ON GROUPS INTERPRETABLE IN VARIOUS VALUED FIELDS

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ABSTRACT. We study infinite groups interpretable in V-minimal, power bounded T-convex or certain expansions of p-adically closed fields. We show that every such group G has unbounded exponent and if G is dp-minimal then it is abelian-by-finite.

Along the way, we associate with any infinite interpretable group an infinite type-definable subgroup which is definably isomorphic to a group in one of four distinguished sorts: the underlying valued field K, its residue field **k** (when infinite), its value group Γ, or $K/\mathcal{O}$, where $\mathcal{O}$ is the valuation ring.

Our work uses and extends techniques developed in [8] to circumvent elimination of imaginaries.

CONTENTS

1. INTRODUCTION

We continue our work from [8], where we studied interpretable fields in a variety of valued fields, and extend our investigation to interpretable groups. To recall, in [8] we considered interpretable objects, namely quotients of definable sets by definable equivalence relations, in valued fields, with a focus on interpretable fields in three families of valued fields: (i) V-minimal (e.g. algebraically closed valued fields of residue characteristic 0), (ii) power bounded T-convex expansions of o-minimal structures (e.g. real closed valued fields) and (iii) certain expansions of p-adically closed fields (e.g. finite extensions of $\mathbb{Q}_p$). As a corollary of our work here, we obtain:

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{intro-1}

Theorem 1.1 (Section 7)**.** *Let K be a valued field of characteristic 0 and $\mathcal{K}$ an expansion of K that is either (i) V-minimal, (ii) power bounded T-convex, or (iii) p-adically closed. Then every dp-minimal group interpretable in $\mathcal{K}$ is abelian-by-finite.*

This means, in particular, that 1-dimensional groups definable in any of these fields are abelian-by-finite, extending the work of Pillay and Yao, [20] in $\mathbb{Q}_p$ and of Onshuus and Vicaria [18, Theorem 1.1] in models of Presburger Arithmetic. It also complements the work of Simonetta, [27], who gave an example of a dp-minimal group interpretable in an algebraically closed valued field of characteristic p that is nilpotent of class 2 (so not abelian-by-finite).

By a result of Simon, [23, Proposition 3.1], if G is dp-minimal then it has a definable abelian normal subgroup, H, such that G/H has bounded exponent. Thus, it is sufficient to prove that every infinite dp-minimal group in our setting has unbounded exponent (i.e. G/H must be finite). The advantage of this approach is that unbounded exponent can be detected locally, namely it will suffice – given a dp-minimal group G – to find a subgroup $H \leq G$ of unbounded exponent. So our strategy relies on finding in G a (type) definable subgroup that can be better studied than G itself, and show that it has unbounded exponent.

The key idea from [8] is to bypass results on elimination of imaginaries and replace them with a reduction to four *distinguished* sorts: the valued field K itself, the residue field **k** (when infinite), the value group Γ, and the quotient $K/\mathcal{O}$, where $\mathcal{O}$ is the valuation ring. Applying techniques similar to those developed in [8] we show that our interpretable group G has a (type) definable subgroup ν definably isomorphic to a (type) definable group in one of these sorts. This part of the analysis does not require dp-minimality of the group G, and we show:

{intro-2}

Theorem 1.2 (Section 7)**.** *Let K be a valued field of characteristic 0 and $\mathcal{K}$ an expansion of K that is either (i) V-minimal, (ii) power bounded T-convex, or (iii) p-adically closed.*

Let G be an infinite group interpretable in $\mathcal{K}$. Then, after possibly replacing G with a quotient by a finite normal subgroup, there is an infinite type-definable subgroup $\nu \leq G$ of unbounded exponent such that one of the following holds:

(1) *ν is definably isomorphic to a group type definable in either K, or **k**, or*
(2) *ν is definably isomorphic to a type definable* subgroup *of $\langle \Gamma^n, + \rangle$, or to a* definable subgroup *of $\langle (K/\mathcal{O})^n, + \rangle$.*

In practice, we first construct ν satisfying conditions (1) or (2) of the theorem, and exploit this information (as well as some properties of the construction) to deduce that ν has unbounded exponent. In the appendix we show that, in fact, when $\mathcal{K}$ is power-bounded T-convex and G is dp-minimal, ν is a type-definable ordered group.

Studying groups instead of fields somewhat complicates the situation with respect to our work in [8]. For instance, when studying groups in p-adically closed fields, we are forced to consider the sorts Γ and $K/\mathcal{O}$, that were easily eliminated when we were interested in fields only. Consequently, the distinguished sorts, break into three types: topological (such as the valued field sort K in all settings), discrete (such as Γ and $K/\mathcal{O}$ in the p-adically closed setting) and strongly minimal (**k** in the V-minimal setting).

Finding a uniform environment, allowing us to avoid repetition of proofs as we move across the different settings and different distinguished sorts, turned out to be one of the challenges of the present work. In [8], much of the work was carried out in a setting (introduced by Simon and Walsberg in [26]) of dp-minimal uniform structures, that we call SW-uniformities. This setting fits the sorts K, Γ and $K/\mathcal{O}$ in all non p-adically closed fields considered here. In the p-adically closed case, the natural topology on the sorts $K/\mathcal{O}$ and Γ is discrete and the quantifier $\exists^\infty$ is not eliminable. They are, therefore, not SW-uniformities. However, both admit a non-definable, non-Hausdorff topology (generated by the image of balls of non-standard, infinite, negative radius) which is still useful for our purposes. In Section 3.1 we prove the necessary results for $K/\mathcal{O}$, and then give an *ad hoc* axiomatization covering both SW-uniformities and the discrete sorts in the p-adically closed case, $K/\mathcal{O}$ and Γ (the strongly minimal case still has to be treated separately).

As in [8], our method is local, in the following sense: given an infinite interpretable group, G, we find a definable infinite $X \subseteq G$ of maximal dp-rank with respect to the property that, after possibly passing to a quotient by a finite normal subgroup, there is a definable injection from X into D^n for one of the distinguished sorts, D (we call roughly such sets D-sets, see Definition 4.16 for a precise formulation). We show that there is at least one such D and a corresponding D-set $X \subseteq G$ of positive dp-rank.

When such a set X exists, we call (roughly) the group a D-group. Within our axiomatic framework, we develop in Section 5 the basics of D-groups. Using methods in the spirit of [8] we associate with a D-group an infinite type-definable subgroup $\nu_D(G) \leq G$ definably isomorphic to a type-definable group within the sort D. In some cases, we can even find a definable such subgroup of G. When G is dp-minimal and D is an SW-uniformity, this gives rise to an SW-uniform structure on G. In general, however, the type-definable subgroup $\nu_D(G)$ is not necessarily a topological group, and our analysis depends on the distinguished sort D as well as on the class of valued fields we are studying.

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1.1. Previous work on groups in valued fields.

We list briefly additional work on groups in valued fields, beyond what has been mentioned above. Definable groups in $\mathbb{Q}_p$ were first studied by Pillay, [19]. Hrushovski and Pillay, [11], discuss connections between definable groups in local fields and algebraic groups. In [2] Acosta gives an exhaustive list of all 1-dimensional groups definable in $\mathbb{Q}_p$. In [1] he extends this result to 1-dimensional *commutative* groups definable in algebraically closed valued fields[1] Johnson and Yao, [14], study definable, definably compact, groups in $\mathbb{Q}_p$. Montenegro, Onshuus and Simon, [17, Theorem 2.19], work under general assumptions, applicable in several examples of our setting (e.g., $\mathrm{ACVF}_{0,0}$, RCVF, p-adically closed fields). A classification of definably simple groups definable in henselian fields of characteristic 0 (or, more generally, in 1-h-minimal fields of characteristic 0) is obtained in an unpublished work of Gismatullin, Halupczok and Macpherson.

While the above results mostly study definable groups, the work of Hrushovski and Rideau-Kikuchi, [12], covers interpretable, stably dominated groups in algebraically closed valued fields,

[1]By Theorem 1.1 the commutativity assumption is redundant in algebraically closed valued fields of equi-characteristic 0.

using the results on elimination of imaginaries in such fields. In a recent pre-print, Johnson [13] studies topology on groups interpretable in p-adically closed fields.

Finally, although we do not make use of these here, we note the seminal papers of [9], [16], and [10], where elimination of imaginaries with appropriate sorts is proved for algebraically closed valued fields, real closed valued fields and p-adically closed fields, respectively.

1.2. **Structure of the paper.** In Section 2 we review the notion of SW-uniformities and the distinguished sorts introduced in [8] and collect some of their useful properties. In Section 3.1, we study the geometry of $K/\mathcal{O}$ in some P-minimal fields, isolating (Section 4.1) key geometric properties true in all the cases we are studying. Based on these properties we introduce in Section 4.2 the framework of *vicinic structures* where most of our work is taking place. It is a generalisation of SW-uniformities encompassing also the distinguished sorts of P-minimal fields.

The rest of Section 4 is dedicated to collecting the tools needed for the construction in Section 5 of infinitesimal subgroups of interpretable groups. Much of the work in Section 4 is devoted to dealing with a technical issue not arising in the study of fields: in [8] we have shown that every interpretable field can be locally injected into one of the distinguished sorts. In the present setting we are required to work with finite-to-one functions, and this requires some additional work. Also, in the V-minimal case, the sort $D = \mathbf{k}$ is a pure algebraically field, and thus has a different nature. It is treated in Section 6.

Finally, in Section 7 we collect all of the results of previous sections to prove Theorem 1.1 and Theorem 1.2. In the final section of the paper we examine several natural examples of interpretable groups in light of the results of this paper.

2. Background and preliminaries

S: background}

2.1. **Notation, terminology and some preliminaries.** Throughout, structures will be denoted by calligraphic letters, $\mathcal{M}, \mathcal{N}, \mathcal{K}$ etc., and their respective universes by the corresponding Latin letters, M, N and K. We reserve $\mathcal{K}$ to denote expansions of valued fields, and K will always be a valued field. All structures may be multi-sorted. A valued field $\mathcal{K} = (K, v, \dots)$ is always considered with a single home sort (for the ground field K) with all other sorts coming from $\mathcal{K}^{eq}$. All sets are definable using parameters unless specifically mentioned otherwise.

Tuples from a structure $\mathcal{M}$ are denoted by small Roman characters $a, b, c, \dots$. Tuples are always assumed to be finite, we apply the standard model theoretic abuse of notation writing $a \in M$ for $a \in M^{|a|}$. Variables will be denoted $x, y, z, \dots$ with the same conventions as above. We do not distinguish notationally tuples and variables belonging to different sort, unless some ambiguity can arise. Capital Roman letters $A, B, C, \dots$ usually denote small subsets of parameters from M. As is standard in model theory, we write Ab as a shorthand for $A \cup \{b\}$. In the context of definable groups we will, whenever confusion can arise, distinguish between, e.g., $Agh := A \cup \{g, h\}$ and $Ag \cdot h := A \cup \{g \cdot h\}$.

Valued fields When $(K, v, \dots)$ is an expansion of a valued field, we let $\mathcal{O}_K$ (or just $\mathcal{O}$, if the context is clear) denote its valuation ring. Its maximal ideal is $\mathbf{m}_K$ (or $\mathbf{m}$) and $\mathbf{k} := \mathcal{O}/\mathbf{m}$ its residue field. The value group is Γ_K (or just Γ). As in [8], K, $\mathbf{k}$, Γ and $K/\mathcal{O}$ are the *distinguished sorts*. We shall occasionally (especially in Section 8) also use the sorts $\mathrm{RV}_\gamma := K^\times/(1+\mathbf{m}_\gamma) \cup \{0\}$ where, for a non-negative $\gamma \in \Gamma$ we denote $\mathbf{m}_\gamma := \{x \in K : v(x) > \gamma\}$. We also let $\mathrm{rv}_\gamma : K \to \mathrm{RV}_\gamma$ denote the quotient map extended by 0 at 0.

A closed ball in K is a set of the form $B_{\geq\gamma}(a) := \{x \in K : v(x-a) \geq \gamma\}$ and γ is its (valuative) radius. Open balls are defined similarly and denoted by $B_{>\gamma}(a)$. Throughout by "ball" we mean either an open or a closed ball. An open (closed) ball in K^n is a product of n open (closed) balls of **equal radius**, i.e., $B_1 \times \cdots \times B_n$ where $B_i := B_{>r}(a_i)$ (resp. $B_i := B_{\geq r}(a_i)$) for some $r \in \Gamma$ and $a_i \in K$. A ball in K^n is either a closed or an open ball, it has radius r if it is a product of n closed (or n open) balls in K each of radius r.

Throughout this section $\mathcal{K}$ is either a V-minimal field, a T-convex expansion of a power bounded o-minimal T or a 1-h-minimal P-minimal field (see [8, Section 2.3] and references therein for the definitions and basic properties of such fields). We complement these with the definition of 1-h-minimal fields of mixed characteristic:

{D: 1-h-min.}

Definition 2.1. A theory T of valued fields of characteristic 0 is *1-h-minimal* if for any set of parameters $A \subseteq K \cup \mathrm{RV}_n$ for some positive integer n and any A-definable $f : K \to K$ the following hold:

(1) There exists a finite A-definable C and a positive integer m such that for any ball B m-next to C there exists $\gamma_B \in \Gamma$ such that or all $x, y \in B$ we have
$$\gamma_B - v(m) \leq v(f(x) - f(y)) - v(x-y) \leq \gamma_B + v(m).$$
(2) The set $\{y \in K : |f^{-1}(y)| = \infty\}$ is infinite.

The above geometric definition is equivalent to the model theoretic definition by [6, Theorem 2.2.7]. It will only be used as a black box (see also Definition 3.13 for the notion of a ball being m-next to C).

We also assume the reader familiar with the notion of dp-rank, see [8, §2.1] for the definition and a quick survey of dp-rank and its main properties. It may help the reader to note (and this will be proved in some cases below) that in every distinguished sort D in our setting the dp-rank of $a \in D^n$ equals its acl-dimension, even if acl does not satisfy exchange.

Finally, when saying that a valued field has *definable Skolem functions*, we mean that the valued field sort, K, has definable Skolem functions. Power bounded T-convex valued fields have definable Skolem function by [31, Remark 2.4] (after adding a constant). So do p-adically closed fields [30, Theorem 3.2] and their expansion by all sub-analytic sets [7, Theorem 3.6].

2.2. **Almost strong internality.** The starting point of the present work is the following result that can be deduced from [8, Proposition 5.5] (see also §7, *loc. sit.* for the details). By *a finite-to-finite correspondence between X_1 and X_2* we mean a definable relation $C \subseteq X_1 \times X_2$ such that both projections $\pi_i : C \to X_i$ ($i = 1, 2$) are sujective with finite fibres.

{F: 5.5}

Fact 2.2. *Let $\mathcal{K}$ be an either P-minimal, power bounded T-convex or C-minimal expansion of a valued field. If X is infinite and interpretable in $\mathcal{K}$ then there exist infinite $T \subseteq X$, $S \subseteq D$ (possibly defined over additional parameters) and a definable finite-to-finite correspondence $C \subseteq T \times S$ where D is either K, $\boldsymbol{k}$, Γ or $K/\mathcal{O}$.*

When – in the above notation – X was a field, we were able to eliminate the discrete sorts $K/\mathcal{O}$ and Γ in the P-minimal case, and the remaining sots turned out to be rather tame topological structures (that we called SW-uniformities – to be described in more detail below). In that setting (and for X a field) we were able to strengthen Fact 2.2 to obtain a definable bijection between T and a subset of one of the distinguished sorts. In the present setting some extra work is needed to obtain a weaker result, i.e. a finite-to-one function from a definable subset of our group G into one of the distinguished sorts. Extending our terminology from [8] we define:

ef: internal}

Definition 2.3. A definable set X is *A-almost strongly internal to a definable set D* if there exists an A-definable finite-to-one function $f : X \to D^k$, for some $k \in \mathbb{N}$. The set X is *locally almost strongly internal to* D if there exists an infinite A-definable $X' \subseteq X$ that is almost strongly internal to D.

Recall that X is *strongly internal* to D if we can find an injective $f : X \to D^k$. It is somewhat inconvenient that, in general, a set X locally almost strongly internal to D need not be locally strongly internal to D. The upshot of this situation is that we have to develop the theory, in parallel, for subsets of G almost strongly internal to a distinguished sort D, and for subsets of G strongly internal (possibly to the same sort D). Since the statements and the proofs are, as a rule, similar in both cases we usually state both results simultaneously; e.g.

> "If X is locally (almost) strongly internal to D ... then Y is (almost) strongly internal..."

with the convention that either all brackets are included or all brackets are omitted. I.e., the two statements included in the above formulation are "If X is locally strongly internal to D ... then Y is strongly internal..." and "If X is locally almost strongly internal to D ... then Y is almost strongly internal...".

-uniformities}

2.3. **Simon-Walsberg uniformities and group topologies.** Though the setting of the present paper requires that we take into account the discrete sorts Γ and $K/\mathcal{O}$ (in P-minimal fields), SW-uniformities still have a significant role to play, and even in the discrete sorts some arguments are modelled after the analogous arguments in the topological setting. So we remind:

Definition 2.4. A definable set D in a structure $\mathcal{M}$ is an SW-uniformity if:

1. D is dp-minimal.
2. D has a definable uniform structure (or, uniformity) giving rise to a Hausdorff topology. I.e., there exists a formula $\varphi(x, y, z)$ and a definable set S such that $\{\varphi(D^2, s) : s \in S\}$ is a uniform structure on D, and the intersection of all $\varphi(D^2, s)$, for $s \in S$, is the diagonal.
3. Every definable subset of D has non-empty interior with respect to the uniform topology.
4. D has no isolated points in the uniform topology.

One of the most important tools in [8] was the technical fact below. We shall prove here an a analogue of this result in other settings as well, and eventually include a variant of it as one of the axioms of our *ad hoc* setting of vicinic sorts.

{Gen-Os in SW}

Fact 2.5 ([8, Corollary 3.13])**.** *Let D be an SW-uniformity. For every definable $X \subseteq D^n$, $Y \subseteq X$ a definable subset, and a in the relative interior of Y in X, and for every $b \in D^k$, and A a small set of parameters, there exists $B \supseteq A$ and a B-definable open subset $U = U_1 \times \cdots \times U_n \subseteq D^n$ such that $a \in U \cap X \subseteq Y$ and* $\text{dp-rk}(a, b/B) = \text{dp-rk}(a, b/A)$.

We will also require the following result (implicit in [8]):

om surjective}

Lemma 2.6. *Let $\mathcal{M}$ be some structure and let D be an SW-uniformity, W a definable set, and $f : Z \to W$ a definable finite-to-one surjection.*

(1) *If Z is strongly internal to D then there exists a a definable $W_1 \subseteq W$, with $\operatorname{dp-rk}(W_1) = \operatorname{dp-rk}(W)$ such that W_1 is strongly internal to D.*
(2) *If Z is almost strongly internal to D then there exists a definable $W_1 \subseteq W$, $\operatorname{dp-rk}(W_1) = \operatorname{dp-rk}(W)$, such that W_1 is almost strongly internal to D.*

Proof. Assume everything is definable over some parameter set A.

(1) Since Z is strongly internal to D we may assume that $Z \subseteq D^k$ for some k. We fix $z_0 \in Z$ such that $\operatorname{dp-rk}(z_0/A) = \operatorname{dp-rk}(Z)$. We first find a definable open set $V \subseteq D^k$ such that $|[z_0]_f \cap V| = 1$, where $[z_0]_f = f^{-1}(f(z_0))$. By Fact 2.5 there exists an open set U, $z_0 \in U \subseteq V$, definable over some $B \supseteq A$, such that $\operatorname{dp-rk}(z_0/A) = \operatorname{dp-rk}(Z)$, and $[z_0]_f \cap U = \{z_0\}$. But now, the set $Z_1 = \{z \in Z : [z]_f \cap U = \{z\}\}$ is defined over B and contains z_0 hence $\operatorname{dp-rk}(Z_1) = \operatorname{dp-rk}(Z) = \operatorname{dp-rk}(W)$. Now $f \restriction Z_1$ is injective, and its image W_1 is our desired set.

