Tame complex analysis and o-minimality

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Goal

Describe the development of complex analysis within the o-minimal framework. In the setting of a real closed field and its algebraic closure one obtains analogues of classical results, as well as strong variants, due to the o-minimality assumption.

- In the first part we give an overview of some definitions and results from the general theory.
- In the second part we outline in details a particular classical complex analytic construction, and point out how it can be viewed within the o-minimal setting.

O-minimal structures

An **o-minimal structure** is an expansion $\widetilde{R} = \langle R, <, +, \cdot, \cdots \rangle$ of a real closed field R such that every first-order definable (with parameters) subset of R is a finite union of intervals with endpoints in $R \cup \{\pm\infty\}$.

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$R_{an,exp}$

$R_{exp} = \langle R, <, +, \cdot, e^x \rangle$

$R_{an} = \langle R, <, +, \cdot, \{f|[a,b]^n\} \rangle$
f real analytic on $U \supseteq [a,b]^n$

$\overline{R} = \langle R, <, +, \cdot \rangle$

Figure: O-minimal structures over $R = \mathbb{R}$

Some o-minimality

O-minimal structures offer a tame setting for various areas of mathematics.

Some features:

- Topology: Order topology on R and product topology on R^n. It might be totally disconnected, but **definably** connected.
- Dimension: For definable $A \subset R^m$, $\dim(A)$ is the maximal $n \leqslant m$ s. t. the projection of A onto n coordinates contains an open set.

Finiteness

Definable subsets of R^n have finitely many definably connected components, uniformly in parameters.

There are no definable infinite discrete sets !

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Complex analysis and o-minimality

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Let $\widetilde{R} = \langle R, <, +, \cdot, \cdots \rangle$ be an o-minimal expansion of a real closed field. From now on, "definable" means "definable in $\widetilde{R}$".

Let $K = R(\sqrt{-1})$ be the algebraic closure of R.

Since $[K : R] = 2$, after fixing $i = \sqrt{-1}$, the field K can be identified with R^2. It makes K a topological field (e.g $\mathbb{C}$ and $\mathbb{R}$)).

By a definable subset of K^n, we mean a definable subset of R^{2n} (under the above identification). A definable function from K^n to K is a function whose graph is a definable subset of $R^{2n} \times R^2$.

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K-holomorphic functions

Goal: Develop analytic theory for functions from K^n to K which are definable in the o-minimal structure $\widetilde{R}$.

Definition

Let $U \subseteq K$ be open, $z_0 \in U$. A function $f : U \to K$ is called K-holomorphic at z_0 if $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists in K.

We only consider K-holomorphic functions which are definable in $\widetilde{R}$

Examples

- Over any real closed R: Every K-polynomial is definable in $\langle R, <, +, \cdot \rangle$ and K-holomorphic.
- Over $\mathbb{R}$ and $\mathbb{C}$: Locally, every holomorphic function is definable in $\mathbb{R}_{an}$.

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Some analogues of classical results

- The derivative of a definable K-holomoprhic function is K-holomorphic.
- The Maximum Principle: A definable continuous f on a closed disc, which is K-holomorphic on the interior, attains $\max |f(z)|$ on the boundary.
- The Identity Theorem: If f and all its derivatives at 0 vanish then f vanishes in a a neighborhood of 0.

Main idea: Instead of power series and integration (not available!), we use “Topological Analysis”.

Tameness

Key feature

Definable K-holomorphic functions have no essential singularities.

Namely, if f is a definable K-holomorphic function on the punctured unit disc then there is an $n \in \mathbb{N}$ so that $z^n f(z)$ is K-holomorphic at 0.

Corollaries

1. Every definable K-holomorphic $f : K \to K$ is a K-polynomial.
2. (Uniformity) If $\{f_t : t \in T\}$ is a definable family of K-holomorphic functions on the punctured unit disc, then there is a fixed $n \in \mathbb{N}$ such that for all $t \in T$, the function $z^n f_t(z)$ is K-holomorphic at 0.

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Advanced theory

- Define K-holomoprhic functions of several variables.
- A definable K-manifold: A definable set M which is endowed with a finite and definable K-atlas.
- A definable K-analytic subset of a K-manifold M: A (definable) subset of M which around every point of M is given as the zero set of finitely many definable K-holomorphic functions.

Examples

- Both K^n and $\mathbb{P}_n(K)$ are K-manifolds. Every affine (or projective) algebraic variety over K is a K-analytic set. All are definable in $\langle R, <, +, \cdot \rangle$.
- Every compact complex manifold is isomorphic to a definable $\mathbb{C}$-manifold in the o-minimal $\mathbb{R}_{an}$.

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Removal of singularities

Let M be a definable K-manifold, $F \subseteq M$ a definable closed set, A is a definable K-analytic subset of $M \setminus F$.

1. If, locally, $\dim_{\tilde{R}}(F) \leqslant \dim_{\tilde{R}}(A) - 2$, then $Cl(A)$ is K-analytic in M.
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These are strong variants of similar classical results. They fail without the definability assumption.

