Topological groups and stabilizers of types

(w. Sergei Starchenko)

Kobi Peterzil

Department of Mathematics
University of Haifa

Logic Colloquium 2015, Helsinki

Topological groups within model theory

A topological group

A group G admitting a Hausdorff topology τ, with $(g, h) \mapsto g \cdot h$ and $g \mapsto g^{-1}$ continuous.

Within model theory

Given a structure $\mathcal{M}$,

• A group G is **a definable in** $\mathcal{M}$ if its domain, and group operation are definable in the structure $\mathcal{M}$.

• We say that G is **a definable topological group in** $\mathcal{M}$ if in addition, there is a basis $\mathcal{B}$ for τ (equivalently, for neighborhoods of e), such that every $U \in \mathcal{B}$ is definable in $\mathcal{M}$.

• G has **a uniformly definable topology** if $\mathcal{B}$ is given as a uniformly definable family of sets: $\mathcal{B} = \{\phi(G, a) : a \in T\}$.

Examples of definable topological groups

• $\mathcal{M}$ arbitrary, G a definable group, with discrete topology.

• $\mathcal{M} = (\mathbb{R}, <, +, \cdot, \ldots.)$, $G = GL(n, \mathbb{R})$, or any real algebraic subgroup.

Take $\mathcal{B} = \{U_\epsilon = \{A : ||A - I|| < \epsilon\} : \epsilon > 0\}$, uniformly definable.

• Start with G an arbitrary topological group.

$$\text{Let } \mathcal{M} = \mathbb{G}^{\text{set}} = (G; \cdot, \mathcal{P}(G)).$$

Clearly, G is a definable topological group in $\mathcal{M}$.

Side Q: When is it also uniformly definable?

Complete G-types

Assume that G is a definable (topological) group in $\mathcal{M}$.
A G-formula is a formula that defines a subset of G. A collection of G-formulas is called **a G-type**.

The space of complete G-types

Let $S_G(M)$ be the set of complete G-types over M.

The Stone topology on $S_G(M)$: For $\phi(x)$ a G-formula, $U_\phi = \{p \in S_G(M) : \phi \in p\}$ is a basic open set.

- The space $S_G(M)$ is compact Hausdorff.
- G acts on $S_G(M)$ **by homeomorphisms**: $g \cdot p = \{\phi(g^{-1}x) : \phi \in p\}$.
- G embeds, as a G-set, onto a dense, discrete, subspace of $S_G(M)$, via $g \mapsto \{x = g\}$.

Drawback

The action of G is not continuous with respect to the product topology on $G \times S_G(M)$, unless G is discrete.

Topological dynamics and model theory

Topological dynamics is the study of dynamical systems, given by a continuous action of a topological group G on a (usually compact) space X.

In Model Theory, it was Ludomir Newelski who began investigating definable groups (viewed as discrete groups), via their action on the compact space $S_G(M)$. Notions from topological dynamics such as *minimal sub-flows*, *Ellis semi-group* were related to model theoretic counter-parts, one shedding light on the other.

Additional feature in the current approach

Bringing into play the topology of the group G

The infinitesimal subgroup μ

Let G be a definable topological group in $\mathcal{M}$.

$$\mu := \{\theta(x) \in \mathcal{L}_M : \theta \text{ defines an open neighborhood of } e\}.$$

As a collection of formulas, μ is **a partial** G-type, defining a basis for the open neighborhoods of e.

Some properties of the partial type μ

• If $\mathcal{M} \prec \mathcal{N}$ then $\mu(\mathcal{N})$ is a group, called **the infinitesimal subgroup** of $G(\mathcal{N})$. It is normalized by all $g \in G(\mathcal{M})$.

• Consider the partial type $\Sigma(x, y) = \{\theta(x^{-1} \cdot y) : \theta \in \mu\}$.

Model theoretically, it is a type-definable equivalence relation on G (defining the μ-left-coset-relation on $G(\mathcal{N})$).