(2) Since Z is almost strongly internal to D there exists $\sigma : Z \to D^k$ with finite fibres. Let

$$C = \{(w, y) \in W \times D^k : y \in \sigma(f^{-1}(w))\};$$

it is a finite-to-finite correspondence. Choose $y_0 \in \pi_2(C)$ (the projection of C into D^k) with $\operatorname{dp-rk}(y_0/A) = \operatorname{dp-rk}(\pi_2(C))$. Since C is finite-to-finite the set of all $y \in D^k$ such that $C^y \cap C^{y_0} \neq \emptyset$ is finite, so there exists an open $V \ni y_0$ such that for all $y \in V$, $C^y \cap C^{y_0} = \emptyset$.

As above, we may replace V with an open U defined over B such that $\operatorname{dp-rk}(y_0/B) = \operatorname{dp-rk}(y_0/A)$. Then $C \cap (W \times U)$ is a graph of a finite-to-one function, F. Since y_0 is in the range of that function and $\operatorname{dp-rk}(y_0/A) = \operatorname{dp-rk}(W)$, necessarily $\operatorname{dp-rk}(\operatorname{dom}(F)) \geq \operatorname{dp-rk}(W)$, so equality must hold, as required. □

Much of Section 3.1 is dedicated to making sense of these results for $K/\mathcal{O}$ in P-minimal fields and proving an appropriate analogue. Similar results for $\mathbb{Z}$-groups are easier and follow from known properties of such groups.

When G is locally strongly internal to an SW-uniformity, the group ν we obtain in Theorem 1.2 admits a definable group topology. This will allow us to topologise G, using the following easy observation.

{L:from H a u

Lemma 2.7. *Let G be a group and H a subgroup endowed with a Hausdorff group topology, with $\mathcal{B}_H$ a basis for the e neighbourhoods in H. For $V \in \mathcal{B}_H$, let $U_V = \{(x, y) \in G \times G : x^{-1}y \in V\}$.*

Then the collection $\mathcal{U}^G = \{U_V : V \in \mathcal{B}_H\}$ is a left-invariant uniformity on G extending the associated uniformity on H.

The uniformity $\mathcal{U}^G$ induces a topology on G, call it τ^G, whose basis is the collection of all sets $U_V(h, G)$, as h varies in G and V varies in $\mathcal{B}_H$. Though this need not in general be a group topology, for every $g \in G$ the map $x \mapsto gx$ is continuous. We can show, however:

{L: top from

Lemma 2.8. *Assume that $H \leq G$ is a topological group such that,*

(i) *For every $g \in G$ there is an open $V \subseteq H$ such that $V \subseteq H^g \cap H$.*

(ii) *For every $g \in G$, the function $x \mapsto x^g = gxg^{-1}$, from H into H^g, is continuous at e with respect to the topology on H (this makes sense due to (i)).*

Then the uniformity $\mathcal{U}^G$ defined above makes G a topological group.

Proof. Let $\mathcal{B}_H$ denote a neighbourhood base for H at e. We may assume that each $W \in \mathcal{B}_H$ is symmetric. Note also that by Lemma 2.7 for each $g \in G$, the family $\{gV : V \in \mathcal{B}_H\}$ is a basis for the τ^G-neighbourhoods at g.

We prove first that group inverse is continuous. Assume that $g^{-1}V$ is an open neighbourhood of g^{-1}. We need to find an open neighbourhood gW of g such that $W^{-1}g^{-1} \subseteq g^{-1}V$. By our assumptions, the map $x \mapsto gxg^{-1}$ is continuous at e and hence there is $W \in \mathcal{B}_H$ (in particular $W^{-1} = W$), such that $gWg^{-1} \subseteq V$. It follows that $Wg^{-1} \subseteq g^{-1}V$.

To prove continuity of multiplication let $g_1, g_2 \in G$, and assume that $g = g_1g_2$, and gV is a basic neighbourhood of g and fix some $V' \in \mathcal{B}_H$ such that $V' \subseteq H^g \cap H$, as provided by (i). Using the continuity at e of $x \mapsto x^g$, we can shrink V' so that $V' \subseteq V^g$. Using the continuity of $x \mapsto x^{g_1}$ and the fact that H is a topological group, we can find $W_1, W_2 \in \mathcal{B}_H$, such that

$$(g_1W_1g_1^{-1})(gW_2g^{-1}) \subseteq V'.$$

It follows that $g_1W_1g_2W_2g^{-1} \subseteq gVg^{-1}$, and hence $g_1W_1g_2W_2 \subseteq gV$, as needed. Thus $\mathcal{U}^G$ gives rise to a group topology. □

{S:P-minimal}

3. Some results on P-minimal fields

An important tool in [8] is the analysis of interpretable fields via subsets that are strongly internal to SW-uniform sorts. Such an analysis is not available to us when studying interpretable groups in a P-minimal field, K, as neither the value group nor $K/\mathcal{O}$ are SW-uniform sorts. In the present section we endow $K/\mathcal{O}$ with a topology – neither uniform nor Hausdorff – where, nonetheless, many of the tools available in SW-uniformities can still be applied. We isolate three of the geometric properties that suffice to put the machinery in gear, and verify that they hold in models of Presburger Arithmetic. This will allow, in later sections, a uniform treatment of a wide variety of examples.

In the present section $\mathcal{K} = (K, +, \cdot, \dots)$ is a P-minimal field. We study the structure of $K/\mathcal{O}$ from a topological perspective. One of the main results of this section (also needed in the sequel) is Corollary 3.34 asserting that if $\mathcal{K}$ is p-adically closed then definable functions $f : (K/\mathcal{O})^r \to K/\mathcal{O}$ are generically locally affine (generalising an analogous result from [8]).

We remind that as a pure valued field any P-minimal field $\mathcal{K}$ is p-adically closed and as such it is elementarily equivalent to a finite extension, denoted $\mathbb{F}$, of $\mathbb{Q}_p$. We work in a large saturated P-minimal valued field, $\mathcal{K} = (K, v, \dots)$, and so we may assume that (K, v) is an elementary extension (in the reduct to the pure valued field language) of $(\mathbb{F}, v)$. We add constants for all elements of $\mathbb{F}$; since its value group $\Gamma_\mathbb{F}$ is isomorphic to $\mathbb{Z}$ as an ordered abelian group for simplicity of notation we identify $\Gamma_\mathbb{F}$ with $\mathbb{Z}$, and note that $\mathbf{k}_K = \mathbf{k}_\mathbb{F}$. It is well known that, as an abelian group, $\mathbb{F}/\mathcal{O}_\mathbb{F}$ is isomorphic to $\bigoplus_{i=1}^n \mathbb{Z}(p^\infty)$, where $\mathbb{Z}(p^\infty)$ is the Prüfer p-group and $n = [\mathbb{F} : \mathbb{Q}_p]$. Throughout, we will use without further reference the fact that v is well-defined on $K/\mathcal{O} \setminus \{0\}$ and denote $v(0) = \infty$ (for $0 \in K/\mathcal{O}$). As usual v extends to $(K/O)^n$ by setting $v(a_1, \dots, a_n) = \min\{v(a_i) : i = 1, \dots, n\}$.

-radius balls}

Fact 3.1. (1) *For every $a \in K$ with $v(a) \in \mathbb{Z}$, there is an element $r \in \mathbb{F}$ with $v(a-r) > \mathbb{Z}$.*

(2) *For every $b \in K/\mathcal{O}$, if $v(b) \in \mathbb{Z} \cup \{\infty\}$ then $b \in \mathbb{F}/\mathcal{O}_{\mathbb{F}} \leq K/\mathcal{O}$.*

(3) *For all $n \in \mathbb{N}$, there are only finitely many $b \in K/\mathcal{O}$ such that $nb = 0$; in fact any such element is in $\mathbb{F}/\mathcal{O}_{\mathbb{F}}$.*

Proof. (1) This follows from the completeness of $\mathbb{F}$ as a valued field. Indeed, for $t \in \mathbb{F}$ with $v(at) = 0$ we can find $c \in \mathbb{F}$ with $v(at - c) > 0$ thus $v(a - ct^{-1}) > -v(t)$; continuing inductively we construct a pseudo-Cauchy sequence in $\mathbb{F}$, converging, by completeness, to a limit in $\mathbb{F}$. This limit has the desired properties.

(2) Every $b \in K/\mathcal{O}$ with $v(b) \in \mathbb{Z} \cup \{\infty\}$ is of the form $b = B_{\geq 0}(a)$ for some $a \in K$ with $v(a) \in \mathbb{Z} \cup \{\infty\}$. By (1) there is $r \in \mathbb{F}$ with $v(a-r) > \mathbb{Z} \geq 0$, so $r + \mathcal{O} = a + \mathcal{O}$ as required.

(3) It suffices to show that if $nb = 0$ then $b \in \mathbb{F}/\mathcal{O}_{\mathbb{F}}$. If $nb = 0$ and $b' + \mathcal{O} = b$ then $v(nb') = v(n) + v(b') \geq 0$. So $v(b') \in \mathbb{Z}$, i.e., $b \in \mathbb{F}/\mathcal{O}_{\mathbb{F}}$. □

We note that Clause (3) above is not needed for what we prove in the present section. It will, however, play a role later in the proof of the main results of the paper (Section 7).

{S: local K/O

3.1. Local analysis in $K/\mathcal{O}$.

As noted above, the valuation on K is well-defined on $K/\mathcal{O}$, but the topology it induces on $K/\mathcal{O}$ is discrete. For that reason we introduce a coarser (non-Hausdorff) topology retaining some properties of SW-uniformities. We first need some preliminaries:

Definition 3.2. A ball $B \subseteq K^n$ is *large* if $r(B)$, the valuative radius of B, satisfies $r(B) < \mathbb{Z}$.

Let $\pi : K \to K/\mathcal{O}$ be the quotient map, we also write $\pi : K^n \to (K/\mathcal{O})^n$ for any integer n.

Definition 3.3. A *ball* $U \subseteq (K/\mathcal{O})^n$ is the image under π of a large ball in K.

Note that balls in $(K/\mathcal{O})^n$ are precisely images of large balls in K. We consider the topology on $(K/\mathcal{O})^n$ whose basis is the family of balls. Note that (for $n > 1$) this topology is precisely the product topology on $(K/\mathcal{O})^n$.

Remark 3.4. The ball topology on $K/\mathcal{O}$ is not Hausdorff, in fact, not even T_0: for $b_1 \neq b_2 \in K/\mathcal{O}$, if $v(b_1 - b_2) \in \mathbb{Z}$ then every open set containing b_1 also contains b_2. In addition, the ball topology has no definable basis (since the set of balls of infinite radius in K is not definable),

Despite the above remark, some amount of local analysis in $K/\mathcal{O}$ is possible using the lemma below. For $a = (a_1, \dots, a_n) \in (K/\mathcal{O})^n$, we let $\dim_{\mathrm{acl}}(a)$ be the maximal k such that some subtuple a' of a of length k is algebraically independent and $a \in \mathrm{acl}(a')$ (this is well-defined even when acl does not satisfy exchange).

{L:generic ov

Lemma 3.5. *Let $a = (a_1, \dots, a_m)$ be a tuple of elements in K or $K/\mathcal{O}$ and A an arbitrary parameter set such that $\dim_{\mathrm{acl}}(a/A) = n$. Then for any $\gamma_0 < \gamma_1 < \mathbb{Z}$ with $\gamma_0 - \gamma_1 < \mathbb{Z}$ there exists $\gamma \in \Gamma$ such that $\gamma_0 < \gamma < \gamma_1 < \mathbb{Z}$ and $\dim_{\mathrm{acl}}(a/A\gamma) = n$.*

Proof. By passing to a subtuple of a which is acl-independent over A, we may assume that $|a| = n$.

By replacing γ_0 with $\gamma_0 - \gamma_1 < \mathbb{Z}$ it is enough to find $\gamma \in \Gamma$, $\gamma_0 < \gamma < \mathbb{Z}$ such that $\dim_{\mathrm{acl}}(a/A\gamma) = n$.

For $1 \le i \le n$, let $\hat{a}_i := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$. Assume towards a contradiction that for every infinite $\gamma > \gamma_0$, there is $1 \le i \le n$ such that $a_i \in \operatorname{acl}(\hat{a}_i, A, \gamma)$. We may assume, without loss of generality, that $a_1 \in \operatorname{acl}(\hat{a}_1, A, \gamma)$ for arbitrarily large γ below $\mathbb{Z}$. We now consider all formulas $\varphi(x, y)$ over $\hat{a}_1, A$ with y a Γ-variable and x a $K/\mathcal{O}$-variable.

By compactness, there are formulas $\varphi_1(x, y), \dots, \varphi_k(x, y)$ over $\hat{a}_1, A$, and some $k \in \mathbb{N}$, such that $|\varphi_i(K/\mathcal{O}, \gamma)| \le k_i$ for every $i \le k$, and for every $\gamma_1 \in \Gamma$, if $\gamma_0 < \gamma_1 < \mathbb{Z}$ then there exists $\gamma_1 < \gamma < \mathbb{Z}$, such that $\bigvee_{i=1}^{k} \varphi_i(a_1, \gamma)$.

Without loss of generality we may assume that for $i = 1$ the set of $\gamma > \gamma_0$ such that $\models \varphi_1(a_1, \gamma)$, is cofinal below $\mathbb{Z}$. Since Γ is a pure $\mathbb{Z}$-group in P-minimal structures, we may further assume that $\varphi_1(a_1, \Gamma)$ is a definable set of the form $\{t \in \Gamma : \alpha < t < \beta \wedge x \equiv_N c\}$ for some $N \in \mathbb{N}$ and $0 \le c < N$. By the cofinality assumption, we cannot have $\beta < \mathbb{Z}$, so there exists $m \in \mathbb{Z}$, such that $\varphi_1(a_1, m)$, contradicting the assumption that $a_1 \notin \operatorname{acl}(A, a_2, \dots, a_n)$. $\square$

In terms of the large ball topology, the above lemma says that if $a \in (K/\mathcal{O})^n$ is acl-generic then any neighbourhood of a contains a ball of "generic radius". In Proposition 3.8 we strengthen this to provide a "generic neighbourhood" of a (inside any neighbourhood of a). But we first need some preliminary results.

terior in K/O}

Lemma 3.6. *Assume that $b \in (K/\mathcal{O})^n$ is such that $\dim_{\mathrm{acl}}(b/A) = n$, for some parameter set A.*

(1) *For every A-definable $X \subseteq (K/\mathcal{O})^n$, if $b \in X$ then b is in the interior of X.*
(2) *If $p = \operatorname{tp}(b/A)$ then $p(\mathcal{K})$ is open in $(K/\mathcal{O})^n$*
(3) $\operatorname{dp-rk}(b/A) = n$.

Proof. Let $a \in K^n$ be such that $\pi(a) = b$.

(1) We proceed by induction on n. For $n = 1$, we first prove that every A-definable set Y containing a must contain a large ball. Indeed, since $b \in \pi(Y)$ is non-algebraic Y intersects infinitely many 0-balls. Since K is P-minimal, by [8, Proposition 5.8, Lemma 6.26], Y contains a large ball.

Let $q = \operatorname{tp}(a/A)$; by compactness we conclude that $q(\mathcal{K})$ contains a large ball, but then, as $\mathbb{Z}$ is automorphism invariant, every $\alpha \models q$ belongs to a large ball $B \subseteq q(\mathcal{K})$. In particular, $a \in B \subseteq Y$ for some large ball B.

Now assume that $b \in X$ for an A-definable set $X \subseteq K/\mathcal{O}$ and apply the above to $a \in Y = \pi^{-1}(X)$. We conclude that there is a ball $U \subseteq K/\mathcal{O}$ with $b \in U \subseteq X$.

For $n > 1$, write $b = (b', b_n) \in X$ for $b' \in (K/\mathcal{O})^{n-1}$. Let $a = (a', a_n) \in K^n$ be such that $\pi(a) = b$. Then $\dim_{\mathrm{acl}}(a) = n$ and $\dim_{\mathrm{acl}}(a'/a_n A) = n - 1$. Let $Y = \pi^{-1}(X)$.

Because $b_n \notin \operatorname{acl}(b'A)$, it follows that $X_{b'} = \{(b', t) \in X : t \in K/\mathcal{O}\}$ is infinite, so by the case $n = 1$, b_n is in the interior of $X_{b'}$. It follows that a_n is in the interior of $Y_{a'} = \{(a', t) \in Y : t \in K\}$ in the large ball topology. Since $\dim_{\mathrm{acl}}(a'/a_n A) = n - 1$, by Lemma 3.5 we can find $\gamma < \mathbb{Z}$ such that $\dim_{\mathrm{acl}}(a'/a_n A\gamma) = n - 1$ and $B_{\ge\gamma}(a_n) \subseteq Y_{a'}$.

We consider the $a_n A\gamma$-definable set $W = \{x \in K^{n-1} : B_\gamma(a_n) \subseteq Y_x\}$. It contains a', so by induction it contains a large ball $B_{\ge\gamma'}(a')$. Denoting $\widehat{\gamma} = \max\{\gamma, \gamma'\}$ the ball $B_{\ge\widehat{\gamma}}(a') \times B_{\ge\widehat{\gamma}}(a_n)$ is centred at a and contained in Y. It follows that $b = \pi(a)$ is in the interior of X.

(2) Follows by compactness from (1).

(3) By (2), for $p = \mathrm{tp}(b/A)$, $p(\mathcal{K})$ contains a ball U. Such a ball is a Cartesian product of n balls, each of which has dp-rank 1. So $\text{dp-rk}(p) \geq n$, and equality must hold. $\square$

We conclude:

{C: dpr=acl-d

Corollary 3.7. *For $b \in (K/\mathcal{O})^n$ and any parameter set A we have $\text{dp-rk}(b/A) = \dim_{\mathrm{acl}}(b/A)$.*

Proof. By sub-additivity of the dp-rank we just need to show that $\text{dp-rk}(b/A) \geq \dim_{\mathrm{acl}}(b/A)$. If $\dim_{\mathrm{acl}}(b/A) = k$ then by passing to a subtuple we may assume that $\dim_{\mathrm{acl}}(b_1, \dots, b_k/A) = k$. By Lemma 3.6(3) we have that $\text{dp-rk}(b_1, \dots, b_k/A) = k$ but then $\text{dp-rk}(b/A) \geq k$, as needed. $\square$

We can now prove the desired generalisation of Lemma 3.5:

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Proposition 3.8. *Let $b \in (K/\mathcal{O})^n$, A any set of parameters and $c \in (K/\mathcal{O})^m$.*

(1) *Assume that $\text{dp-rk}(b/A) = n$ and that $b \in W$ for some ball W. Then there is a ball $V \subseteq W$, defined over $B \supseteq A$ such that $b \in V$ and $\text{dp-rk}(b/B) = \text{dp-rk}(b/A) = n$.*
(2) *For any $\gamma < \mathbb{Z}$ there exists $B \supseteq A$ and a B-definable ball $V \subseteq B_{\geq\gamma}(b)$ such that $b \in B$ and $\text{dp-rk}(b, c/A) = \text{dp-rk}(b, c/B)$.*
(3) *Assume that $b \in X \subseteq (K/\mathcal{O})^n$ for some B-definable X with $\text{dp-rk}(b/B) = n$ then there exists $C \supseteq A$ and a C-definable ball $V \subseteq X$ such that $b \in V$ and $\text{dp-rk}(b, c/C) = \text{dp-rk}(b, c/A)$.*

Proof. In all that follows, we repeatedly use – without further mention – Corollary 3.7, i.e., that $\dim_{\mathrm{acl}} = \text{dp-rk}$.