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Variations of Chow's Theorem

1. If A is a definable K-analytic subset of K^n (or $\mathbb{P}_n(K)$) then it is a K-algebraic variety.

2. (Uniform Algebraicity) If $\{A_t : t \in T\}$ is a definable family of K-analytic subsets of K^n (or $\mathbb{P}_n(K)$) then it is contained in an algebraic family of algebraic varieties.

Algebraicity

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In the second part of this talk we shall consider a particular family of K-manifolds, the family of complex tori, and see how some of the above machinery can be applied.

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Tori and abelian varieties

Let $g \in \mathbb{N}^{>0}$. For $\Omega = (\omega_1, \ldots, \omega_{2g})$ a tuple of $2g$ vectors in $\mathbb{C}^g$, linearly independent over $\mathbb{R}$, let $\Lambda_\Omega \subset \mathbb{C}^g$ be the lattice $\mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{2g}$.

The quotient group $\mathcal{E}_\Omega = (\mathbb{C}^g, +)/(\Lambda_\Omega, +)$ is a g-dimensional complex torus. The matrix $\Omega \in M_{g\times 2g}(\mathbb{C})$ is called a period matrix for $\mathcal{E}_\Omega$.

Every $\mathcal{E}_\Omega$ is a compact complex-analytic group, and has a semialgebraic atlas.

Fact

Every projective abelian variety over $\mathbb{C}$ is biholomorphic with a torus.

If R is any real closed field and $K = R(\sqrt{-1})$ then for a tuple Ω of $2g$ vectors in K^g, linearly independent over R, we have a definable K-torus $\mathcal{E}_\Omega$.

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Theorem (2010)

Let $\widetilde{R} = (R, \dots) \succcurlyeq \mathbb{R}_{\text{an,exp}}$ and $K = R(\sqrt{-1})$. Every abelian variety over K is definably K-biholomorphic with a K-torus.

Observation

Every $\widetilde{R}$-definable K-manifold M comes from a definable family of $\mathbb{C}$-manifolds: there is a formula $\varphi(\bar{x}, \bar{y})$such that $\varphi(\bar{x}, \bar{a})$ defines M for some $\bar{a} \in R^m$, and $\varphi(\bar{x}, \bar{b})$ defines a $\mathbb{C}$-manifold for every $\bar{b} \in \mathbb{R}^m$.

Similarly, every definable K-holomorphic map $F \colon M \to N$ comes from a definable family of $\mathbb{C}$-holomorphic maps.

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Thus the theorem follows from the following

Theorem (Uniform version)

In the structure $\mathbb{R}_{\mathrm{an,exp}}$:
Let A_t, $t \in T$, be a definable family of g-dimensional abelian varieties over $\mathbb{C}$. Then there is a definable map $\alpha \colon T \to M_{g\times 2g}(\mathbb{C})$ and a definable family of biholomorphisms $\Phi_t \colon A_t \to \mathcal{E}_{\alpha(t)}$, $t \in T$.

In the rest of the talk we will outline the proof of the theorem.

For simplicity we consider only one dimensional abelian varieties, i.e. elliptic curves:
smooth projective varieties isomorphic to projective cubics.

From now on:

- We work in the structure $\mathbb{R}_{\mathrm{an,exp}}$.
- A torus means a one-dimensional torus.

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Definability of tori

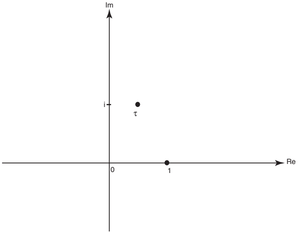
Let $\mathcal{H} = \{\tau \in \mathbb{C} \colon Im(\tau) > 0\}$ be the upper half plane and $\tau \in \mathcal{H}$.

Let $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ and $\mathcal{E}_\tau = (\mathbb{C}, +)/(\Lambda_\tau, +)$ be the corresponding torus.

The parallelogram

$$F_\tau = \{t_1 + \tau t_2 \colon 0 \leqslant t_1, t_1 < 1\}$$

contains exactly one representative from each Λ_τ-coset, and we will identify the underlying set of $\mathcal{E}_\tau$ with F_τ.



The family F_τ, $\tau \in \mathcal{H}$, is definable, and we obtain a definable family of complex tori $\mathcal{E}_\tau$, $\tau \in \mathcal{H}$.

Definability of tori

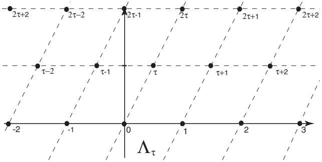
Let $\mathcal{H} = \{\tau \in \mathbb{C} \colon Im(\tau) > 0\}$ be the upper half plane and $\tau \in \mathcal{H}$.

Let $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ and $\mathcal{E}_\tau = (\mathbb{C}, +)/(\Lambda_\tau, +)$ be the corresponding torus.

The parallelogram

$$F_\tau = \{t_1 + \tau t_2 \colon 0 \leqslant t_1, t_1 < 1\}$$

contains exactly one representative from each Λ_τ-coset, and we will identify the underlying set of $\mathcal{E}_\tau$ with F_τ.