Set-wise, it is a collection of subsets of $G \times G$, which make up **a uniformity** on G.

The space of μ-types I

For two G-formulas θ, ϕ, let $(\theta \cdot \phi)(M) := \theta(M) \cdot \phi(M)$.

For a complete type $p(x) \in S_G(M)$, let

$$\mu \cdot p = \{(\theta \cdot \phi)(x) : \theta \in \mu \ , \ \phi \in p\}.$$

- $\mu \cdot p$ is a **partial** G-type, which we call **a μ-type.** Clearly, $p \vdash \mu \cdot p$.
- For any complete G-types p, q, if $\mu \cdot q \cup \mu \cdot p$ is consistent then $\mu \cdot q = \mu \cdot p$.
- We obtain an equivalence relation on $S_G(M)$: $p \sim_\mu q \Leftrightarrow \mu \cdot p = \mu \cdot q$.

The $\sim_\mu$-classes are of the form $[p]_\mu = \{q : q \vdash \mu \cdot p\}$.

Example

Consider $G = (\mathbb{R}^2, +)$ definable in $\mathcal{M} = (\mathbb{R}, <, +, \cdot)$.
We consider three G-types:

1. Let $p(x, y) := \{y = 0\} \cup \{x > r : r \in \mathbb{R}\}$. It is the (one dimensional) type at $+\infty$ of the curve $\gamma(t) = (t, 0)$.

2. Let $q(x, y)$ be the (one dimensional) type at $+\infty$ of the curve $\sigma(t) = (t, 1/t)$.

3. Let $r(x, y)$ be the (two-dimensional) type of $(\alpha, 1/\beta)$ with $\mathbb{R} < \alpha << \beta$.

We have $p \sim_\mu q \sim_\mu r$.

The space of μ-types II

Let $S_G^\mu(M) := S_G(M)/\sim_\mu$, with the quotient topology, identified with the set of partial types $\{\mu \cdot p : p \in S_G(M)\}$.

Proposition

- The quotient space $S_G^\mu(M)$ is compact Hausdorff.
- The action of G on $S_G^\mu(M)$, given by $g \cdot \mu p = \mu \cdot gp$ is continuous.
- The map $g \mapsto \mu \cdot tp(g/M)$ is **a topological** G-equivariant embedding of $G(M)$ onto a dense subset of $S_G^\mu(M)$.

Connection to the Samuel compactification

Thus, $S_G^\mu(M)$ is a compactification of the group G, as a G-space. It is basically the "Samuel compactification" of G with respect to the uniformity given by μ.
In the case of $\mathbb{G}^{\mathrm{set}}$, the two objects are identical.

Some Examples

Example

1. If G is an $\mathcal{M}$-definable discrete topological group then $S^{\mu}_{G}(M) = S_G(M)$.
   In particular, if G is a discrete group, viewed as a $\mathbb{G}^{\text{set}}$-definable group then $S^{\mu}_{G}(M) = \beta G$, the Stone-Čhech comapctification of G.

2. If $G = (\mathbb{R}, +)$ viewed as a topological group definable in the field of real numbers, then the space $S^{\mu}_{G}(\mathbb{R})$ is homeomorphic to $\mathbb{R} \cup \{\pm\infty\}$ with the obvious action of G.

3. If G is a compact topological group, definable in some $\mathcal{M}$ then $S^{\mu}_{G}(M) = G$.

Stabilizers of types

An important tool in model theory of groups is the stabilizer subgroup of $p \in S_G(M)$. Namely, the group $Stab(p)$ of all $g \in G$ such that $g \cdot p = p$.

Roughly speaking, the group $Stab(p)$ measures how close p is to being “the type” of a left-coset of some sub-group.

• In $\mathcal{M} = (\mathbb{C}, +, \cdot)$, every connected algebraic group G has a (unique) type $p \in S_G(M)$, such that $Stab(p) = G$.