(1) By Lemma 3.6(2) $\mathrm{tp}(b/A)$ is open and thus contains a ball, Z, centred at b. Without loss of generality, $Z = W$. By Lemma 3.5 we can find γ, $r(W) < \gamma < \mathbb{Z}$, such that $B_{\geq\gamma}(b) \subseteq W \vdash \mathrm{tp}(b/A)$ and $\text{dp-rk}(b/A\gamma) = n$. We let $s = B_{\geq\gamma}(b)$, $r(s) = \gamma$ its radius and $[s]$ its code.

Claim 3.8.1. *For any parameter set A, any $b \in (K/\mathcal{O})^n$ and any $K/\mathcal{O}$-ball s containing b, if $r(s) \in \mathrm{dcl}(A)$ and $\text{dp-rk}(b/A) = n$ then $\text{dp-rk}(b/A[s]) = n$.*

Proof. Fix A, b, s as in the statement and let $\gamma = r(s)$. By Lemma 3.6(2), $q := \mathrm{tp}(b/A)$ is open in $(K/\mathcal{O})^n$. It follows that q contains a large ball U, that we may assume to be contained in s.

The proof is based on the case $n = 1$ which we prove first: Since U is large, there are $b_m \in U \subseteq s$, $m \in \mathbb{N}$, such that $b_{m'} \neq b_m$ for $m' \neq m$. Obviously, $b_m \models q$ for all m. Thus, for every b_m, there is $\sigma_m \in \mathrm{Aut}(\mathcal{K}/A)$ mapping b to b_m. Because $b_m \in s = B_{\geq\gamma}(b)$ and $\gamma \in \mathrm{dcl}(A)$ also $\sigma_m(B_{\geq\gamma}(b)) = B_{\geq\gamma}(b_m) = B_{\geq\gamma}(b)$; so $\sigma_m(s) = s$. As a result, we get $b \notin \mathrm{acl}(A[s])$; i.e. $\text{dp-rk}(b/A[s]) = 1$.

Assume now $b = (b_1, \dots, b_n)$, and $s \subseteq (K/\mathcal{O})^n$ is a ball containing b as above. We shall see that for every $i = 1, \dots, n$, b_i is not in the algebraic closure of $A[s]b'$, where $b' = (b_1, \dots, \widehat{b_i}, \dots, b_n)$. By Corollary 3.7, this implies $\text{dp-rk}(b/A[s]) = n$. For simplicity of notation, we treat the case $i = n$, so $b' = (b_1, \dots, b_{n-1})$.

We write $s' = B_{\geq\gamma}(b')$ and $s_n = B_{\geq\gamma}(b_n)$. By applying the case $n = 1$ to $\mathrm{tp}(b_n/Ab')$, we get that $\text{dp-rk}(b_n/Ab'[s_n]) = 1$. Because $r(s') = r(s) \in \mathrm{dcl}(A)$ (by assumption) we get $[s'] \in \mathrm{dcl}(Ab')$ and as $s = s' \times s$ also $[s] \in \mathrm{dcl}(Ab'[s_n])$. Therefore $\text{dp-rk}(b_n/Ab'[s]) = 1$, as required. $\square$ (claim)

To conclude, recall that we have fixed some $\gamma < \mathbb{Z}$ such that $\text{dp-rk}(b/A\gamma) = n$ and such that $B_{\geq\gamma}(b) \subseteq W$. Applying the claim to b over $A\gamma$ we obtain the desired conclusion.

(2) To simplify notation, assume that $A = \emptyset$. We start with the case where $m = 0$, i.e. there is no c. Let $b = (b_1, \ldots, b_n)$ and $k = \text{dp-rk}(b)$; without loss of generality, assume that $\text{dp-rk}(b_1, \ldots, b_k) = k$. Fix any $\gamma < \gamma' < \mathbb{Z}$ such that $\gamma - \gamma' < \mathbb{Z}$. Choose γ_1 be as provided by Lemma 3.5 applied to γ', so $\gamma < \gamma_1$, $\text{dp-rk}(b/\gamma_1) = k$ and $\gamma - \gamma_1 < \mathbb{Z}$. Let $U_1 = B_{\geq\gamma_1}(b)$ and let $b' = (b_1, \ldots, b_k, b'_{k+1}, \ldots, b'_n) \in U_1$ be such that $\text{dp-rk}(b'/\gamma_1) = n$.

By (1), there exists $B \supseteq A$ and a B-definable ball $V \subseteq U_1$ with $V \ni b'$ and $\text{dp-rk}(b'/B) = n$. Since $\text{dp-rk}(b'/B) = n$ and $\gamma - \gamma_1 < \mathbb{Z}$ we get by Lemma 3.5 that there exists γ_2, $\gamma < \gamma_2 < \gamma_1$ with $\text{dp-rk}(b'/B\gamma_2) = n$. Consider the ball

$$V' = \{x \in (K/\mathcal{O})^n : v(x - y) \geq \gamma_2 \text{ for some } y \in V\}.$$

Observe that $b' \in V'$ as $v(b - b') \geq \gamma_1 > \gamma_2$. Our choice of b', γ_2 assures that $\text{dp-rk}(b'/B\gamma_2) = n$ and that $\text{dp-rk}(b/B\gamma_2) = k$. Since $V' \subseteq B_{\geq\gamma}(b)$ it satisfies the requirements.

Now, given $c \in (K/\mathcal{O})^m$ and a ball $U \ni b$, we apply the above result to the tuple (b, c) and the open set $U \times (K/\mathcal{O})^m$ to obtain the desired conclusion.

(3) By Lemma 3.6(1), b is in the interior of X. Hence, there exists some $B_{\geq\gamma}(b) \subseteq X$ for $\gamma < \mathbb{Z}$. By (2), there exists $C \supseteq A$ and a C-definable ball $V \subseteq (K/\mathcal{O})^n$, $b \in V \subseteq B_{\geq\gamma}(b)$, with $\text{dp-rk}(b, c/C) = \text{dp-rk}(b, c/A)$. □

We conclude this section with a couple of applications of the tools developed thus far. The first result is known for SW-uniformities (see [8, §6.2]). The present proof is similar, but some extra care is needed, since the ball topology on $K/\mathcal{O}$ is not Hausdorff.

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Lemma 3.9. (1) *Let $\{X_t : t \in T\}$ be a definable family of finite subsets of $(K/\mathcal{O})^n$, such that for all $b_1 \neq b_2 \in X_t$ we have $v(b_1 - b_2) \in \mathbb{Z}$ and every $b \in (K/\mathcal{O})^n$ belongs to finitely many X_t. Then there is a finite-to-one function from T to $(K/\mathcal{O})^n$.*

(2) *Let $f : X \to T$ be a definable finite-to-one surjective map, and assume that X is almost strongly internal to $K/\mathcal{O}$. Then there exists a definable subset $T_1 \subseteq T$ with $\text{dp-rk}(T_1) = \text{dp-rk}(T)$ that is almost strongly internal to $K/\mathcal{O}$.*

Proof. (1) By saturation,

$$Z := \{v(b_1 - b_2) : b_1, b_2 \in X_t, t \in T\}$$

is finite. Let $m_0 = \min\{Z\} \in \mathbb{Z}$; then for every $t \in T$ the set $\bigcup X_t$ is contained in a single ball of radius m_0.

Let $U \subseteq K^n$ be a closed ball of valuational radius m_0 centred at 0. Since (K, v) is p-adically closed, for each $a \in K^n$, $a + U$ contains only finitely many balls of radius 0. Hence, the map sending t to the unique coset of U containing X_t is finite-to-one, so we have constructed a finite-to-one definable map from T into K^n/U. Since $(K/\mathcal{O})^n$ and K^n/U are in definable bijection, we are done.

(2) Let $g : X \to (K/\mathcal{O})^n$ be a finite-to-one definable map. For any $t \in T$ let $X_t = g(f^{-1}(t))$. Then $\{X_t : t \in T\}$ is a definable family of finite subsets of $(K/\mathcal{O})^n$ such that each $b \in (K/\mathcal{O})^n$ belongs to only finitely many X_t.

By saturation, there is a uniform bound m on $|X_t|$ for $t \in T$. We proceed by induction on m. If $m = 1$, we are done; so assume otherwise. For simplicity, assume everything is defined over $\emptyset$.

Let $t_0 \in T$ be with $\text{dp-rk}(t_0) = \text{dp-rk}(T)$ and assume first that there are $b \neq b' \in X_t$ with $v(b-b') < \mathbb{Z}$; so there exists a ball $U \ni b$ containing b and not b'. By Proposition 3.8(2), there exists a ball $V \subseteq U$ containing b defined over parameters B satisfying that $\text{dp-rk}(b/B) = \text{dp-rk}(b)$. As b and t_0 are inter-algebraic over $\emptyset$, we also have $\text{dp-rk}(t_0/B) = \text{dp-rk}(t_0) = \text{dp-rk}(T)$. As $b' \notin V$, clearly $|X_{t_0} \cap V| < |X_{t_0}|$.

The set $T_1 = \{t \in T : |X_t \cap V| < |X_t|\}$ is defined over B and contains t_0, thus $\text{dp-rk}(T_1) = \text{dp-rk}(T)$. We many now replace each X_t with $X_t \cap V$ and proceed by induction. This completes the proof in the case where $v(b - b') < \mathbb{Z}$ for some $b, b' \in X_{t_0}$.

So we assume now that $v(b - b') \in \mathbb{Z}$ for all $b \neq b' \in X_{t_0}$. By saturation there exists $m \in \mathbb{Z}$ such that $v(b - b') \geq m$ for all $b \neq b' \in X_{t_0}$. Thus, the set

$$T_1 = \{t \in T : (\forall b \neq b' \in X_t) v(b - b') \geq m\}$$

contains t_0 so $\text{dp-rk}(T_1) = \text{dp-rk}(T)$, and we conclude by (1). □

We end this section with a small observation that will be used later on.

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Lemma 3.10. (1) *For any definable subgroup* $H \subseteq (K/\mathcal{O})^n$ *of full dp-rank,* $\text{Tor}(H) = (\mathbb{F}/\mathcal{O}_\mathbb{F})^n$.
(2) *Every definable subgroup* $H \subseteq (K/\mathcal{O})^r$ *has non-trivial torsion.*

Proof. (1) The fact that every torsion element of H is included in $(\mathbb{F}/\mathcal{O}_\mathbb{F})^n$ is Fact 3.1(3). For the other direction, assume that H is A-definable and let $b \in H$ with $\text{dp-rk}(b/A) = 1$. By Lemma 3.6, there exists some $\gamma < \mathbb{Z}$ with $B_{>\gamma}(b) \subseteq H$; but then $B_{>\gamma}(0) = B_{>\gamma}(b) - b \subseteq H$ as well. Now obviously, $\mathbb{F}/\mathcal{O}_\mathbb{F} \subseteq B_{>\gamma}(0) \subseteq H$.

(2) Let $H \subseteq (K/\mathcal{O})^r$ be a definable subgroup, which we may assume to be infinite, and let $n = \text{dp-rk}(H)$. Since by Lemma 3.7 the dp-rank is given by the acl-dimension, there exists a finite-to-one coordinate projection $\tau : H \to (K/\mathcal{O})^n$, with $\text{dp-rk}(\tau(H)) = n$. By (1) $\tau(H)$ has non-trivial torsion, and since $\ker(\tau)$ is a finite group H has non-trivial torsion as well. □

{ss: functio

3.2. **Definable functions in $K/\mathcal{O}$.** In this subsection, **we assume that $\mathcal{K}$ is a sufficiently saturated 1-h-minimal P-minimal field,** e.g. a p-adically closed field.

We show that definable functions in $K/\mathcal{O}$ are locally affine at generic points. With the tools developed in the previous subsection, the proof is quite similar to the one in [8, §6.2]. All results remain true in $\mathbb{Q}_p^{an} := (\mathbb{Q}_p, \mathcal{L}_{an})$, the expansion of $\mathbb{Q}_p$ by all convergent power series $f : \mathcal{O}^n \to K$, since those are P-minimal with definable Skolem functions [33] and [7, Theorem 3.6], respectively.

Definition 3.11. A partial type $P \subseteq K^n$ over a set of parameters A is *large* if $\text{dp-rk}(\pi_* P) = n$, where $\pi : K \to K/\mathcal{O}$ is the quotient map.

By Lemma 3.6 (and Corollary 3.7), a partial type P is large if and only if $(\pi_* P)(\mathcal{K})$ has non-empty interior, if and only if P contains a large ball. Thus, a ball in K is large (as a partial type) if and only if $r(B) < \mathbb{Z}$, i.e., if it is a large ball.

For the following, recall that by [8, Example 6.2] $K/\mathcal{O}$ is opaque,[2] thus by [8, Lemma 6.4] for every complete type q concentrated of $(K/\mathcal{O})^n$ with $\text{dp-rk}(q) = n$ there exists a unique complete type p concentrated on K^n with $\pi_* p = q$; moreover $\text{dp-rk}(p) = n$ as well.

{op large ball}

Lemma 3.12. *Let $\mathcal{K} = (K, v, \dots)$ be a P-minimal field and A an arbitrary set of parameters.*

1. *A partial type $P \vdash K^n$ over A is large if and only if there is a completion p of P (over A) that is large.*
2. *If a partial type $P \vdash K^n$ is large, then $\text{dp-rk}(P) = n$.*
3. *If $\text{tp}(a_1, \dots, a_n/A)$ is large, then so is $\text{tp}(a_1/Aa_2, \dots, a_n)$.*
4. *Let $B \subseteq K^n$ be a large open ball. Then $B + \mathcal{O}^n = B$.*

Proof. (1) We only need to show that if P is large, it has a large completion. Because $\text{dp-rk}(\pi_* P) = n$ it has a completion q of full dp-rank, and since $K/\mathcal{O}$ is opaque, by [8, Lemma 6.4] there is a unique complete type p such that $\pi_*(p) = q$. Since dp-rank can only decrease under definable maps, necessarily $\text{dp-rk}(p) = n$.

The rest is as in [8, Lemma 6.9]. □

We need some results from the theory of 1-h-minimal fields in mixed characteristic, as developed in [6]. Since, as in [8], we apply 1-h-minimality as a black box, we will not dwell on it beyond the definition already given (Definition 2.1). It suffices for us that p-adically closed fields, as well as the above-mentioned sub analytic expansions, are P-minimal and 1-h-minimal ([5, Lemma 6.2.7]).

{D: m-next to}

Definition 3.13. [5, Definition 2.1.6] For an integer $m \in \mathbb{N}$, a ball B in K is m-next to $c \in K$ if $B = \{x \in K : \text{rv}_m(x - c) = t\}$ for some non-zero $t \in \text{RV}_m$.

A ball B is m-next to a finite non-empty set $C \subseteq K$ if $B = \bigcap_{c \in C} B_c$ where B_c is a ball m-next to c for all $c \in C$.

{rem:next}

Remark 3.14. If $x_0 \neq c$ then a ball B is m-next to c if and only if $B = B_{>r}(x_0)$, for some x_0 with $r = v(x_0 - c) + m$. Indeed, since $\text{rv}_m(x) = \text{rv}_m(y) \iff v(x - y) > v(y) + m$, we have that $\{x \in K : \text{rv}_m(x - c) = \text{rv}_m(x_0 - c)\} = \{x \in K : v(x - x_0) > v(x_0 - c) + m\}$.

The main result concerning 1-h-minimality that we need is:

{F: VJP}

Fact 3.15. [6, Corollary 3.1.3] *Let T be a 1-h-minimal theory, $K \models T$ and $f : K \to K$ an A-definable function ($A \subseteq K \cup \text{RV}_n$). Then there exists an A-definable finite set C, and $m \in \mathbb{N}$ such that for any ball B m-next to C, f is differentiable on B and $v(f')$ is constant on B. Moreover:*

1. *For all $x, x' \in B$,*
$$v(f(x) - f(x')) = v(f'(x)) + v(x - x')$$
2. *If $f' \neq 0$ on B then for any open ball $B' \subseteq B$ the image $f(B')$ is an open ball of radius $v(f') + r(B')$ where $r(B')$ is the valuative radius of B'.*

To apply this result, we need the following observation:

{L: m-next}

Lemma 3.16. *Let $p \in S_n(A)$ be a large type in K^n, $C \subseteq K^n$ a finite A-definable set. Then for any fixed $m \in \mathbb{N}$ and any $b \models p$ there exists a large ball B, $b \in B \subseteq p(\mathcal{K})$, and a large ball $B' \supseteq B$ that is m-next to C.*

[2]Opacity is needed only in order to invoke [8, Lemma 6.4], so we omit the definition. See [8, §6.1] for details.

Proof. Fix $b \models p$, $m \in \mathbb{N}$ and a finite A-definable set C; by Lemma 3.6, there exists an open large ball $B_{>r_0}(b) \subseteq p(\mathcal{K})$. Since C is A-definable, $p(\mathcal{K}) \cap C = \emptyset$ so $v(b-c) \leq r_0$ for any $c \in C$. Let $c_0 \in C$ with $v(b-c_0) \geq v(b-c)$ for any $c \in C$. It is enough to show that $B_{>r_0+m}(b)$ is contained in a large ball B that is m-next to c_0. As $v(b-c_0) \leq r_0$, we also have $v(b-c_0)+m \leq r_0+m$ and obviously $B_{>r_0+m}(b) \subseteq p(\mathcal{K})$. The desired conclusion follows from Remark 3.14. □

Recall the following from [8].

Definition 3.17. A (partial) function $f : K^n \to K$ *descends* to $K/\mathcal{O}$ if $\mathrm{dom}(f) + \mathcal{O}^n = \mathrm{dom}(f)$ and for every $a, b \in \mathrm{dom}(f)$, if $a - b \in \mathcal{O}^n$, then $f(a) - f(b) \in \mathcal{O}$. The function f *descends to* $K/\mathcal{O}$ *on some (partial) type* $P \vdash \mathrm{dom}(f)$ if $f \restriction P$ descends to $K/\mathcal{O}$.

Conversely, a function $F : (K/\mathcal{O})^n \to K/\mathcal{O}$ *lifts* to K, if it is the image under the natural quotient map of a definable function $f : K^n \to K$ descending to $K/\mathcal{O}$.

The following basic example will play an important role:

{example-desc

Example 3.18. When $a \in \mathcal{O}$ the linear function $\lambda_a : x \mapsto a \cdot x$ descends to an endomorphism $\tilde{\lambda}_a : (K/\mathcal{O}, +) \to (K/\mathcal{O}, +)$. If $a \in \mathbf{m}$, then $\tilde{\lambda}_a$ has an infinite kernel.

The next technical lemma is an analogue of [8, Lemma 6.14]. The proof is similar with minor adaptations. For $f : K^n \to K$ we let f_{x_i} denote the partial derivative with respect to x_i.

{L:derivtives

Lemma 3.19. *Let $p \in S_n(A)$ be a large type, $p \vdash \mathrm{dom}(f)$ for some A-definable $f : K^n \to K$. Then:*

(1) *f is differentiable on p.*
(2) *If f descends to $K/\mathcal{O}$ on p then $(f_{x_1}, \ldots, f_{x_n})(a) \in \mathcal{O}^n$ for all $a \models p$.*
(3) *Assume that $\mathrm{Im}(f) \subseteq \mathcal{O}$. Then for every $a \models p$ there exists a large ball $B \ni a$ contained in $p(\mathcal{K})$ such that for all $b \in B$ and $1 \leq i \leq n$*
$$v(f_{x_i}(b)) + 2r(B)) > 0.$$
In particular $f_{x_i}(a) \in \mathfrak{m}$ for all $a \models p$.

Proof. (1) By [6, Proposition 3.1.1], every definable function is generically differentiable and p is a generic type in K^n, so the result follows.