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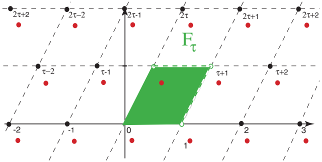

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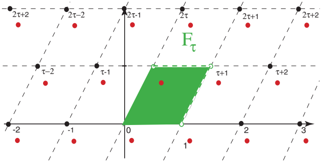

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- A map $\phi\colon F_\tau \to \mathbb{P}_n(\mathbb{C})$ is holomorphic on $\mathcal{E}_\tau$, iff $\phi = \Phi \restriction F_\tau$ for some holomorphic and Λ_τ-invariant $\Phi\colon \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ (i.e. $\Phi(z+\lambda) = \Phi(z)$ for any $\lambda \in \Lambda_\tau$).
- Since F_τ is a bounded subset of $\mathbb{C}$, the restriction $\Phi \restriction F_\tau$ is definable (even in $\mathbb{R}_{\mathrm{an}}$) for any holomorphic $\Phi\colon \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$.

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The collection of Riemann's theta functions $\vartheta_{a,b}(z,\tau) \colon \mathbb{C} \times \mathcal{H} \to \mathbb{C}$ is a family of holomorphic maps, parameterized by $a, b \in \mathbb{R}$.

Important Properties

1. The map

$$\Theta(z,\tau) = \left(\vartheta_{0,0}(2z,\tau) : \vartheta_{0,\frac{1}{2}}(2z,\tau) : \vartheta_{\frac{1}{2},0}(2z,\tau) : \vartheta_{\frac{1}{2},\frac{1}{2}}(2z,\tau)\right)$$

   is a holomorphic map from $\mathbb{C} \times \mathcal{H}$ into $\mathbb{P}_3(\mathbb{C})$.

2. For every $\tau \in \mathcal{H}$ the map $\Theta_\tau(z) : z \mapsto \Theta(z,\tau)$ is Λ_τ-invariant on $\mathbb{C}$ and induces an embedding of $\mathcal{E}_\tau$ into $\mathbb{P}_3(\mathbb{C})$.

Remark

Each embedding $\Theta_\tau : \mathcal{E}_\tau \to \mathbb{P}_3(\mathbb{C})$ is definable (even in $\mathbb{R}_{\mathrm{an}}$), but the whole family $\Theta_\tau : \mathcal{E}_\tau \to \mathbb{P}_3(\mathbb{C})$, $\tau \in \mathcal{H}$, is not definable (in any o-minimal structure).

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The group $SL(2,\mathbb{Z})$ acts on $\mathcal{H}$ and two tori $\mathcal{E}_\tau, \mathcal{E}_{\tau'}$ are bihilomorphic iff τ and τ' are in the same $SL(2,\mathbb{Z})$-orbit. (In other words, the quotient $\mathcal{H}/SL(2,\mathbb{Z})$ is a moduli space of complex tori.)

Fact

$\mathfrak{F} = \{\tau \in \mathcal{H}\colon -\frac{1}{2} \leqslant Re(\tau) \leqslant \frac{1}{2}, |\tau| \geqslant 1\}$ *contains a representative from every* $SL(2,\mathbb{Z})$*-orbit.*

(Notice: $\mathfrak{F}$ is a semialgebraic subset of $\mathbb{C}$.)

Theorem

For all $a, b \in \mathbb{R}$ *the restriction of the function* $\vartheta_{a,b}(2z,\tau)$ *to the set*

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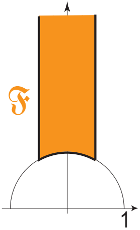

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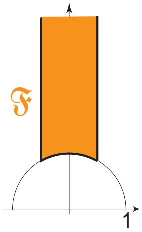

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Corollary

The family of embeddings $\Theta_\tau : \mathcal{E}_\tau \to \mathbb{P}_3(\mathbb{C})$, $\tau \in \mathfrak{F}$, *is definable.*

Proposition

There is a definable set $\mathfrak{F}_0 \subset \mathcal{H}$ containing $\mathfrak{F}$, such that

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   There is a family of elliptic curves $\mathcal{C}_x$, $x \in \mathcal{X}$, definable in $(\mathbb{C}, +, \cdot)$, and a definable surjective map $\Psi : \mathfrak{F}_0 \to \mathcal{X}$ such that $\Theta_\tau(\mathcal{E}_\tau) = \mathcal{C}_{\Psi(\tau)}$.

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Let $\mathcal{A} = \{A_t \colon t \in T\}$, be a definable family of elliptic curves.

We need: a definable map $\alpha \colon T \to \mathfrak{F}_0$ and a definable family of biholomorphisms $\Phi_t \colon A_t \to \mathcal{E}_{\alpha(t)}$, $t \in T$.

- Using Uniform Algebraicity, we may replace $\mathcal{A}$ with a family definable in $(\mathbb{C}, +, \cdot)$.
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Summary

We have:

- Over arbitrary R and K: Analogues of classical results for definable "holomorphic" objects.
- Over $\mathbb{R}$ and $\mathbb{C}$:
  - Strong uniform variants of classical theorems for those complex-analytic objects which are definable in o-minimal structures.
  - O-minimality of some families of classical complex-analytic objects.