• In $\mathcal{M} = (\mathbb{R}, <, +, \cdot)$ the situation is very different. E.g. in $G = SL(2, \mathbb{R})$, the stabilizer of every type $p \in S_G(M)$ is contained in a conjugate of the upper triangular group (so the stabilizer is solvable).
But “most” G-types have trivial stabilizer.

In general, $Stab(p)$ is not a definable group. But if we restrict to definable types, we improve the situation:

Definable types

Definition

A (possibly partial) type p is **definable** if for every formula $\phi(x, \bar{y})$, the set $\{\bar{a} \in M^n : p \vdash \phi(x, \bar{a})\}$ is definable in $\mathcal{M}$.

- In stable theories every complete type is definable.

- In ordered (e.g. o-minimal structures), not every complete type is definable.

Example

- In $\mathcal{M} = (\mathbb{R}, <, +, \cdot)$, the type $x >> \mathbb{R}$ is definable, since the set $\{a \in \mathbb{R} : p \vdash x > a\} = \mathbb{R}$.
- In $\mathcal{M} = (\mathbb{Q}, <, +)$ the cut of π in $\mathbb{Q}$ is not a definable type in $\mathcal{M}$.
- In $\mathbb{G}^{\text{set}}$ every type is (trivially) definable, because every subset is definable.

Stabilizers of definable types

Fact

If p is a definable (partial) G-type over M, then $Stab(p)$ can be written as the intersection of (infinitely many) definable subgroups of G.

Proof For $\varphi(x) \in \mathcal{L}_M$ a G-formula, let $G_\varphi(p) = \{g \in G : p \vdash \varphi(g^{-1}x)\}$. Since p is a definable type this is a definable set. Let

$$Stab_\varphi(p) = \{h \in G : h \cdot G_\varphi(p) = G_\varphi(p).\}.$$

This is clearly a definable subgroup of G.

We have $Stab(q) = \bigcap_{\varphi \in \mathcal{L}_M} Stab_\varphi(q)$. □

Corollary- back to topological groups

If G has **a uniformly definable topology** and $p \in S_G(M)$ is a definable type then $Stab(\mu \cdot p)$ is the intersection of definable subgroups.

Proof The partial type $\mu \cdot p$ is definable and hence the above holds. □

Groups in o-minimal structures I

Recall

• $\mathcal{M} = (M, <, +, \cdot, \ \cdots)$ is an o-minimal structure if every definable subset of M is a finite union of intervals with endpoints in M.

• Main examples are real closed fields and their expansions (over the reals) by the real exponential, restricted analytic functions, and many more.

• A non-example is $(\mathbb{R}, <, +, \cdot, \sin x)$

• The order topology in M, and box-topology in M^n yield a uniformly definable topology.

• Definable subsets of M^n have a finite decomposition into manifold-like sets called **cells**, resulting in a good theory of dimension.

• Rich theory of definable groups (examples are complex algebraic, real algebraic groups, compact Lie groups and more):

Groups in o-minimal structures II

For simplicity by an o-minimal structure we mean an o-minimal expansion of a real closed field. Let G be a group definable in $\mathcal{M}$.

Fact (Pillay)

*1. There is a definable injection $f\colon G \hookrightarrow M^n$ such that $f(G)$ is a closed subset of M^n and the group operations are continuous on $f(G)$ (with respect to the topology induced from M^n). **So G has a uniformly definable topology**.*

2. (DCC)If $G \supseteq G_1 \supseteq G_2 \supseteq \cdots G_n \supseteq \cdots$ is a descending chain of definable groups then there exists N such that $G_n = G_N$ for all $n \geqslant N$.

Corollary

If $p \in S_G(M)$ is a definable type then $Stab(p)$ and $Stab(\mu \cdot p)$ are **definable** groups.

What are the μ-stabilizer subgroups?