(2) For simplicity, assume that $A = \emptyset$. Fix $a = (a_1, \ldots, a_n) \models p$. We show that $f_{x_1}(a) \in \mathcal{O}$. Set $g(t) = f(t, a_2, \ldots, a_n)$. By Fact 3.15 there is a finite $(a_2, \ldots, a_n)$-definable set, C, and a natural number m such that g' is constant on any ball B m-next to C. Since $p_1 := \mathrm{tp}(a_1/a_2, \ldots, a_n)$ is large, it follows from Lemma 3.16 that there is a large ball $B \subseteq p_1(\mathcal{K})$ containing a_1 where g' is constant. By Lemma 3.12, $B + 1 = B$ and therefore by Fact 3.15 we have:
$$v(g(a_1+1) - g(a_1)) = v(g'(a_1)) = v(f_{x_1}(a)).$$
Since f descends to $K/\mathcal{O}$ on p we get that $v(f(a_1+1, a_2, \ldots, a_n) - f(a)) \geq 0$, so that $v(f_{x_1}(a)) \geq 0$, as required.

(3) Let $B_0 \subseteq p(\mathcal{K})$ be a large ball containing a, $r_0 := r(B_0)$. By $\aleph_1$-saturation of $\mathcal{K}$ there exists $r_0 < r < \mathbb{Z}$ such that $r_0 - r < \mathbb{Z}$. For $B := B_{>r}(a)$, let $b \in B$; we claim that $v(f_{x_i}(b)) + r \geq 0$.

To simplify the notation we show it for $i = 1$. Let $g(t) := f(t, b_2, \ldots, b_n)$. Note that since $b \in B_{>r_0}(a)$, for all $r' > r_0$ we have that $B_{>r'}(b_1) \subseteq p_1(\mathcal{K})$ where $p_1 := \mathrm{tp}(b_1/b_2, \ldots, b_n)$.

In particular, if C is the $(b_2, \ldots, b_n)$-definable finite set provided by Fact 3.15 (applied to g) and $m \in \mathbb{N}$ is the corresponding natural number, then $B_{>r+m}(b_1) \subseteq p_1(\mathcal{K})$ and therefore $v(b_1 - c) < r_0 + m$ for all $c \in C$. In particular, $B_{>r}(b_1)$ is contained in a ball m-next to C. So $v(g')$ is constant on $B_{>r}(b_1)$. If $g'(t) \equiv 0$, the claim holds trivially. Otherwise, by Fact 3.15 $g(B_{>r}(b_1))$ is an open ball of radius $v(g'(b_1)) + r$. Since $g(B_{>r}(b_1)) \subseteq \mathcal{O}$, the claim follows.

Replacing r with $r - 1$, if needed, we may assume that r is even. Applying the claim to $r/2$ the result follows. $\square$

affine in K/O}

Lemma 3.20. *Let $f : K \to K$ be an A-definable partial function and $p \vdash \mathrm{dom}(f)$ a complete large type over A. If f descends to $K/\mathcal{O}$ on p then for every $a \models p$ there is a large ball B, $a \in B \subseteq p(\mathcal{K})$, such that for all $x \in B$.*

$$f(x) - f(a) - f'(a)(x - a) \in \boldsymbol{m}.$$

Proof. By Lemma 3.19(2), $f'(c) \in \mathcal{O}$ for all $c \models p$. We may thus assume that $f'(c) \in \mathcal{O}$ for all $c \in \mathrm{dom}(f)$.

For every finite A-definable set $C \subseteq K$, $p(\mathcal{K}) \cap C = \emptyset$. Let $a \models p$ and $B_0 \subseteq p(\mathcal{K})$ be a large ball containing a. Fix $r < \mathbb{Z}$ such that $r(B_0) - r < \mathbb{Z}$. We let $B = B_{>r}(a)$, then for any $m \in \mathbb{N}$ we see (Remark 3.14) that B is contained in a ball m-next to C. By Taylor's theorem [6, Theorem 3.1.2] , for every $x \in B$.

$$v(f(x) - f(a) - f'(a)(x - a)) = v(\frac{1}{2}f''(a)(x - a)^2) \tag{1}$$

As $f'(B) \subseteq \mathcal{O}$, Lemma 3.19(3), applied to f', gives a large open ball B', $a \in B' \subseteq p(\mathcal{K})$, such that for all $b \in B'$,

$$v(f''(b)) + 2r(B') > 0.$$

Thus for $B_1 = B \cap B'$, $v(f''(a)) + 2r(B_1) > 0$ and for every $x \in B_1$, $v((x - a)^2) > 2r(B_1)$. Since all of the above remains true if we replace r with $r+1$ we can ignore the contribution of $v(\frac{1}{2})$. Thus, it follows from the above equation that $v(f(x) - f(a) - f'(a)(x - a)) > 0$, as required. $\square$

To conveniently apply the result of the last lemma we make the following definition:

Definition 3.21. A definable function $\lambda : K/\mathcal{O} \to K/\mathcal{O}$ is a *scalar-endomorphism* of $K/\mathcal{O}$ if there exists $a \in \mathcal{O}$ such that $\lambda(x + \mathcal{O}) = \lambda_a(x + \mathcal{O}) := \pi(ax)$ where $\pi : K \to K/\mathcal{O}$ is the quotient map. More generally, $\lambda : (K/\mathcal{O})^n \to (K/\mathcal{O})$ is a scalar endomorphism if $\lambda(x_1, \ldots, x_n) = \sum \lambda_i(x_i)$ where λ_i are scalar-endomorphisms in one variable.

We now turn to proving that definable functions on $(K/\mathcal{O})^n$ are generically given by translates of scalar endomorphisms. We start by observing that the germs of definable scalar-endomorphism are $\emptyset$-definable (recall that $\mathbb{F} \subseteq \mathrm{dcl}(\emptyset)$):

f endo in K/O}

Lemma 3.22. *For any scalar endomorphism $\lambda : K/\mathcal{O} \to K/\mathcal{O}$, there exists $r \in \mathcal{O}_{\mathbb{F}}$ such that $\lambda(x) = \lambda_r(x)$ for all x in some open ball around 0.*

Proof. By assumption $\lambda = \lambda_a$ for some $a \in \mathcal{O}$. By Fact 3.1, there exists $a' \in \mathcal{O}_{\mathbb{F}}$ such that $v(a - a') > \mathbb{Z}$ thus, $\lambda_{a-a'}(b) = 0$ for any $b \in K/\mathcal{O}$ with $v(b) \in \mathbb{Z}$. By compactness, λ and $\lambda_{a'}$ agree on some open ball around 0. $\square$

When $\mathcal{K}$ has definable Skolem functions, every definable function $f:(K/\mathcal{O})^n\to K/\mathcal{O}$ lifts to a function $F:K^n\to K$. So the assumptions in the next proposition (and until the end of this section) are naturally met in those contexts. This is our only use of definable Skolem functions.

The proof of [8, Proposition 6.16] (for unary functions) goes through unaltered using Lemmas 3.19 and 3.20 giving the following.

{P: K/O-affin

Proposition 3.23. *Let $f:K/\mathcal{O}\to K/\mathcal{O}$ be an A-definable partial function with $\operatorname{dom}(f)$ open in the ball topology. If f lifts to K then for every $a\in\operatorname{dom}(f)$ with $\text{dp-rk}(a/A)=1$ there exist a ball $U\subseteq\operatorname{dom}(f)$ with $a\in U$ and a scalar endomorphism $L:(K/\mathcal{O})\to(K/\mathcal{O})$ such that $f(x)=L(x-a)+f(a)$ for all $x\in U$.*

Corollary 3.24. *Let $f:K/\mathcal{O}\to K/\mathcal{O}$ be a partial A-definable function lifting to K on some non-algebraic $p\in S_1(A)$. Then there is $t\in\mathcal{O}_{\mathbb{F}}$ such that $f(x)-\lambda_t(x)$ is locally constant on p.*

Proof. By Proposition 3.23, if $a\models p$ then there is some scalar endomorphism λ (that may depend on a) such that $f(x)=f(a)+\lambda(x-a)$ for all x in some ball $B\subseteq p(\mathcal{K})$. By Lemma 3.22, we may assume that $\lambda=\lambda_t$ for some $t\in\mathcal{O}_{\mathbb{F}}$. Hence, $f(x)=\lambda_t(x)+d$ on some large sub-ball of B, for some $d\in K/\mathcal{O}$ depending on a. As $t\in\mathcal{O}_{\mathbb{F}}$ and $\mathbb{F}\subseteq\operatorname{dcl}(\emptyset)$, t is constant on p, and the conclusion follows. □

Summing up all of the above we get.

{P: n-dim aff

Proposition 3.25. *Let $f:(K/O)^n\to K/\mathcal{O}$ be a partial A-definable function lifting to K on some $p\in S_n(A)$ with $\text{dp-rk}(p)=n$. Then there exists a scalar-endomorphism $\lambda:(K/\mathcal{O})^n\to(K/O)$ definable over $\mathcal{O}_{\mathbb{F}}$ such that $f-\lambda$ is locally constant on $p(\mathcal{K})$.*

Proof. Without loss of generality $A=\emptyset$. Let $a=(a_1,\dots,a_n)\models p$ and consider the function $g_1(t):=f(t,a_2,\dots,a_n)$. By what we have shown there is $r_1\in\mathcal{O}_{\mathbb{F}}$ such that $g_1(t)-\lambda_{r_1}$ is locally constant on $\operatorname{tp}(a_1/a_2,\dots,a_n)$. Similarly, for all $1\le i\le n$ we can find λ_{r_i} such that $g_i(t)=f(a_1,\dots,\hat{a}_i,\dots,a_n)$ is locally constant on $\operatorname{tp}(a_i/a_1,\dots,\hat{a}_i,\dots,a_n)$. The result follows. □

Our next goal is to extend the above result to types, not necessarily of full dp-rank in $(K/\mathcal{O})^r$. To do so we introduce a notion that will be of importance in the sequel as well:

Definition 3.26. A set $S\subseteq(K/\mathcal{O})^n$ has *minimal fibres* if $\text{dp-rk}(S)=k$ and there exists a coordinate projection $\pi:S\to(K/\mathcal{O})^k$ and some $m\in\mathbb{N}$ such that for every $y\in(K/\mathcal{O})^k$, $|\pi^{-1}(y)|\le m$ and there is no definable (possibly over additional parameters) $S_1\subseteq S$ such that $\text{dp-rk}(S_1)=\text{dp-rk}(S)$ and for every $y\in(K/\mathcal{O})^k$, $|f^{-1}(y)\cap X_1|<m$.

Remark 3.27. It is easy to see that a set $S\subseteq(K/\mathcal{O})^n$ has minimal fibres if and only if $\text{dp-rk}(S)=k$ and there exists a coordinate projection $\pi:S\to(K/\mathcal{O})^k$ with finite fibres such that for any definable $S'\subseteq S$, $\text{dp-rk}(\pi^{-1}(\pi(S')\setminus S')<\text{dp-rk}(S)$.

Notice that if some $S\subseteq(K/\mathcal{O})^n$ of rank k, projects finite-to-one into $(K/\mathcal{O})^k$ then there exists $S'\subseteq S$, $\text{dp-rk}(S')=\text{dp-rk}(S)$ (possibly defined over additional parameters) such that S' has minimal fibres with respect to the same projection. Indeed, we just choose $S'\subseteq S$ with the property that the fibres of $\pi\restriction S'$ are of minimal size among all definable subsets of S of maximal dp-rank. The following is a local version of this observation:

{L:wma has fi

Lemma 3.28. *Let $X \subseteq (K/\mathcal{O})^r$ be an A-definable subset. For any $a \in X$ with* $\text{dp-rk}(a/A) = \text{dp-rk}(X)$ *there exists $X' \subseteq X$ definable over some set $C \subseteq A$ such that $a \in X'$,* $\text{dp-rk}(a/C) = \text{dp-rk}(a/A)$ *and X' has minimal fibres.*

Proof. Let $p = \text{tp}(a/A)$ and set $\text{dp-rk}(a/A) = \dim_{\text{acl}}(a/A) = \text{dp-rk}(X) = n$. Let $\tau : (K/\mathcal{O})^r \to (K/\mathcal{O})^n$ be a coordinate projection such that $\dim_{\text{acl}}(\tau(a)/A) = \dim_{\text{acl}}(a/A)$. Letting $q = \tau_* p$ it follows that $\text{dp-rk}(q) = n$.

By Proposition 3.8, there exits a large ball $B \subseteq q(\mathcal{K})$ containing $\tau(a)$; so without loss of generality $X = \tau^{-1}(B) \cap p(\mathcal{K})$, as it is a definable set of dp-rank n. Note that $f \restriction X$ has finite fibres and as $\text{dp-rk}(X) = n$, there exists $X' \subseteq X$ satisfying that $\text{dp-rk}(X') = \text{dp-rk}(X)$ with minimal fibres for τ; assume that X' is definable over some parameter set C'.

Let $a' \in X'$ be an element with $\text{dp-rk}(a'/C') = \text{dp-rk}(X')$. Since $a' \models p$ there exists an automorphism σ over A mapping a' to a. Then $\sigma(X') \subseteq X$ contains a and has minimal fibres for τ and is definable over $C := \sigma(C)$. □

For any function $f : X \to Y$, we write $[x]_f = f^{-1}(f(x))$. For the next result recall that we view $\mathbb{F}/\mathcal{O}_{\mathbb{F}}$ as canonically embedded inside $K/\mathcal{O}$.

coset in K/O}

Lemma 3.29. *Let $X \subseteq (K/\mathcal{O})^r$ be an A-definable set with minimal fibres. Assume that* $\text{dp-rk}(X) = n$ *and let $a \in X$ be such that* $\text{dp-rk}(a/A) = n$. *Then there exists an A-definable subset $X_1 \subseteq X$ with $a \in X_1$, a finite subgroup $G_a \subseteq (\mathbb{F}/\mathcal{O}_{\mathbb{F}})^r$ and a coordinate projection $\tau : (K/\mathcal{O})^r \to (K/\mathcal{O})^n$ such that for every $b \in X_1$, $[b]_\tau = G_a + b$.*

Proof. Let $\tau : X \to (K/\mathcal{O})^n$ be a coordinate projection witnessing that X has minimal fibres. For $b \in X$, set $G_b = \{x - b : x \in [b]_\tau\}$.

Let $a \in X$, $\text{dp-rk}(a/A) = n$. We claim that for all $b \in [a]_\tau$, $v(b - a) \in \mathbb{Z}$. Indeed, assume towards a contradiction that there is $b \in [a]_\tau$ with $v(b - a) < \mathbb{Z}$. By Proposition 3.8 we can find a ball U containing a, defined over some parameters C such that $\text{dp-rk}(b/C) = n$ and $b \notin U$. Thus $|[a]_\tau \cap U| < |[a]_\tau|$. Now the set $S' = \{x \in X : |[x]_\tau \cap U| < |[a]_\tau|\}$ contradicts the assumption that X has minimal fibres.

Therefore, $v(G_a) \subseteq \mathbb{Z}$ and by Fact 3.1(2), $G_a \subseteq \mathbb{F}/\mathcal{O}_{\mathbb{F}}$ and so it is $\emptyset$-definable. Thus, replacing X by a subset X_1 of the same dp-rank, we may assume that $G_a = G_b$ for all $b \in X$. In particular, $G_a = G_b$ for any $b \in [a]_\tau$. It easily follows that G_a is a subgroup and $[a]_\tau$ its coset. □

Definition 3.30. For $a \in (K/\mathcal{O})^r$, let

$$Z(a) := \{x \in (K/\mathcal{O})^r : v(x - a) \in \mathbb{Z} \cup \infty\}.$$

For $X \subseteq (K/\mathcal{O})^r$ and $a \in X$ we let $Z_X(a) := Z(a) \cap X$.

Notice that $Z_X(a)$ is a $\bigvee$-definable set. The next lemma shows that $Z_X(a)$ does not depend on the choice of X:

Lemma 3.31. *Let $X_1 \subseteq X \subseteq (K/\mathcal{O})^r$ be A-definable subsets with minimal fibres and $a \in X_1$ with* $\text{dp-rk}(a/A) = n = \text{dp-rk}(X)$. *Then $Z_X(a) = Z_{X_1}(a)$.*

Proof. Let $\tau : X \to (K/\mathcal{O})^n$ be a coordinate projection witnessing minimal fibres. It follows that $[a]_\tau \cap X_1 = [a]_\tau$. By replacing X_1 with a subset of minimal fibres, still of full dp-rank we may assume that for all $x \in X_1$ we have $[x]_\tau \cap X_1 = [x]_\tau$.

By Lemma 3.6(1), we have that $\tau(a)$ is in the interior of $\tau(X_1)$. Thus, there exists a ball $V \subseteq (K/\mathcal{O})^n$, with $\tau(a) \in V \subseteq \tau(X_1)$. By our choice of X_1, we have $\tau^{-1}(V) \subseteq X_1$; the result follows. □

From now on, we further assume that definable function on $K/\mathcal{O}$ lift to definable functions on K. This holds, e.g., if $\mathcal{K}$ has definable Skolem functions.

The next lemma is the main result of this section. It states that definable sets are locally affine at all generic points:

{L:sets are l

Lemma 3.32. *Let $X \subseteq (K/\mathcal{O})^r$ be A-definable with minimal fibres,* $\text{dp-rk}(X) = n$*. Let $a \in X$ be such that* $\text{dp-rk}(a/A) = n$*. Then there exists $C \supseteq A$, with* $\text{dp-rk}(a/C) = n$*, a C-definable ball $U \subseteq (K/\mathcal{O})^r$, $0 \in U$, (hence U is a subgroup), a $\emptyset$-definable subgroup $H \subseteq (K/\mathcal{O})^r$, and a C-definable $X_1 \subseteq X$ containing a, such that $X_1 = a + (H \cap U)$. In particular, $Z_{X_1}(a) = a + H(\mathbb{F}/\mathcal{O}_\mathbb{F})$.*

Proof. By Lemma 3.29, after reducing X and permuting the coordinates, we may assume that the fibres of the projection τ on the first n-coordinates are all cosets of the same finite subgroup $G \subseteq (\mathbb{F}/\mathcal{O}_\mathbb{F})^r$. Since, by definition, $\tau(G) = 0$ we identify G with a subgroup of $(\mathbb{F}/\mathcal{O}_\mathbb{F})^{r-n}$. We also let $\sigma : (K/\mathcal{O})^r \to (K/\mathcal{O})^{r-n}$ be the projection onto the last $r - n$ coordinates.

Let $k = |G|$, $(x, x') \in X$ such that $x \in \tau(X)$, denote

$$f(x) := \sigma\left(\sum_{b \in \tau^{-1}(x)} b\right) \in (K/\mathcal{O})^{r-n},$$

and note that $f(x) = kx'$, as the sum over all elements of G is 0. Let $p = \text{tp}(\tau(a)/A) \vdash (K/\mathcal{O})^n$. Since in $\mathcal{K}$ every definable function $f : (K/\mathcal{O})^n \to K/\mathcal{O}$ lifts to K, we may apply Proposition 3.25 to find some ball $V \subseteq p(\mathcal{K})$ with $\tau(a) \in V$, such that f is a translate of a scalar-endomorphism, on V. By Proposition 3.8(1), we may assume that V is defined over some $C \supseteq A$ with $\text{dp-rk}(\tau(a)/C) = n$. Let $Y := \text{Graph}(f \restriction V)$. The graph of f is a coset of a $\emptyset$-definable subgroup $H_1 \subseteq (K/\mathcal{O})^r$, and thus

$$Y = (b + H_1) \cap (V \times (K/\mathcal{O})^{r-n}),$$

for some $b \in (K/\mathcal{O})^r$. Let $X_1 := \{(x, y) \in X : (x, ky) \in Y\}$. Note that X_1 is definable over C, contains a, and hence $\text{dp-rk}(X_1) = n$.

