(\gamma A \cap \mathfrak{F}) = \dim A$.
For $\gamma \in \Gamma$, we have $\gamma A \subseteq \Theta^{-1}(X)$ (since that set is Γ-invariant).

A-L continues

Let $G(A) = \{g \in G : \dim(gA \cap (\Theta^{-1}(X) \cap \mathfrak{F})) = \dim A\}$.

- As we showed, $\Gamma \cap G(A)$ is infinite.
- By analyticity of $\Theta^{-1}(X)$ and irreducibility of A, if $g \in G(A)$ then $gA \subseteq \Theta^{-1}(X)$.
- The set $G(A)$ is definable in $\mathbb{R}_{an,exp}$.

A counting Lemma (proof omitted)

The set $\{\gamma \in SL(2,\mathbb{Z})^n : \gamma \in G(A)\}$ is large*.

A second use of Pila-Wilkie

By PW, $G(A)$ contains a semi-algebraic connected curve σ.

End of proof of A-L

- So, we have $G(A) = \{g \in G : \dim(gA \cap (\Theta^{-1}(X) \cap \mathfrak{F})) = \dim A\}$, containing a semi-algebraic curve σ.

- The set $\sigma \cdot A \subseteq \Theta^{-1}(X)$ is a semi-algebraic set containing (a translate of) A.

- **By the maximality of A, $\sigma \cdot A = A$, hence the group $Stab_G(A)$ is infinite.**

- Consider the real algebraic group $Stab_G(A) \subseteq G$. It is thus infinite and contains infinitely many Γ points (by a finer use of Pila-Wilkie).

- Let H be the Zariski closure of $G(A) \cap \Gamma$. It is a real algebraic group defined over $\mathbb{Q}$ which stabilizes A. Using induction and decomposition of Shimura varieties, one can show that A is an orbit of H and that $\Theta(A)$ is algebraic, hence A is weakly special.

- It follows that X contains a (weakly) special variety.

End of Pila's Theorem.

Further work around Pila-Zannier

André Oort for $\mathcal{A}_g$ for $g = 2$ (Pila Tsimerman)

Theorem The André- Oort conjecture holds for $\mathcal{A}_2$, the moduli space of abelian surfaces.

I. Definability: P-Starchenko.

II. Number Theory: Uses results of Tsimerman.

III. A-L: using strongly the low dimension of $\mathcal{A}_2$ ($\dim \mathcal{A}_2 = 3$).

Status of General André-Oort

Recent work of Klingler, Yafaev and Ullmo (2013)

I. The restriction of the Baily-Borel embedding of any Shimura variety to the Siegel fundamental set is definable in $\mathbb{R}_{an,exp}$ (!).
III. Ax-Lindemann holds for arbitrary Shimura varieties.

what is missing?

The number Theory part