For $p \in S_G(M)$, we call $Stab(\mu \cdot p)$ **the μ-stabilizer of p**. Notice that $Stab(p) \subseteq Stab(\mu \cdot p)$.

Example

Let $G = (\mathbb{R}^2, +)$, definable in $\mathcal{M}$ the field of reals and $\sigma(t) = (t, t^2)$. Let p be the type of σ at $+\infty$. Then the μ-stabilizer of p is:

$$Stab(\mu \cdot p) = \{0\} \times \mathbb{R}.$$

• More generally, given an unbounded definable curve $\sigma(t) \subseteq \mathbb{R}^2$, the μ-stabilizer of its type at $+\infty$ is a one-dimensional subspace of $\mathbb{R}^2$ whose slope is the limit tangent of σ at $+\infty$.

• Consider the two-dimensional type $p(x, y) = \{x >> \mathbb{R}\ ,\ \ y >> \mathbb{R}(x)\}$. Its μ-stabilizer (as well as its stabilizer) is the whole of $\mathbb{R}^2$.

The main theorem

Theorem

Let $G \subseteq M^n$ be a definable group in an o-minimal structure, $p \in S_G(M)$ a definable type. Let

$$d = \min\{dim(q) : q \sim_\mu p\}.$$

Then

1. $Stab(\mu \cdot p)$ is a definable subgroup of G (we already saw).
2. It is solvable, torsion-free.
3. It has dimension d.

In particular, if p is unbounded in M^n then $dim(Stab(\mu \cdot p)) > 0$.

In dimension one the above result (in a different formulation) was proved by Pe-Steinhorn in 1999.

The standard part map (Marker-Steinhorn's work)

Assume that $p(x) \in S(M)$ is a definable type.
Let α be a realization of $p(x)$ and $\mathcal{N} = \mathcal{M}\langle\alpha\rangle$ a prime model over $M \cup \{\alpha\}$.
Let $\mathcal{O}_{\mathcal{N}} \subseteq N$ be the convex hull of M in N.

Fact

There is "a standard part map" $\mathrm{st}\colon \mathcal{O}_{\mathcal{N}} \to M$ defined as:
$\mathrm{st}(n)$ is the unique $m \in M$ such that for every open M-definable V containing m, we have $n \in V$

We extend it to $\mathrm{st}\colon \mathcal{O}_{\mathcal{N}}^k \to M^k$, coordinate-wise, and instead of $\mathrm{st}(X \cap \mathcal{O}_{\mathcal{N}}^k)$ we will just write $\mathrm{st}(X)$.

Fact

In the above setting $\mathrm{st}(X)$ is definable in $\mathcal{M}$ for any $\mathcal{N}$-definable set X.

On the proof of the theorem- a different definition of $Stab(\mu \cdot p)$

Let $G \subseteq M^k$ be a group definable in o-minimal $\mathcal{M}$ and $p \in S_G(M)$ a definable type.

Let q be a definable type of minimal dimension d in the $\sim_\mu$-class of p (exists!).

Let $\alpha \models q$ and $\mathcal{N} = \mathcal{M}(\alpha)$.

Theorem

There exists an M-definable set $X \ni \alpha$ such that
(1) $Stab(\mu \cdot p) == Stab(\mu \cdot q) = \mathrm{st}(X\alpha^{-1})$
(2) $\dim(\mathrm{st}(X\alpha^{-1})) = d$

The idea: X is “almost linear”, so it is infinitesimally close to a coset of a definable group. Translating on the right by α^{-1} and taking standard part we get the desired μ-stabilizer group.

Generalizations and future goals

• **Fact:** Much of the above can be carried out in the setting of definable G-actions.

• **Goal:** Understand (hermitian) symmetric spaces in the above setting.

• **Ambitious goal:** Understand certain compactifications of symmetric spaces via the space of μ-types.

• **Very ambitious goal:** Understand (compactification of) locally symmetric spaces via the space of μ-types.