Because the map $(x, y) \mapsto (x, ky)$ is a $\emptyset$-definable endomorphism of $(K/\mathcal{O})^n \times (K/\mathcal{O})^{n-r}$, the pre-image of $b + H_1$ under the map $(x, y) \to (x, ky)$ is of the form $a + H$ for some $\emptyset$-definable subgroup H, and the pre-image of V is a ball $V' \ni a$ in $(K/\mathcal{O})^r$, which we may write $V' = a + U$, for some ball $U \ni 0$. Thus, $X_1 = a + H \cap a + U = a + (H \cap U)$. By Fact 3.1(2), we now get that $Z_{X_1}(a) = a + H(\mathbb{F}/\mathcal{O}_\mathbb{F})$. □

Remark 3.33. In the statement of the previous lemma we cannot require that the ball U is $\emptyset$-definable. Indeed, if X itself was a ball around 0, which is smaller than all $\emptyset$-definable balls, then we cannot find a $\emptyset$-definable such U.

{C: KO affine

Corollary 3.34. *If* $f:(K/\mathcal{O})^r\to K/\mathcal{O}$ *is an* A*-definable partial function and* $a\in \operatorname{dom}(f)$ *is such that* $\operatorname{dp-rk}(a/A)=\operatorname{dp-rk}(\operatorname{dom}(f))$ *then there exists* $C\supseteq A$*, a* C*-definable coset* $X:=H+d$ *and a* $\emptyset$*-definable scalar endomorphism* $\ell:(K/\mathcal{O})^r\to K/\mathcal{O}$ *such that*

(1) $\operatorname{dp-rk}(a/C)=\operatorname{dp-rk}(a/A)$.
(2) $a\in X$.
(3) $f\restriction X=\ell(x-a)+\ell(a)$.

Proof. By Lemma 3.28 we may assume that the graph of f has minimal fibres. We can now apply Lemma 3.32 to the graph of f. □

The next example will help us clarify an important distinction arising in Section 4.

{L: not lsi}

Example 3.35. Let $\mathbb{F}$ be a residual quadratic extension of $\mathbb{Q}_p$ and let $\mathcal{K}\succ\mathbb{F}$ be a sufficiently saturated extension. Let $C_p\le K/\mathcal{O}$ be a cyclic subgroup of order p, and $G:=(K/\mathcal{O})/C_p$. Then G is locally almost strongly internal to $K/\mathcal{O}$ but not locally strongly internal to $K/\mathcal{O}$.

On the other hand, letting $H=\{x\in G: px=0\}$ the group G/H is strongly internal to $K/\mathcal{O}$.

Proof. Since the quotient map $\pi:K/\mathcal{O}\to G$ is finite-to-one it follows directly from Lemma 3.9(2) that G is almost strongly internal to $K/\mathcal{O}$.

We now verify that G is not locally strongly internal to $K/\mathcal{O}$. Assume towards a contradiction that there exists a definable injection $f:X\to (K/\mathcal{O})^r$ with X infinite. Let $V=\pi^{-1}(X)$ and $\widehat{f}:V\to(K/\mathcal{O})^r$ the (clearly, definable) lifting of f to $K/\mathcal{O}$. As $\operatorname{dp-rk}(V)=1$, we may assume – shrinking V if needed – that V is an open ball. Assume everything is definable over $\emptyset$. Let $a\in V$ with $\operatorname{dp-rk}(a)=1$. Shrinking V further, we may apply Proposition 3.23, to deduce that $\widehat{f}:=(\widehat{f}_1,\dots,\widehat{f}_r)$ where each $\widehat{f}_i$ is of the form $L_i(x-a)+\widehat{f}_i(a)$ on V, for some scalar endomorphism, L_i. By Fact 3.1(3) and Lemma 3.10, $V+C_p=V$. Because $\widehat{f}$ factors through C_p on V, it follows that the scalar endomorphisms L_i are all invariant under C_p, and therefore $C_p\subseteq\ker(L_i)$ for $i=1,\dots,r$.

Since $\mathbb{F}$ is a residual extension of $\mathbb{Q}_p$, the subgroup $H=\{x\in K/\mathcal{O}: v(x)\ge -1\}$, where $1\in\Gamma$ is the minimal positive element, is readily seen to contain C_p, and $|H|=|\mathbf{k}_K|=p^2$. It is also easy to verify that for every scalar endomorphism λ of $K/\mathcal{O}$ with non-trivial kernel, we have $H\subseteq\ker(\lambda)$, so $C_p\subsetneq H\subseteq\ker(L_i)$ for every L_i.

Thus, the function $\widehat{f}$ is invariant under the group $H\supseteq C_p$, contradicting our assumption that f was injective.

For the final statement, since $\mathbb{F}/\mathcal{O}_{\mathbb{F}}$ is isomorphic to $\mathbb{Z}(p^\infty)\oplus\mathbb{Z}(p^\infty)$, G/H is definably isomorphic to $K/\mathcal{O}$ as witnessed by the map $x\mapsto px$. □

Remark 3.36. It can be shown that if $F\equiv\mathbb{Q}_p$ then $(F/\mathcal{O})^r/H$ is definably isomorphic to $(F/\mathcal{O})^r$ for any finite subgroup H. In particular, in the above example, the group G is isomorphic to $(K/\mathcal{O})^2$, but not definably so.

4. Vicinic sorts and their properties

{S: asi to D}

In the present section we develop an *ad hoc* axiomatic setting allowing us to unify the treatment of all unstable distinguished sorts considered in the sequel. We then apply the technical tools we develop to show that, given a group G almost strongly internal to such a set, D, there exists a finite normal subgroup $H \trianglelefteq G$ such that G/H is strongly internal to D. This suffices for the construction of an infinitesimal subgroup in G/H.

Throughout this section we let $\mathcal{M}$ be a multi-sorted, $|T|^+$-saturated structure and D a dp-minimal definable set.

{ss:key geo p

4.1. **The key geometric properties.** To motivate our axiomatic setup we isolate some key geometric properties shared by SW-uniformities, models of Presburger arithmetic and $K/\mathcal{O}$ for $\mathcal{K} = (K, v, \dots)$ p-adically closed. These properties are, in the latter setting, Corollary 3.7, Proposition 3.8 and Lemma 3.9(2). These suffice for the development of much of the theory of infinitesimal subgroups in both topological and non- topological settings.

For p-adically closed $\mathcal{K}$, the induced structure on $K/\mathcal{O}$ is not geometric: it does not eliminate the quantifier $\exists^\infty$ ([8, Lemma 6.27]) nor does acl have the exchange property. Nevertheless, Corollary 3.7 asserting that $\dim_{\mathrm{acl}} = \text{dp-rk}$ gives dp-rank enough geometric flavour to get us going:

{Gamma:A1}

Fact 4.1. *If D is an SW-uniformity (in some structure) or a model of Presburger arithmetic then* $\dim_{\mathrm{acl}}(a/A) = \text{dp-rk}(a/A)$ *for all $a \in D^n$ and any parameter set A.*

Proof. For SW-uniformities this follows from [26, Propsition 2.4] noting that by Simon's work dp-rank is local in dp-minimal theories, [24, Theorem 0.3]. If D is a model of Presburger Arithmetic then this follows from [24, Theorem 0.3] since D satisfies exchange. ☐

Proposition 3.8, originally proved in the context of SW-uniformities (see Fact 2.5) is crucial for allowing local analysis at a generic point. For Presburger arithmetic, the analogous statement is simpler to prove, using quantifier elimination and the fact that acl satisfies exchange:

{Gamma: A2}

Lemma 4.2. *Let $(D, +, <)$ be a model of Presburger arithmetic. Let A be any set of parameters, $g \in D^n$, $h \in D^m$. For any set of parameters B and B-definable $X \subseteq D^n$, if $g \in X$ and* $\text{dp-rk}(g/B) = n$ *then there exists $C \supseteq A$ and a C-definable set $X_1 \subseteq X$ such that* $\text{dp-rk}(X_1) = n$*, $g \in X_1$ and* $\text{dp-rk}(g, h/A) = \text{dp-rk}(g, h/C)$.

Proof. By [8, Lemma 3.14(2)], we can find a small model $L \supseteq B$ for which $\text{dp-rk}(g/L) = n$. By [18, Lemma 3.4], there exists an L-definable subset $X_0 \subseteq X$ containing $g = (g_1, \dots, g_n)$ of the form $I_1 \times \cdots \times I_n$, where for each $i \leq n$, $g_i \in I_i = \{\alpha_i \leq x \leq \beta_i : x \equiv_{N_i} c_i\}$, for some $N_i \in \mathbb{N}$, where both intervals $[\alpha_i, g_i]$ and $[g_i, \beta_i]$ are infinite and $0 \leq c_i < N_i$.

For simplicity, we prove the result for $n = 1$. The general case follows by induction. We use compactness to find $\alpha_1 < \alpha_1' < g_1$ such that $\text{dp-rk}(\alpha_1'/g_1, hL) = 1$ and $[\alpha_1', g_1]$ infinite and then $g_1 < \beta_1' < \beta_1$ such that $\text{dp-rk}(\beta_1'/g_1, h, \alpha_1' L) = 1$ and $[g_1, \beta_1']$ infinite. We have $g_1 \in I_1' = \{\alpha_1 \leq x \leq \beta_1' : x \equiv_{N_i} c_i\} \subseteq I_1$ and by exchange $\text{dp-rk}(g_1, h/\alpha_1', \beta_1', L) = \text{dp-rk}(g_1, h/L)$. ☐

Lemma 3.9(2) is the most technical of the three geometric properties we need. It would be nice to find a slicker description, e.g., in terms of elimination of finite imaginaries. The next lemma may be a first step in that direction:

{L:asi diagar

Lemma 4.3. *Let $\mathcal{M}$ be any structure, and S a stably embedded definable set. If S has elimination of imaginaries, X is (almost) strongly internal to S and $g : X \to Y$ is a definable surjection then Y is (almost) strongly internal to S.*

Proof. We consider the almost strongly internal case only, as the strongly internal case is similar and easier. Let $f : X \to S^n$ be a finite-to-one definable function. For any $y \in Y$ consider the definable set $W_y = f(g^{-1}(y)) \subseteq S^n$. By elimination of imaginaries in S, there exists a definable function $h : Y \to S^m$ such that $h(y_1) = h(y_2)$ if and only if $W_{y_1} = W_{y_2}$. It will suffice to show that h is finite-to-one. So assume towards a contradiction that this is not the case. I.e., there exists $b \in S$ with $H := h^{-1}(b)$ infinite. Fix $y_1 \in H$. Then for any other $y_2 \in H$, obviously, $g^{-1}(y_1) \cap g^{-1}(y_2) = \emptyset$ (but $g^{-1}(y_2) \neq \emptyset$ because g is onto). Since f is finite to one and H is infinite, for $w \in W_{y_1}$ there is $y_2 \in H$ s.t. $f^{-1}(w) \cap g^{-1}(y_2) = \emptyset$, contradicting the assumption that $W_{y_1} = W_{y_2}$. □

Since Presburger arithmetic eliminates imaginaries this gives the analogue of Lemma 3.9(2) for Presburger Arithmetic. For SW-uniformities this is Lemma 2.6.

{ss: vicinic}

4.2. **Vicinic dp-minimal sets.** We now introduce the basic axiomatic setting we will be interested in, but first:

Definition 4.4. Let X be an A-definable set of finite dp-rank, $a \in X$ and $B \supseteq A$ a set of parameters.

(1) The point a is *B-generic* in X (or, *generic in X over B*) if $\text{dp-rk}(a/B) = \text{dp-rk}(X)$.
(2) For an A-generic $a \in X$, a set $U \subseteq X$ is *a B-generic vicinity of a in X* if $a \in U$, U is B-definable, and $\text{dp-rk}(a/B) = \text{dp-rk}(X)$ (in particular, $\text{dp-rk}(U) = \text{dp-rk}(X)$).

Note that if D is an SW-uniformity then a generic vicinity of a generic point a in a set $X \subseteq D^k$ is, in fact, a neighbourhood of a in the relative topology of X. The existence of generic neighbourhoods in SW-uniformites is given by Fact 2.5.

Definition 4.5. A dp-minimal set D is *vicinic* if it satisfies the following axioms:

(A1) $\dim_{\text{acl}} = \text{dp-rk}$; i.e. for any tuple $a \in D^n$ and set A $\dim_{\text{acl}}(a/A) = \text{dp-rk}(a/A)$.
(A2) For any sets of parameters A and B, for every A-generic elements $b \in D^n$, $c \in D^m$ and any B-generic vicinity, X, of b in D^n, there exists $C \supseteq A$ and a C-generic vicinity of b in X such that $\text{dp-rk}(b, c/A) = \text{dp-rk}(b, c/C)$.

Note that in Axiom (A2) it is crucial that the parameter set B need not contain A. The topological intuition is that if b is in the interior of a B-definable set, X, then we can find a smaller neighbourhood of d defined over a new parameter set, C, that is generic with respect to all the initial data.

Let us note that indeed all distinguished sorts in our various settings are vicinic:

s are vicinic}

Fact 4.6. (1) *Every SW-uniformity is vicinic.*
(2) *If $\mathcal{K} = (K, +, \cdot, v, \dots)$ is either V-minimal, power-bounded T-convex or a P-minimal valued field, then all the distinguished sorts, except **k** in the V-minimal case, are vicinic.*

Proof. When D is an SW-uniformity, Axiom (A1) holds by [26, Proposition 2.4] and Axiom (A2) is Fact 2.5. This proves (1) and (2) follows in all cases except when $D = K/\mathcal{O}$ and $D = \Gamma$ in the P-minimal case.

For $K/\mathcal{O}$, see Corollary 3.7 for (A1), and Proposition 3.8 for (A2). For Γ, see Fact 4.1, for (A1) and Lemma 4.2, for (A2). □

Question 4.7. Does any dp-minimal distal structure satisfy Axiom (A2)? Note, however, that it follows from the above fact that, e.g., the valued field sort of an algebraically closed field of equi-characteristic 0 satisfies Axioms (A2). Thus, Axiom (A2) does not imply distality.

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Remark 4.8. By Axiom (A1), for every parameter set A, every $c \in D^m$ is inter-algebraic over A with an A-generic $c' \in D^k$, for $k = \text{dp-rk}(c/A)$. Thus, (A2) remains true if we drop the genericity assumption from c.

Below we assume that D is vicinic.

We first note an immediate implication of Axiom (A1):

{L:acl-dp in

Lemma 4.9. *Let X be a definable set $a \in X$ and assume that X is almost strongly internal to D over A. Then for any $B \supseteq A$ we have $\text{dp-rk}(a/B) = k$ if and only if there exists $a' \in D^k$ such that $\text{dp-rk}(a'/B) = k$, $a' \in \text{dcl}(aA)$ and $a \in \text{acl}(a'B)$.*

We now start developing the technical tools needed for the construction of infinitesimal groups in the setting of vicinic structures. We first want to show that boxes form vicinity-bases at generic points in the following sense:

{L:genos impl

Lemma 4.10. *For any $A \subseteq \mathcal{M}$, A-generic $b = (b_1, \dots, b_n) \in D^n$ and $c \in D^m$, and any A-generic vicinity X of b in D^n, there exists $C \supseteq A$ and C-definable vicinities $I_i \ni b_i$ for $i = 1, \dots, n$ such that $I_1 \times \cdots \times I_n \subseteq X$ and $\text{dp-rk}(b, c/C) = \text{dp-rk}(b, c/A) = n + m$.*

Proof. We use induction on n, where the case $n = 1$ is an immediate application of (A2) to $b \in D$, and $c \in D^m$.

We now consider $b = (b_1, \dots, b_n)$ and X an A-vicinity of b in D^n. Let $b' = (b_1, \dots, b_{n-1})$, and apply (A2) to $b_n \in X_{b'} \subseteq D$.

We then find $C' \supseteq A$ and a C'-vicinity $I \subseteq X_{b'}$ of b_n in D, such that $\text{dp-rk}(b_n, (b', c)/C') = n + m$. Let

$$X_1 := \{x \in D^{n-1} : (\forall y)(y \in I \to (x, y) \in X)\}.$$

This is a C'-definable set, containing b'. So it is a C'-vicinity of b' in D^{n-1}. By induction (now replacing c with b_n, c), there exist $I_i \subseteq D$, $i = 1, \dots, n-1$, defined over $C \supseteq C'$, such that $b' \in \prod_{i=1}^{n-1} I_i \subseteq X_1$, with $\text{dp-rk}(b', b_n, c/C) = n + m$. The set $\prod_{i=1}^{n} I_i \subseteq X$ is the desired vicinity of b. $\square$

4.3. **Functions with minimal fibres.** The notion of definable sets with minimal fibres, with respect to some finite-to-one projection, appeared already in Section 3. We slightly generalize.

Definition 4.11.
1. For X a definable set of finite dp-rank, a definable function $f : X \to Y$ has *minimal fibres* if there exists some $m \in \mathbb{N}$ such that for every $y \in Y$, we have $|f^{-1}(y)| \leq m$, and there is no definable $X_1 \subseteq X$ (possibly over additional parameters), such that $\text{dp-rk}(X_1) = \text{dp-rk}(X)$ and for every $y \in Y$, $|f^{-1}(y) \cap X_1| < m$.
2. A set $X \subseteq D^n$ *has minimal fibres (in D)* if there exists a coordinate projection $\pi : D^n \to D^m$ such that $\text{dp-rk}(X) = m$, and π has minimal fibres.

{remark1}

Remark 4.12. (1) Notice that an A-definable finite-to-one $f: X \to Y$ has minimal fibres if and only if for every $B \supseteq A$ and every B-generic $a \in X$, all elements of $[a]_f$ satisfy the same type over $f(a)B$.

(2) Every definable set $Z \subseteq D^r$ has a definable subset $Z_1 \subseteq Z$ (possibly over additional parameters) such that $\text{dp-rk}(Z_1) = \text{dp-rk}(Z)$ and Z_1 has minimal fibres.

(3) If $Z \subseteq D^r$ has minimal fibres and $X' \subseteq X$ with $\text{dp-rk}(X') = \text{dp-rk}(X)$ then X' has minimal fibres as well.

Before proceeding to the next lemma we recall that for any function $f : X \to Y$, we write $[x]_f = f^{-1}(f(x))$.

Our next goal is to show that Axiom (A2) can be pulled back via maps with minimal fibres, in the following sense:

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Lemma 4.13. *Let X be definable in $\mathcal{M}$ with $\text{dp-rk}(X) = n$ and $f : X \to D^n$ an A-definable function with minimal fibres. Let $b \in X$ be A-generic and $c \in \mathcal{M}$ inter-algebraic over A with some $d \in D^m$.*

Then, for every parameter set, B, and every B-generic vicinity $Y \subseteq X$ of b, there exists $C \supseteq A$ and a C-generic vicinity $Y_1 \subseteq Y$ of b, such that $\text{dp-rk}(b, c/C) = \text{dp-rk}(b, c/A)$.

Proof. Let B be any set of parameters and Y a B-generic vicinity of b in X. Since has finite fibres (because it has minimal fibres) our assumptions imply that $f(b)$ is A-generic in D^n, and $f(Y)$ is a B-generic vicinity of $f(b)$ in D^n. Thus, by (A2), there exists $C \supseteq A$, and a C-generic vicinity $W \subseteq f(Y)$ of $f(b)$, such that $\text{dp-rk}(f(b), d/C) = \text{dp-rk}(f(b), d/A)$. Hence, $\text{dp-rk}(b, c/C) = \text{dp-rk}(b, c/A)$ as well.

Let m be the size of maximal f-fibres. Since f has minimal fibres, for every C-generic $y \in Y$, we have $|[y]_f| = m$, so $[y]_f \subseteq Y$. Thus, the set $W_1 \subseteq W$, of all $w \in W$ such that $|f^{-1}(w)| = m$ satisfies $\text{dp-rk}(W_1) = \text{dp-rk}(W)$, and we have $Y_1 := f^{-1}(W_1) \subseteq Y$. The set Y_1 is the desired C-generic vicinity of b. □

Now we wish to pull back the conclusion of Lemma 4.10 via functions with minimal fibres. We first note:

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Lemma 4.14. *Assume that $f_i : X_i \to Y_i$, $i = 1, 2$, have minimal fibres. Then $(f_1, f_2) : X_1 \times X_2 \to Y_1 \times Y_2$ has minimal fibres. In particular, if $X_i \subseteq D^{m_i}$ has minimal fibres then so does $X_1 \times X_2 \subseteq D^{m_1} \times D^{m_2}$.*

Proof. Let $f = (f_1, f_2) : X_1 \times X_2 \to Y_1 \times Y_2$ and assume for simplicity that it is $\emptyset$-definable. We apply Remark 4.12(1). Let $(a, b) \in X_1 \times X_2$ be generic over some B. Then, by sub-additivity of dp-rank, a is Bb generic in X_1, so all elements of $[a]_{f_1}$ realize the same type over $f_1(a)bB$, hence also over $f_1(a)f_2(b)bB$. It follows that all elements of $[a]_{f_1} \times \{b\}$ satisfy the same type over $f_1(a)f_2(b)B$. Similarly, all elements in $\{a\} \times [b]_{f_2}$ satisfy the same type over $f_1(a)f_2(b)B$. It easily follows that all elements of $[(a, b)]_f$ realize the same type over $f(a, b)B$, as needed. □

of min fibres}

Corollary 4.15. *For $i = 1, 2$ let X_i be A-definable sets, $\text{dp-rk}(X_i) = n_i$, and $f_i : X_i \to D^{n_i}$ A-definable functions with minimal fibres. Let $(d_1, d_2) \in X_1 \times X_2$ be A-generic. Then for any*

A-generic vicinity $Y \subseteq X_1 \times X_2$ of (d_1, d_2), there exists $B \supseteq A$ and a B-generic vicinity of (d_1, d_2) of the form $I_1 \times I_2 \subseteq Y$, with $I_i \subseteq X_i$.

Proof. Let $f = (f_1, f_2) : X_1 \times X_2 \to D^{n_1} \times D^{n_2}$. By the minimality assumption on f_1, f_2, and Lemma 4.14, we have $[(d_1, d_2)]_f \cap Y = [(d_1, d_2)]_f$. Let

$$W = \{w \in f(Y) : f^{-1}(w) \cap Y = f^{-1}(w)\}.$$

Then, $(f(d_1), f(d_2)) \in W$, so $\text{dp-rk}(W) = \text{dp-rk}(d_1, d_2) = n_1 + n_2$.

By Lemma 4.10, there exists $B \supseteq A$ such that W contains a B-generic vicinity of the form $V_1 \times V_2$ of $(f_1(d_1), f_2(d_2))$ in $f(X_1 \times X_2)$. By definition, $f_1^{-1}(V_1) \times f_2^{-1}(V_2)$ is contained in Y and it is thus a B-generic vicinity of (d_1, d_2) satisfying the requirements. □

4.4. **Critical, almost critical and D-sets.** Let $\mathcal{M}$ and D be as before. Recall the definition of a set *strongly internal* and *almost strongly internal* to D (Definition 2.3). We remind (and expand) a definition from [8]:

{D: D-sets}

Definition 4.16. Let S be a definable set of finite dp-rank.

(1) A definable $X \subseteq S$ is m-internal to D if there exists an m-to-one $f : X \to D^n$,
(2) A definable set $X \subseteq S$ is D-critical (for S) if X is strongly internal to D, and has maximal dp-rank among all such subsets of S. Its dp-rank is the *D-critical rank of S*.
(3) A definable set $X \subseteq S$ is *almost D-critical (for S)* if (i) it is almost-strongly internal to D and has maximal dp-rank among all such subsets of S, and (ii) it is m-internal for minimal m among all sets satisfying (i). We call $\text{dp-rk}(X)$ the *almost D-critical rank of S*.
The set X is called (almost) D-critical *over A* if the corresponding map of X into D^n is defined over A.
(4) A definable set $X \subseteq S$ an (almost) D-set over A, if X is an (almost) D-critical set, witnessed by an A-definable $f : X \to D^n$, such that *in addition* $f(X)$ has minimal fibres.

Notice that the D-critical rank of S is always bounded above by the almost D-critical rank of S, but the ranks need not be equal, as we shall now see. Thus, a D-critical set is not necessarily almost D-critical.

{E: crit dim

Example 4.17. Let $\mathcal{K}$ be an elementary extension of a quadratic residual extension of $\mathbb{Q}_p$, as in Example 3.35. Let $G_0 = (K/\mathcal{O})/C_p$ be as in Example 3.35. Let $G = K/\mathcal{O} \times G_0$. Then G is almost strongly internal to $K/\mathcal{O}$, since both $K/\mathcal{O}$ and G_0 are; so its almost $K/\mathcal{O}$-critical rank is 2.

We claim that the critical $K/\mathcal{O}$-rank of G is 1: The definable set $K/\mathcal{O} \times \{0\}$ is strongly internal to $K/\mathcal{O}$, so we only need to note that G has no definable subset of dp-rank 2 which is strongly internal to $K/\mathcal{O}$. Indeed, if $X \subseteq G$ of dp-rank 2, then, just like in Lemma 3.6, it contains a definable set of the form $Y_1 \times Y_2$, with $Y_2 \subseteq G_0$ of dp-rank 1. So if X were strongly internal to $K/\mathcal{O}$ then G_0 would be locally strongly internal to $K/\mathcal{O}$, contradicting Example 3.35.

Remark 4.18. Let $X \subseteq S$ be an almost D-critical set, witnessed by $f : X \to D^k$. Then there exists $X_1 \subseteq X$, defined over some $B \supseteq A$, such that $\text{dp-rk}(X_1) = \text{dp-rk}(X)$ and X_1 is an almost D-set for S, witnessed by f. If f is injective then we can find such an $X_1 \subseteq X$ which is a D-set.

We will use the following remark implicitly throughout.

{important re

Remark 4.19. (1) If $f : X \to D^n$ witnesses that X is an (almost) D-set for S then, since $f(X)$ has minimal fibres, we may compose f with an appropriate coordinate projection to obtain a map with minimal fibres $\pi \circ f : X \to D^n$, such that $n = \text{dp-rk}(X)$. Thus, in Lemma 4.13 the assumption that $\text{dp-rk}(X) = n$ can be dropped when X is an (almost) D-set. Similarly, Lemma 4.15 holds when X_1, X_2 are (almost) D-sets.

(2) If $X_1, X_2 \subseteq S$ are (almost) D-sets then so is $X_1 \times X_2 \subseteq S^2$. Indeed, by Lemma 4.14 it is sufficient to show that $X_1 \times X_2$ is (almost) D-critical in $S \times S$: If $Y \subseteq S \times S$ is (almost) strongly internal to D then so are the fibres Y_x and Y^x for every $x \in S$. In particular $\text{dp-rk}(Y_x) \leq \text{dp-rk}(X_1)$ and $\text{dp-rk}(Y^x) \leq \text{dp-rk}(X_2) = \text{dp-rk}(X_1)$ for any $x \in S$. By sub-additivity $\text{dp-rk}(Y) \leq 2\text{dp-rk}(X_1)$, so $2\text{dp-rk}(X_1)$ is the (almost) D-critical rank of $S \times S$.

(3) If $X \subseteq S$ is an (almost) D-set and $Y \subseteq X$ with $\text{dp-rk}(Y) = \text{dp-rk}(X)$ then Y is also an (almost) D-set.

Below, if the ambient set is clear from the context or immaterial, we will just refer to (almost) D-critical sets, without explicit mention of S (though such an S of finite dp-rank is always assumed to exist in the background).

We end this section with a result on generic vicinities which will play an important role in the next sections.

{filter base}

Lemma 4.20. *Assume that X is an (almost) D-set over A and $d \in X$ generic over A. Let $X_i \subseteq X$, $i = 1, 2$, be two A_i-generic vicinities of d in X for some $A_i \supseteq A$.*

Then there exists C and a C-definable $U \subseteq X_1 \cap X_2$ which is a C-generic vicinity of d in X.

Proof. By Remark 4.19 (1) we may apply Lemma 4.13 as follows: first apply the lemma to X (viewed as A_2-definable), d and the A_1-generic vicinity X_1 (in the role of Y in the lemma) to obtain $U_1 \subseteq X_1$ defined over some $A_2' \supseteq A_2$ such that $d \in U$ is A_2'-generic. Now $U_1 \cap X_2$ is A_2'-definable, so an A_2'-generic vicinity of d satisfying the requirements. ☐

4.5. ***D*-groups.** We proceed with a series of technical lemmas ultimately allowing us to construct, inside a group that is (almost) strongly internal to D, a definable subset that is both (almost) strongly internal and sufficiently closed under the group operation.

g/gh-abstract}

Lemma 4.21. *Let G be an A-definable group. Let $X, Y \subseteq G$ be A-definable and X (almost) strongly internal to D over A, and fix an A-generic $(g, h) \in X \times Y$.*

If $\text{dp-rk}(g/A, g \cdot h) < \text{dp-rk}(X)$ then there exists a finite-to-one definable function from a subset of $X \times Y$ onto a set $W \subseteq X \cdot Y \subseteq G$ satisfying $\text{dp-rk}(W) > \text{dp-rk}(Y)$.

Proof. For simplicity of notation assume that $A = \emptyset$ and write $k = gh \in G$. Assume that $d := \text{dp-rk}(g/k) < m := \text{dp-rk}(X)$, and let $n = \text{dp-rk}(Y)$.

Because X is (almost) strongly internal to D we may apply Lemma 4.9 to obtain $a \in D^d$, $a \in \text{dcl}(g)$ such that $g \in \text{acl}(a, k)$.

Notice that each two pairs of g, h, k are interdefinable over $\emptyset$. E.g., the map $(x, y) \mapsto (x, xy)$ sends (g, h) to (g, k). Thus, we have $\text{dp-rk}(a, k) = \text{dp-rk}(g, k) = \text{dp-rk}(g, h) = m + n$.

Since $a \in D^d$, we have $\text{dp-rk}(a) \leq d$ so by sub-additivity of dp-rank we have $\text{dp-rk}(k/a) \geq n + m - d > n$.

Let $\varphi(x, k)$ be the formula over a isolating $\text{tp}(g/a, k)$.

Let $l := |\varphi(X, k)|$ and

$$Z = \{(x, y) \in X \times Y : \varphi(x, x \cdot y) \wedge (\exists^{\leq l} z)(z \in X \wedge \varphi(z, x \cdot y)\}.$$

This is an a-definable set containing (g, h), thus its image under $(x, y) \mapsto x \cdot y$, call it W, is also a-definable and contains k, so $\text{dp-rk}(W) \geq \text{dp-rk}(k/a) > n$. Our assumption on φ implies that the restriction of the group multiplication to Z is a finite-to-one map. □

{C:g/gh - abs

Corollary 4.22. *Let G be an A-definable group and assume that*

($\star$) *For any definable finite-to-one surjection $f : X \to Y$ with X (almost) strongly internal to D there exists a definable subset $Y' \subseteq Y$ with $\text{dp-rk}(Y') = \text{dp-rk}(Y)$ (almost) strongly internal to D.*

Let $X_1 \subseteq G$ be (almost) strongly internal to D and $X_2 \subseteq G$ (almost) D-critical, both over A. Then for every A-generic $(g, h) \in X_1 \times X_2$ we have $\text{dp-rk}(g/A, g \cdot h) = \text{dp-rk}(X_1)$.

Proof. Assume towards a contradiction that $\text{dp-rk}(g/A, g \cdot h) < m := \text{dp-rk}(X_1)$; then by Lemma 4.21 there exists a finite-to-one definable function from a subset of $X_1 \times X_2$ onto a definable subset $W \subseteq X_1 \cdot X_2$ with $\text{dp-rk}(W) > \text{dp-rk}(X_2)$.

Since X_1 and X_2 are both (almost) strongly internal to D so is $X_1 \times X_2$, and hence any definable subset. By ($\star$), there exists a definable subset $W_1 \subseteq W$ with $\text{dp-rk}(W_1) = \text{dp-rk}(W)$ and a finite-to-one map from W_1 to some D^p, namely W_1 is almost strongly internal to D. This contradicts the maximality of $\text{dp-rk}(X_2)$. □

The conclusion of Corollary 4.22 is important for much that follows. For the sake of clarity of exposition we isolate this property of groups and define:

{Def: D-grou

Definition 4.23. Let D be a vicinic sort. An A-definable group G is an *(almost) D-group* if its (almost) D-critical rank is at least 1 and for every $X_1 \subseteq G$ (almost) strongly internal to D, every (almost) D-critical set, $X_2 \subseteq G$, both over some $B \supseteq A$, and for every (g, h) generic in $X_1 \times X_2$ over B, we have

$$\text{dp-rk}(g/B, g \cdot h) = \text{dp-rk}(X_1).$$

Below we show that groups interpretable in the valued fields we are interested in are (almost) D-groups (for D one of the distinguished sorts) by showing that they satisfy condition ($\star$) in one of its forms. More precisely:

Remark 4.24. By Corollary 4.22, if D satisfies ($\star$) for almost strongly internal sets then every definable group G locally almost strongly internal to D is an almost D-group. If D satisfies the version of ($\star$) for strongly internal sets then every definable group G which is locally strongly internal to D is a D-group.

To avoid any confusion we point out that though the name may suggest it, it formally need not be the case that a D-group is an almost D-group (Example 4.29).

We can now collect our previous results and conclude:

{E:interp gro

Fact 4.25. (1) *A group of finite dp-rank locally (almost) strongly internal to an SW-uniformity D is an (almost) D-group.*

(2) *Let $\mathcal{K} = (K, +, \cdot, v, \dots)$ be an expansion of a valued field.*
- *If $\mathcal{K}$ is V-minimal then every interpretable group locally (almost) strongly internal to a distinguished sort D except the residue field, **k**, is an (almost) D-group.*
- *If $\mathcal{K}$ is power-bounded T-convex then every interpretable group locally (almost) strongly internal to a distinguished sort D is an (almost) D-group.*
- *If $\mathcal{K}$ is P-minimal then every interpretable group locally almost strongly internal to an infinite distinguished sort D is an almost D-group.*

Proof. (1) follows from 2.6 and 4.6. For (2) first note that by Fact 4.6, all the distinguished sorts in these cases are vicinic.

If the distinguished sort is an SW-uniformity, then by Lemma 2.6 condition ($\star$) holds for both strongly internal and almost strongly internal sets. This covers the first two cases, so we are left with the P-minimal case.

If D is a model of Presburger arithmetic then by elimination of imaginaries both versions of ($\star$) hold by Lemma 4.3. If the distinguished sort is $K/\mathcal{O}$ condition ($\star$) holds of almost strongly internal sets by Lemma 3.9. $\square$

We also have:

tion-abstract}

Lemma 4.26. *Let G be a definable (almost) D-group, and let $X_1 \subseteq G$ be (almost) strongly internal to G, $X_2 \subseteq G$ (almost) D-critical, both over A.*

Assume that $(g_1, g_2) \in X_1 \times X_2$ is A-generic and let $g = g_1 g_2^{-1}$. Then $X_1 \cap gX_2$ is an Ag-generic vicinity of g_1 in X_1. In particular, if X_1 is (almost) D-critical then $X_1 \cap gX_2$ is also (almost) D-critical.

Proof. Assume for simplicity that $A = \emptyset$. As G is an (almost) D-group, it is easy to see that $\text{dp-rk}(g_1/g_1 \cdot g_2^{-1}) = \text{dp-rk}(X_1)$. Since $g_1 \in (g_1 \cdot g_2^{-1} X_2) \cap X_1$ and the intersection is $(g_1 \cdot g_2^{-1})$-definable, it follows that $X_1 \cap g_1 \cdot g_2^{-1} X_2$ is a generic vicinity of g_1 in X_1. $\square$

We can now deduce the main result of this section.

rnal-abstract}

Lemma 4.27. *Let G be a definable (almost) D-group, $X_1, X_2 \subseteq G$ (almost) D-sets over A. Assume that $(g_1, g_2) \in X_1 \times X_2$ is A-generic. Then there exists $B \supseteq A$ and B-definable subsets $X_i' \subseteq X_i$, such that $X_1' \times X_2'$ is a B-generic vicinity of (g_1, g_2) in $X_1 \times X_2$ and such that $X_1' \cdot X_2' \subseteq X_1 \cdot g_2$. Moreover, $X_1' \cdot X_2' \subseteq X_1 \cdot g$ for any $g \in X_2'$ and thus $X_1' \cdot X_2'$ is an (almost) D-set over Bg.*

Proof. We prove the lemma in case the X_i are almost D-sets (and G is an almost D-group). The proof for the case where the X_i are D-sets and G is a D-group is similar. For simplicity, we assume $A = \emptyset$.

Let $k = g_1 \cdot g_2 \in G$ and let $Y_2 = \{x_2 \in X_2 : k \in X_1 \cdot x_2\}$. Note that Y_2 is k-definable and contains g_2. Since G is an almost D-group, $\text{dp-rk}(g_2/Ak) = n$, hence $\text{dp-rk}(Y_2) = n$. Since g_1 is inter-algebraic over A with some element in D^l we can apply Lemma 4.13 and Remark 4.19(1) to obtain a definable subset $Y_2' \subseteq Y_2$ containing g_2 that is C-definable for some parameter set C such that $\text{dp-rk}(g_1, g_2/C) = \text{dp-rk}(g_1, g_2) = 2n$. It follows that $\text{dp-rk}(g_i, k/C) = 2n$, for $i = 1, 2$.

We let

$$Z = \bigcap_{y \in Y_2'} X_1 y.$$

It is C-definable, containing k, hence $\text{dp-rk}(Z) = n$. Since $g_2 \in Y_2'$, we also have $Z \subseteq X_1 g_2$. Finally, we consider

$$S = \{(x_1, x_2) \in X_1 \times X_2 : x_1 \cdot x_2 \in Z\}.$$

It is definable over C, and contains (g_1, g_2), thus $\text{dp-rk}(S) = 2n$. By Corollary 4.15, there exists $X_1' \times X_2'' \subseteq S$, a B-generic vicinity of (g_1, g_2) in $X_1 \times X_2$, for some $B \supseteq C$. Now let $X_2' = X_2'' \cap Y_2'$; it is still a C-definable vicinity of g_2. Note that for any $g \in X_2'$, we have

$$X_1' \cdot X_2' \subseteq X_1' \cdot X_2'' \subseteq Z \subseteq X_1 \cdot g,$$

where the latter follows from the definition of Z and the fact that $g \in X_2'$. □

{R:variant re

Remark 4.28. A symmetric proof would give that we can find X_1' and X_2' such that $X_1' \cdot X_2' \subseteq g_1 X_2$

The next example shows that interpretable almost D-groups need not be D-groups:

{E:almost d-g

Example 4.29. Let $\mathcal{K}$ be a sufficiently saturated elementary extension of a quadratic residual extension of $\mathbb{Q}_p$ as in Example 4.17 and $G = K/\mathcal{O} \times G_0$, where $G_0 = (K/\mathcal{O})/C_p$ the group from Example 4.17. Below $D = K/\mathcal{O}$.

As in Example 4.17, G_0 is locally almost strongly internal to D, thus so is G. By Fact 4.25, it is an almost D-group, in particular, $\text{dp-rk}(g/B, g \cdot h) = \text{dp-rk}(X_2)$ for any B-definable almost D-critical X_1, X_2 and B-generic $(g, h) \in X_1 \times X_2$.

By Example 4.17, the D-critial rank of G is 1. To see that G is not a D-group we consider two D-sets in G: The subgroups $X_1 = K/\mathcal{O} \times \{0\}$ and $X_2 = \{(h, h + C_p) : h \in K/\mathcal{O}\}$ are in bijection with $K/\mathcal{O}$, thus they are D-sets in G. However, the two subgroups generate G, so if we pick $(g, h) \in X_1 \times X_2$ generic then $\text{dp-rk}(g/g + h) = 0$.

4.6. **From almost strong internality to strong internality.** In order to put the machinery for the construction of infinitesimal subgroups in gear we need to work inside a group locally strongly internal to a vicinic sort D. As we have seen in Example 3.35, in the case $D = K/\mathcal{O}$, it may happen that a group is locally almost strongly internal to D, but not locally strongly internal. In this section we show that modding out a finite normal subgroup resolves this problem. The proof is inspired by a result of a similar nature due to Hrushovski and Rideau-Kikuchi, [12, Lemma 2.25].

We need (also for later use) an elementary fact from group theory:

{F: finding g

Fact 4.30. *Let G be a group $A, B \subseteq G$ arbitrary subsets, $a \in A$, $b \in B$. Assume that*

$$a \cdot B = A \cdot B = A \cdot b.$$

Then there is a subgroup $H \leq G$ such that $A = aH$ and $B = Hb$.

{L:finding a

Lemma 4.31. *Let G be a definable group of finite dp-rank and let D be any definable set. Let $X_1, X_2 \subseteq G$, $f_i : X_i \to D^k$ and $h : X_1 \cdot X_2 \to D^p$ be generically m-to-one definable functions,* *with f_1, f_2, h defined over A. Assume, moreover, that no $X_i' \subseteq X_i$ of the same dp-rank is n-internal to D for any $n < m$.*

Then for every generic $(a, b) \in X_1 \times X_2$ over A, there is a finite subgroup $H(a, b) \subseteq G$ such that $[a]_{f_1} = a \cdot H(a, b)$ and $[b]_{f_2} = H(a, b) \cdot b$.

Proof. By the assumption on X_1, for any $B \supseteq A$, any B-definable function $F : X_1 \to D^q$ (any q) with finite fibres is, generically, at least m-to-one.

Fix $(a, b) \in X_1 \times X_2$ generic over A. Consider the function $F : X_1 \to D^{k+p}$ given by $x \mapsto (f_1(x), h(x \cdot b))$. Notice that $[x]_F = [x]_{f_1} \cap ([x \cdot b]_h \cdot b^{-1}))$, where $[x]_F = F^{-1}(F(x))$.

Since $[a]_F \subseteq [a]_{f_1}$, the assumption that $\text{dp-rk}(a/Ab) = \text{dp-rk}(X_1)$ and the minimality of m forces $|[a]_F| \geq m$, so it must be that, in fact, $[a]_F = [a]_{f_1}$. It follows that $[a]_{f_1} \subseteq [a \cdot b]_h \cdot b^{-1}$. By assumption $|[a \cdot b]_h| \leq m$, so necessarily $[a]_{f_1} = [a \cdot b]_h \cdot b^{-1}$, and so $[a]_{f_1} \cdot b = [a \cdot b]_h$. If $a' \in [a]_{f_1}$ then, as a, a' are interalgebraic, we have $\text{dp-rk}(a', b/A) = \text{dp-rk}(a, b/A)$, so we also get $[a']_{f_1} \cdot b = [a' \cdot b]_h$, but $[a']_{f_1} = [a]_{f_1}$ so $[a' \cdot b]_h = [a \cdot b]_h$. Since the roles of a and b are symmetric we conclude also that $[a \cdot b']_h = [a \cdot b]_h$ and so $[a]_{f_1} \cdot b' = [a \cdot b]_h$ for all $b' \in [b]_{f_2}$. Therefore

$$[a]_{f_1} \cdot b = [a]_{f_1} \cdot [b]_{f_2} = a \cdot [b]_{f_2}$$

and we may conclude the proof by Fact 4.30. □

For the main result of this section we need the following facts. The first was proved in the context of SW-uniformities in [8, Lemma 3.14]. It remains valid for vicinic structures as well:

utual generic}

Fact 4.32. *Let $\mathbb{U} \succ \mathcal{M}$ a monster model, $b_1, \dots, b_n$ some tuples in $\mathbb{U}$. For every $\mathcal{M}$-definable $X \subseteq D^r$ with finite fibres, there exists an M-generic $a \in X$ such that* $\text{dp-rk}(a, b_i/M) = \text{dp-rk}(a/M) + \text{dp-rk}(b_i/M)$ *for all* $1 \leq i \leq n$.

Proof. This was proved in [8] under the assumption that D is an SW-uniformity. However, the proof only uses the fact that, in the notation of the above statement, there is a finite-to-one projection $\pi : X \to M^k$, with $k = \text{dp-rk}(X)$, and an M-generic box in its image (i.e., a product of k dp-minimal subsets). In the vicinic setting this is implied by Axiom (A1) and Corollary 4.15. □

The second of the facts appearing in [8, Lemma 3.14] was proved for a single type. The same proof works for two (or more) types:

over a model}

Fact 4.33. *Let $\mathcal{M}$ be a structure of finite dp-rank and $\mathbb{U} \succ \mathcal{M}$ a monster model.*

For $A \subseteq \mathbb{U}$ and $a, b \in \mathcal{M}^n$, there exists a small model $A \subseteq \mathcal{N} \prec \mathcal{M}$, such that $\text{dp-rk}(a/A) = \text{dp-rk}(a/N)$ *and* $\text{dp-rk}(b/A) = \text{dp-rk}(a/N)$.

Proof. Let $\langle I_t : t < k_1 \rangle$ be mutually indiscernible sequences over A witnessing that $\text{dp-rk}(a/A) \geq k_1$, i.e. each I_t is not indiscernible over Aa, and let $\langle I_t' : t < k_2 \rangle$ be mutually indiscernible sequences over A witnessing that $\text{dp-rk}(b/A) \geq k_2$, i.e. each I_t' is not indiscernible over Ab. Let $\mathcal{M}'$ be some small model with $A \subseteq M'$.

By [25, Lemma 4.2], there exists a mutually indiscernible sequence $\langle J_t : t < k_1 + k_2 \rangle$ over M' such that

$$\text{tp}(\langle I_t : t < k_1 \rangle ^\frown \langle I_t' : t < k_2 \rangle / A) = \text{tp}(J_t : t < k_1 + k_2 / A).$$

Let σ be an automorphism of $\mathbb{U}$ fixing A and mapping the sequence of the J_t to the sequence of the I_t and the I_t'. It follows that $\langle I_t : t < k_1 \rangle$ are mutually indiscernible over $N := \sigma(M')$ and each one is still not indiscernible over Aa so not over Na as well; likewise $\langle I_t : t < k_1\rangle$ are mutually indiscernible over N an each one is not indiscernible over Na as well. □

{P: G/H s.i.}

Proposition 4.34. *Let G be an A-definable group of finite dp-rank locally almost strongly internal to a definable vicinic set D, and assume that G is an almost D-group.*

1. *If $X \subseteq G$ is an almost D-set over A then there exists a finite subgroup $H_X \leq G$, definable over A, and an A-definable subset $X' \subseteq X$, with $\text{dp-rk}(X') = \text{dp-rk}(X)$, such that X'/H_X is strongly internal to D, in particular, G/H_X is locally strongly internal to D.*
2. *The group H_X is normal and does not depend on the choice of X.*
3. *The D-critical rank and the almost D-critical rank of the quotient G/H agree and consequently G/H is a D-group.*

Proof. For simplicity assume that $A = \emptyset$.

(1) Let $X \subseteq G$ be an almost D-set, witnessed by $f : X \to D^p$ with fibres of size m. If $m = 1$ then we take $H = \{e\}$, the trivial subgroup, so assume this is not the case. Note that for any almost D-critical X', any definable function $g : X' \to D^n$ (defined over arbitrary parameters) and any generic $x' \in X$ we have $|[x']_g| \geq m$.

To any generic $(a, b) \in X \times X$ we associate a finite subgroup $H(a, b) \leq G$, such that generic fibres of f are both left and right cosets of $H(a, b)$, as follows: By Lemma 4.27 there is some parameter set B and respective B-generic vicinities X_1, X_2 of a and b in X, and $c \in X_2$ such that $X_1 \cdot X_2 \subseteq Xc$, and $\text{dp-rk}(a, b/Bc) = 2n$. Let h be the map on $X_1 \times X_2$, $z \mapsto f(z \cdot c^{-1})$. Then X_1, X_2 and $X_1 \cdot X_2$ are m-strongly internal to D, witnessed by $f_1 = f \restriction X_1$, $f_2 = f \restriction X_2$ and h, respectively. By minimality of m, $[a]_{f_1} = [a]_f$, $[b]_{f_2} = [b]_f$ and $[a \cdot b]_{h'} = [a \cdot b]_h$, and in addition (a, b) is generic in $X_1 \times X_2$ over Bc. So Lemma 4.31 provides us with a finite subgroup $H(a, b) \leq G$ such that $[a]_f = a \cdot H(a, b)$ and $[b]_f = H(a, b) \cdot b$.

We show that $H(a, b)$ does not depend on the choice of (a, b):

Claim 4.34.1. *For any $(a', b') \in X^2$ generic over A, $H(a, b) = H(a', b')$.*

Proof. By Fact 4.33 there exists a small model $\mathcal{N}$ such that everything we used up until now is defined over N, $\text{dp-rk}(a, b/N) = \text{dp-rk}(a, b)$ and $\text{dp-rk}(a', b'/N) = \text{dp-rk}(a', b')$. Using Fact 4.32 we can find $(c, d) \in X^2$ such that $\text{dp-rk}(a, b, c, d/N) = \text{dp-rk}(a', b', c, d/N) = 4n$. Thus,

$$H(a, b) = a^{-1} \cdot [a]_f = H(a, d) = [d]_f \cdot d^{-1} = H(a', d) = (a')^{-1} \cdot [a']_f = H(a', b'),$$

as needed. □ (claim)

Let $H_X := H(a, b)$. As, by the claim, $[a]_f = a \cdot H(a, b) = H(b, a) \cdot a$ we conclude that for all generic $a \in X$, $[a]_f = a \cdot H_X = H_X \cdot a$. As H_X is an invariant definable set (any automorphism preserves generic points), it is definable over A. Consequently, setting $X' = \{x \in X : [x]_f = H_X \cdot x = x \cdot H_X\}$ we have $\text{dp-rk}(X') = n$.

As $X' \subseteq X \subseteq G$ we may consider the image X'/H_X of X' under the natural quotient (viewing G/H_X as a G-space). Because, as we have just shown, $[x]_f = H_X x$ the function f induces on X'/H_X an injective function witnessing local D-strong internality of G/H_X.

(2) Assume that $X_0 \subseteq G$ is any other almost D-set. By Lemma 4.26, there exists $X_1 \subseteq X_0$, $\text{dp-rk}(X_0) = \text{dp-rk}(X)$, such that X_0 is contained in a translate gX. It is easy to see that $H_{gX} =$

H_X, but then $H_{X_0} = H_{X_1} = H_{gX} = H_X$. Thus H_X does not depend on the almost D-critical set X and on the function f. We denote it H.

Any conjugate X^g is also an almost D-set, thus $H_{X^g} = (H_X)^g = H$, so H is normal.

(3) Let $f : G \to G/H$ be the quotient map. Let $X \subseteq G$ be an almost D-critical set in G and $X' \subseteq X$ as in (1). Let $Y \subseteq G/H$ be an almost D-critical set in G/H. As H is finite, $f^{-1}(Y)$ is almost strongly internal to D, so $\text{dp-rk}(Y) = \text{dp-rk}(f^{-1}(Y)) \leq \text{dp-rk}(X) = \text{dp-rk}(X'/H)$.

On the other hand, the definable set X'/H is strongly internal so $\text{dp-rk}(X'/H) \leq \text{dp-rk}(Y)$ by the choice of Y. It follows that $\text{dp-rk}(X'/H) = \text{dp-rk}(Y)$, so the almost D-critical rank of G/H must equal its D-critical rank, and they are both equal to the almost D-critical rank of G.

It remains to show that G/H is a D-group (Definition 4.23). So let $X_1, X_2 \subseteq G/H$ with X_1 strongly internal to D and X_2 D-critical. To simplify notation, we assume that everything is definable over $\emptyset$. Let also $(g_1, g_2) \in X_1 \times X_2$ be generic. Since the quotient map $f : G \to G/H$ is a group homomorphism with finite fibres taking $g_i' \in f^{-1}(g_i)$ we get that $f^{-1}(X_i)$ is almost D-strongly internal (for $i = 1, 2$) and

$$\text{dp-rk}(g_1'/g_1' \cdot g_2')) = \text{dp-rk}(g_1/g_1 \cdot g_2).$$

Thus, since G is an almost D-group, to show that G/H is a D-group it will suffice to show that $X_2' := f^{-1}(X_2)$ is almost D-critical. By what we have just shown $\text{dp-rk}(X_2') = \text{dp-rk}(X_2)$ is the almost D-critical rank of G, and if $g : X_2 \to D$ witnesses that X_2 is D-strongly internal then $g \circ f$ witnesses that X_2' is $|H|$-internal to D. The construction of H assures that no subset of G of the same rank is m-internal to D for $m < |H|$ so that, indeed, X_2' is almost D-critical as needed. □

4.7. **Maříková's Method.** Definable functions in the distinguished sorts in our various settings are generically well- behaved. For example, when D is an SW-uniformity, definable functions are generically continuous, and in some contexts differentiable with respect to an underlying field. In the P-minimal setting, when $D = K/\mathcal{O}$ or Γ, we saw that definable functions are generically given by translates of endomorphisms. Our aim is to make use of this generically tame behaviour in order to show that if a group G is locally strongly internal to D then G has a (type) definable subgroup with similar properties (e.g. topological, differentiable, linear). The argument goes back to Weil's group-chunk theorem, first cast in a model theoretic setting independently by v.d. Dries [29] and Hrushovski [22, Theorem 4.13]. The specific technique used below is due to Maříková, [15] (this was similarly used in [8]).

For the rest of this subsection, let D be vicinic and let G be a definable D-group which is locally strongly internal to D.

g before Jana}

Remark 4.35. In the following proof we will frequently use some earlier results, for ease we collect them here.

Assume that $X, Y \subseteq G$ are D-sets over A and let $(g, h) \in X \times Y$ be A-generic. Then

(1) There exists $B' \supseteq A$ and a D-set Z over B' containing gh, such that $\text{dp-rk}(g, h/B') = 2\text{dp-rk}(X)$.
(2) For $Z \subseteq G$ a D-set over A and $f : X \times Y \to Z$ an A-definable function, if $f(g, h)$ is A-generic in Z, then for every A-generic vicinity $V \subseteq Z$ of $f(g, h)$ there is $B_2 \supseteq B$ and a B_2-generic vicinity of (g, h) of the form $X_1 \times Y_1 \subseteq X \times Y$ such that $f(X_1 \times Y_1) \subseteq V$.

For (1), let X', Y' and B be as provided by Lemma 4.27. Choose $h' \in Y'$ with $\text{dp-rk}(g, h, h'/B) = 3\text{dp-rk}(X)$ and apply Lemma 4.27 again to conclude that $Y' \cdot h'$ has the desired properties.

For (2), let $S \subseteq X \times Y$ be the A-definable set $\{(x, y) \in X \times Y : f(x, y) \in V\}$; by assumption $\text{dp-rk}(S) = 2\text{dp-rk}(X)$ so we may conclude by Corollary 4.15.

We make the following *ad hoc* definition:

Definition 4.36. Assume that $\bar{a} = (a_1, \ldots, a_n) \in X_1 \times \cdots \times X_n$ and

$$F = (F_1, \ldots, F_m) : X_1 \times \cdots \times X_n \to Y_1 \times \ldots \times Y_m$$

is an A-definable function.

We say that $\bar{a}$ is *sufficiently generic for* F over A if each F_i is a function of a sub-tuple $(x_{i_1}, \ldots, x_{i_r})$ of $(x_1, \ldots, x_n)$ and the corresponding sub-tuple of $\bar{a}$ is generic over A.

For example, if $\text{dp-rk}(a, b/A) = 2\text{dp-rk}(G)$ then the tuple (a, b, a) is sufficiently generic for the map $(x, y, z) \mapsto (xy, z)$.

The main result here is:

{L:Jana1}

Lemma 4.37. *Assume that $Y \subseteq G$ is a D-set over A, $d \in Y$ an A-generic point, and consider $F(x, y, z) = xy^{-1}z$ at (d, d, d). There is $B \supseteq A$, with $\text{dp-rk}(d/B) = \text{dp-rk}(d/A)$, and there are B-definable maps $\psi_1, \psi_2, \psi_3, \psi_4$ whose domain and range are D-sets over B, such that*

$$F \restriction \text{dom}(\psi_1) = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1,$$

$\text{Im}(\psi_i) \subseteq \text{dom}(\psi_{i+1})$*, and for every $i = 0, \ldots, 3$ (with $\psi_0 = id$) we have $\psi_i \circ \cdots \circ \psi_0(d, d, d)$ sufficiently generic for ψ_{i+1}, over B. In addition, we may choose $\text{dom}(\psi_1)$ to be of the form $(Y_0)^3$, where Y_0 is a B-generic vicinity of d.*

Proof. Assume that $\text{dp-rk}(Y) = n$. Fix $b \in Y$ such that $\text{dp-rk}(d, b/A) = 2n$. We first use an auxiliary variable w and write $G(w, x, y, z) = xy^{-1}z$ as a composition of the following four functions:

$$\varphi_1(w, x, y, z) = (w, wx, y^{-1}, z)\,;\ \varphi_2(w, x, y, z) = (w, xy, z)$$
$$\varphi_3(w, x, y) = (w^{-1}, xy)\,;\ \varphi_4(x, y) = xy.$$

A direct computation shows that $\varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1(w, x, y, z) = xy^{-1}z$, and we have

$$\varphi_1(b, d, d, d) = (b, bd, d^{-1}, d)\,;\ \varphi_2\varphi_1(b, d, d, d) = (b, b, d)\,;\ \varphi_3\varphi_2\varphi_1(b, d, d, d) = (b^{-1}, bd).$$

We now need to restrict the domain and range of the φ_i to appropriate D-sets so that $\text{Im}(\varphi_i) \subseteq \text{dom}(\varphi_{i+1})$, and for each $i = 0, \ldots, 3$, $\varphi_i \circ \cdots \circ \varphi_0(b, a, a, a)$ is sufficiently generic for φ_{i+1}, over the defining parameters.

In order to ensure that for each i, $\text{Im}(\varphi_i) \subseteq \text{dom}(\varphi_{i+1})$, we start from φ_4 and work backwards.

- By Remark 4.35 (1), there is $B_1 \supseteq A$ and a D-set Z over B_1, containing bd, such that $\text{dp-rk}(b, d/B_1) = 2n$.
- By Remark 4.35 (2), there is $B_2 \supseteq B_1$ and a B_2-generic vicinity $Y_1^{-1} \times Z_1 \subseteq Y^{-1} \times Z$, of (b^{-1}, bd) such that $Y_1^{-1} \cdot Z_1 \subseteq Y$.
- Again, by Remark 4.35 (2), there is $B_3 \supseteq B_2$ and B_3-generic vicinity $Y_2 \times Y_3 \subseteq Y \times Y$ of (b, d) such that $Y_2 \cdot Y_3 \subseteq Z_1$.

- Similarly, there is $B_4 \supseteq B_3$ and a B_4-generic vicinity $Z_1 \times Y_5^{-1} \subseteq Z \times Y^{-1}$ of (bd, d^{-1}) such that $Z_1 \cdot Y_5^{-1} \subseteq Y_2$.
- Finally, there is $B \supseteq B_4$ and a B-generic vicinity $Y_6 \times Y_7 \subseteq Y \times Y$ of (b, d) such that $Y_6 \cdot Y_7 \subseteq Z_1$. Furthermore, we may assume that $Y_6 \subseteq Y_1$.

We now restrict the φ_i to the appropriate domains and obtain:

$$\varphi_1 : Y_6 \times Y_7 \times Y_5 \times Y_3 \to Y_6 \times Z_1 \times Y_5^{-1} \times Y_3\,;\ \varphi_2 : Y_6 \times Z_1 \times Y_5^{-1} \times Y_3 \to Y_6 \times Y_2 \times Y_3,$$

$$\varphi_3 : Y_6 \times Y_2 \times Y_3 \to Y_6^{-1} \times Z_1\,;\ \varphi_4 : Y_6^{-1} \times Z_1 \to Y.$$

(for φ_4, we used the fact that $Y_6 \subseteq Y_1$).

The appropriate tuples are sufficiently generic for the φ_i, since all coordinate functions are $\emptyset$-definable, and we chose the Z_i and Y_j, so that the points remain generic in them.

We can now write $xy^{-1}z$ as $\psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1$, where

$$\psi_1(x, y, z) = \varphi_1(b, x, y, z)\,;\ \psi_2(x, y, z) = \varphi_2(b, x, y, z)\,;\ \psi_3(x, y) = \varphi_3(b, x, y)\,;\ \psi_4(x) = \varphi_4(b^{-1}, x).$$

It is easy to verify that (d, d, d) satisfies the requirements.

To obtain $\mathrm{dom}(\varphi_1)$ of the form Y_0^3, we may take $Y_0 = Y_7 \cap Y_5 \cap Y_3$. □

infint groups}

5. Infinitesimal groups

In the present section we develop the notion of infinitesimal subgroups for groups locally strongly internal to a vicinic sort D. This generalises analogous results from [8] in the context of SW-uniformities. Since in the present context we do not have an underlying topology we have to start by developing the notion of *infinitesimal vicinities*. The notation and terminology are intended to maintain the topological intuition.

As above $\mathcal{M}$ is a $|T|^+$-saturated structure, D a vicinic sort and G a definable D-group of finite dp-rank locally strongly internal to D.

5.1. Infinitesimal vicinities.

Definition 5.1. Let $Z \subseteq G$ be a D-set over A and $d \in Z$ an A-generic point. The *the infinitesimal vicinity of* d *in* Z, denoted $\nu_Z(d)$, is the partial type consisting of all B-generic vicinities of d in Z, as B varies over all small parameter subsets of $\mathcal{M}$.

We think of $\nu_Z(d)$ both as a collection of formulas and a set of realization of the partial type in some monster model.

containing A}

Remark 5.2. In the definition of $\nu_Z(d)$ there is no harm in restricting to B-generic vicinities for $B \supseteq A$. Indeed, if X is any B-generic vicinity of d in Z then by Lemma 4.13 there exists $C \supseteq A$ and a C-generic vicinity $X_1 \subseteq X$ of d such that $\text{dp-rk}(d/A) = \text{dp-rk}(d/C)$.

By Lemma 4.20 and Remark 5.2, we have:

m filter base}

Lemma 5.3. *The collection of definable sets* $\nu_Z(d)$ *is a filter base, namely if* $X, Y \in \nu_Z(d)$ *then there exists* $W \subseteq X \cap Y$ *in* $\nu_Z(d)$.

For the following recall that by Remark 4.19(2), a cartesian product of D-sets is a D-set.

product of nu}

Lemma 5.4. *If $Z_1, Z_2 \subseteq G$ are D-sets over A and $(d_1, d_2) \in Z_1 \times Z_2$ is A-generic then $\nu_{Z_1\times Z_2}(d_1, d_2) = \nu_{Z_1}(d_1) \times \nu_{Z_2}(d_2)$.*

Proof. That $\nu_{Z_1\times Z_2}(d_1, d_2) \dashv \nu_{Z_1}(d_1) \times \nu_{Z_2}(d_2)$ follows from Corollary 4.15.

For the other direction let $X_i \subseteq Z_i$ be B_i-generic vicinities of d_i in Z_i, for $i = 1, 2$. By Lemma 4.13, twice, we first find $B \supseteq A$ and a B-generic vicinity $X_1' \subseteq X_1$ of d_1 with $\text{dp-rk}(d_1, d_2/B) = \text{dp-rk}(d_1, d_2/A)$, and then $C \supseteq B$ and a C-generic vicinity $X_2' \subseteq X_2$ of d_2 with $\text{dp-rk}(d_1, d_2/C) = \text{dp-rk}(d_1, d_2/B)$. Thus $X_1' \times X_2'$ is a C-generic vicinity of (d_1, d_2) in $Z_1 \times Z_2$, as needed. ☐

The next lemma provides a substitute for generic continuity of definable functions:

{L:definable

Lemma 5.5. *Assume that G_1, G_2 are D-groups over A, $Z_i \subseteq G_i$ are D-sets for G_i over A ($i = 1, 2$), and $f : Z_1 \to Z_2$ is an A-definable function. If c is A-generic in Z_1 and $f(c)$ is A-generic in Z_2 then $f(\nu_{Z_1}(c)) \vdash \nu_{Z_2}(f(c))$.*

Proof. Let $Y \subseteq Z_2$ be a B-generic vicinity of $f(a)$ in Z_2. Because c is A-interalgebraic with some element of D^n (some n) we apply Lemma 4.13 to $f(c)$ to conclude that there is $C \supseteq A$ and a C-generic vicinity $Y' \subseteq Y$ of $f(c)$, such that $\text{dp-rk}(c, f(c)/A) = \text{dp-rk}(c, f(c)/C)$. Since $f(c) \in \text{dcl}(c)$, it follows from sub-additivity of dp-rank that: $\text{dp-rk}(c/A) \leq \text{dp-rk}(f(c)/C, c) + \text{dp-rk}(c/C) \leq \text{dp-rk}(c/A)$, hence $f^{-1}(Y')$ is a C-generic vicinity of c in Z_1.

It follows that $f(\nu_{Z_1}(c)) \vdash \nu_{Z_2}(f(c))$. ☐

As a final result, we show that the above definition of $\nu_D(G)$ agrees with the definition given in [8] when the definable set is strongly internal to an SW-uniformity.

{P: generic n

Proposition 5.6. *Assume that D is an SW-uniformity. Let $Z \subseteq G$ be a D-set over A and let $g : Z \to D^m$ be a definable injection witnessing it. For any A-generic $d \in Z$ the partial type $\nu_Z(d)$ is equivalent to*

$$\{g^{-1}(U) : U \subseteq D^m \text{ open } M\text{-definable containing } g(d)\}.$$

Proof. Let $g(d) \in U \subseteq D^m$ be open and definable over some $B \supseteq A$. By [8, Proposition 3.12] there is some $g(d) \in V \subseteq U$ open, definable over $C \supseteq A$ with $\text{dp-rk}(g(d)/C) = \text{dp-rk}(g(d)/A)$; thus $d \in g^{-1}(V)$.

For the other inclusion, let $Y \in \nu_Z(d)$ be B-definable for some $B \supseteq A$. Let $X := g(Y) \subseteq g(Z)$ be $B \supseteq A$ definable with $g(d) \in X$ and $\text{dp-rk}(d/B) = \text{dp-rk}(Z)$. By [8, Corollary 4.4], $g(d)$ is in the relative interior of X in $g(Z)$ so there is some open $U \subseteq D^m$ definable over B such that $g(d) \in U \cap X \subseteq g(Z)$. I.e., $g^{-1}(U \cap X) \in \nu_Z(d)$ and $g^{-1}(U \cap X) \subseteq Y$, as required. ☐

5.2. **Groups of infinitesimals vicinities.** We are finally ready to introduce infinitesimal subgroups, associated with D-subsets of definable groups. Recall that throughout G is a D-group. First, we show that the infinitesimal vicinities constructed in the previous section are cosets of a type-definable subgroup:

{L:generic vi

Lemma 5.7. *If $X, Y \subseteq G$ are D-sets over A and $(c, d) \in X \times Y$ is A-generic then*

$$c \cdot \nu_Y(d) = \nu_X(c) \cdot \nu_Y(d) = \nu_X(c) \cdot d.$$

In particular, $\nu_X(c)$ and $\nu_Y(d)$ are right and left cosets, respectively, of the same group: $\nu_Y(d)d^{-1} = c^{-1}\nu_X(c)$.

Proof. Let $n = \text{dp-rk}(X) = \text{dp-rk}(Y)$. By Remark 4.35, there are $B \supseteq A$, and $X_1 \times Y_1 \subseteq X \times Y$, a B-generic vicinity of (c, d), and $Z \subseteq G$, a D-set over B containing $c \cdot d$ such that $X_1 \cdot Y_1 \subseteq Z$ and $\text{dp-rk}(c, d/B) = 2n$. We fix Z, and by Lemma 5.3 we may assume that $X = X_1$, $Y = Y_1$. For simplicity assume that $A = B = \emptyset$.

Let us see that $c \cdot \nu_Y(d) = \nu_Z(cd)$:

The function $y \mapsto c \cdot y$ takes Y into Z, thus, since d is generic in Y over c and cd is generic in Z over c, it follows from Lemma 5.5 that $c \cdot \nu_Y(d) \subseteq \nu_Z(cd)$. The map $z \mapsto c^{-1}z$ takes $cY \subseteq Z$ into Y. It follows from Lemma 5.5 again that $c^{-1} \cdot \nu_{cY}(cd) \subseteq \nu_Y(c)$. However, $cY \subseteq Z$ is a c-generic vicinity of cd in Z and we have $\nu_{cY}(cd) = \nu_Z(cd)$, so $c^{-1}\nu_Z(cd) \subseteq \nu_Y(d)$, and we have $c \cdot \nu_Y(d) = \nu_Z(cd)$. Similarly, we have $\nu_X(c) \cdot d = \nu_Z(cd)$, hence we can conclude

$$c \cdot \nu_Y(d) = \nu_Z(cd) = \nu_X(c) \cdot d.$$

Now consider the definable function $(x, y) \mapsto x \cdot y$. By Lemma 5.5, it maps $\nu_X(c) \times \nu_Y(d)$ into $\nu_Z(cd)$, i.e. $\nu_X(c) \cdot \nu_Y(d) \subseteq \nu_z(cd)$. By the above we conclude that $\nu_X(c) \cdot \nu_Y(d) = \nu_Z(cd)$, and thus

$$c \cdot \nu_Y(d) = \nu_X(c) \cdot \nu_Y(d) = \nu_X(c) \cdot d.$$

That both $\nu_X(c)$ and $\nu_Y(d)$ are cosets of the same group now follows from Fact 4.30. □

We can finally prove the main results concerning infinitesimal groups in D-sets.

{P: nu}

Proposition 5.8. *Let G be a D-group locally strongly internal to D.*

(1) *Assume that $X \subseteq G$ is a D-set over A, then for every A-generic $a, b \in X$, we have $\nu_X(a)a^{-1} = \nu_X(b)b^{-1} = a^{-1}\nu_X(a)$. Call this group ν_X.*
(2) *If $X, Y \subseteq G$ are D-sets over A then $\nu_X = \nu_Y$, and we can call it $\nu = \nu_D(G)$.*
(3) *For any $g \in G(M)$ we have $g\nu g^{-1} = \nu$.*

Proof. Let $n = \text{dp-rk}(X)$.

(1) By Fact 4.32, we find $c \in X$ such that $\text{dp-rk}(a, c/A) = \text{dp-rk}(b, c/A) = 2n$. Applying Lemma 5.7 to (a, b) and (a, c) we have $\nu_X(a)a^{-1} = c^{-1}\nu_X(c) = \nu_X(b)b^{-1}$, so the group $\nu_X(a)a^{-1}$ does not depend on the choice of an A-generic point in X. Similarly, the group $a^{-1}\nu_X(a)$ does not depend on the choice of a.

Applying this now to a and c with $\text{dp-rk}(a, c/A) = 2n$. we see that $\nu_X(a)a^{-1} = c^{-1}\nu_X(c) = a^{-1}\nu_X(a)$, and we may now denote this group by ν_X.

(2) It follows from Lemma 5.7 that if X, Y are two D-sets then $\nu_X = \nu_Y$.

(3) Let $g \in G(M)$ and X be any D-set; say over some A with $A \ni g$. Let $d \in X$ be such that $\text{dp-rk}(d/A) = n$, so $\text{dp-rk}(g \cdot d/A) = n$ as well. The function $x \mapsto gx$ sends X to gX, that is also a D-set, and by Lemma 5.5, it sends $\nu_X(d)$ to $\nu_{gX}(gd)$. Hence,

$$g\nu g^{-1} = g\nu_X(d)d^{-1}g^{-1} = \nu_{gX}(gd)(gd)^{-1} = \nu,$$

as needed. □

5.3. The case of D locally linear.

Definition 5.9. A vicinic definable set D is *locally linear* if it expands an abelian group $(D,+)$ and for any parameter set A, A-definable partial function $f : D^n \to D$ and A-generic $a \in \operatorname{dom}(f)$, there exists $B \supseteq A$, a B-generic vicinity X of a in $\operatorname{dom}(f)$ and a definable endomorphism $\lambda : D^n \to D$ such that $f \restriction X = (\lambda(x-a) + f(a)) \restriction X$.

{E:examples o

Fact 5.10. *The following are locally linear:*

1. *The sort $K/\mathcal{O}$, where K is either a V-minimal or power bounded T-convex valued field is locally linear-see* [8, Section 6.3, Proposition 6.16]. *Recall that in this setting, $K/\mathcal{O}$ is an SW-uniformity.*
2. *If D is either a pure ordered abelian group or a pure ordered vectors space over an ordered field then it is locally linear (see* [4, Corollary 1.10]*, and* [32, Chapter 1, Corollary 7.6]*). In particular, the value group Γ is locally linear in all of the above valued fields.*
3. *The sort $K/\mathcal{O}$, where K is a p-adically closed field, is locally linear (by Corollary 3.34).*

{P:infini_vic

Proposition 5.11. *Let D be a locally linear vicinic definable set and G a definable D-group, locally strongly internal to D. Then there exists a definable group isomorphism between the associated type definable $\nu \subseteq G$ and a type-definable subgroup of $(D^r,+)$ for some natural number r.*

Proof. Fix $X \subseteq G$ a D-set over some parameter set A and $d \in X$ generic over A. Without loss of generality, $X \subseteq D^r$ has minimal fibres. Write $F(x,y,z) = xy^{-1}z$ around (d,d,d) as a composition of four maps, as provided by Lemma 4.37, i.e. we find $B \supseteq A$, $Y_0 \subseteq D$ a B-generic vicinity of d and B-definable maps $\psi_1,\dots,\psi_4$ such that $F \restriction (Y_0)^3 = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1$ and $\psi_i \circ \cdots \circ \psi_1(d,d,d)$ is sufficiently generic for ψ_{i+1}. Therefore ψ_{i+1} is a translate of a definable endomorphism in a vicinity of $\psi_i \circ \cdots \circ \psi_1(d,d,d)$. We can thus find a C-generic vicinity $Y_1 \subseteq Y_0$ of d such that F takes values in X and equals on Y_1^3 to $\lambda(x-d, y-d, z-d) + d$ for some $(D,+)$-endomorphism λ.

Since $d = F(x,d,d) = \lambda(x-d,0,0) + d$, we have $\lambda(x-d,0,0) = x-d$ for any $x \in Y_1$. Similarly, $\lambda(0,0,z-d) = z-d$ for any $z \in Y_1$. Since $F(x,x,d) = d$ it follows that $\lambda(0,y-d,0) = d-y$ for any $y \in Y_1$. As λ is additive we conclude that $\lambda(x-d,y-d,z-d) = x-y+z-d$ on Y_1; thus

$$F(x,y,z) = xy^{-1}z = (x-d) - (y-d) + (z-d) + d = x - y + z.$$

Since, by Proposition 5.8, $\nu_X(d)$ is a coset of a subgroup of G it is closed under F; therefore it is also closed under $x-y+z$. It follows that $\nu_X(d)$ is a coset of a subgroup of $(D^r,+)$. The function $x \mapsto (x \cdot_G d) -_D d$. is an isomorphism of the groups $\nu_X(d) \cdot d^{-1}$ and $\nu_X(d) - d$. Hence, $\nu_D(G)$ is isomorphic to a type definable subgroup of $\langle D^r, + \rangle$. □

A similar proof gives the following:

{P:def group

Proposition 5.12. *Let $\mathcal{K} = (K, v, \dots)$ be an expansion of valued field that is either V-minimal, power-bounded T-convex or p-adically closed and set $D = K/\mathcal{O}$. Let G be a definable D-group. Then there is a definable subgroup $G_1 \leq G$ that is D-critical and definably isomorphic to a definable subgroup of $(K/\mathcal{O})^r$, for some natural number r.*

Proof. By Fact 5.10, D is a locally linear vicinic sort. We start as in Proposition 5.11: fix some B-generic d is some B-definable D-set $Y_0 \subseteq G$ such that after identifying Y_0 with a subset of D^r, $xy^{-1}z$ coincides with $x - y + z - d$ on $(Y_0)^3$. We identify Y_0 with a subset of D^r

We claim that Y_0 may be taken to be of the form $d + H$ for some definable subgroup H of $(K/\mathcal{O})^r$. If K is p-adically closed, this is Lemma 3.32. When D is an SW-uniformity (i.e. when $\mathcal{K}$ is V-minimal or power-bounded T-convex) we proceed as follows:

By Fact 4.33, there is exists $\mathcal{K}_0 \prec \mathcal{K}$, containing C, with $\text{dp-rk}(d/K_0) = \text{dp-rk}(d/C)$. By [26, Proposition 4.6] (and using Lemma 4.37 again) we may further assume that $r = \text{dp-rk}(d/K_0)$. By [8, Proposition 3.12], by passing to a definable subset, we may assume that Y_0 is a C'-generic vicinity of d in D^r for some $C' \supseteq \mathcal{K}_0$. Since d is in the interior of Y_0 there exists a ball around 0, $B_0 \subseteq D^r$ such that $d + B_0 \subseteq Y_0$; so take $H = B_0$ (and thus $Y_0 = d + H$).

As a result $(Y_0)^3$ is closed under the function $x - y + z$ hence also under the function $xy^{-1}z$. It follows that $Y_0 d^{-1}$ is a subgroup, $G_1 \leq G$, and hence $Y_0 = G_1 d = H + d$, and the function $x \mapsto (x \cdot_G d) -_D d$ is a group isomorphism between G_1 and H. □

5.4. **The case of D an SW-uniformity.** Although some of the results in this section can probably be proved in a higher level of generality, we restrict to the case where D is an SW-uniformity. Recall that every SW-uniformity is vicinic and that every definable group locally (almost) strongly internal to D is an (almost) D-group; consequently the results of the previous sections hold in this setting.

Let D be an SW-uniformity, definable in $\mathcal{M}$. By Proposition 5.6, we may use the definition of the infinitesimal group ν as given in [8], i.e., the intersection of all definable open neighbourhoods of 0.

u-topological}

Proposition 5.13. *Let G be a definable group, $Y \subseteq G$ a D-critical subset, witnessed by a function $f : Y \to D^n$, all definable over some A. Fix some A-generic $c \in Y$ and $\widehat{\mathcal{M}} \succ \mathcal{M}$, an $|M|^+$-saturated model.*

(1) *The weak topology induced by $x \mapsto f(x \cdot c)$ on $\nu(\widehat{\mathcal{M}})$ turns it into a topological group.*
(2) *The topology on ν obtained in the previous clause does not depend on the choice of the D-critical set Y, the function f, or the choice of the point c.*
(3) (i) *For every $g \in G(\widehat{\mathcal{M}})$ there is an open $V \subseteq \nu(\widehat{\mathcal{M}})$ such that $V \subseteq \nu(\widehat{\mathcal{M}})^g \cap \nu(\widehat{\mathcal{M}})$.*
    (ii) *For every $g \in G(\widehat{\mathcal{M}})$, the function $x \mapsto x^g = gxg^{-1}$, from $\nu(\widehat{\mathcal{M}})$ into $\nu(\widehat{\mathcal{M}})^g$, is continuous at e with respect to topology on $\nu(\widehat{\mathcal{M}})$.*

Proof. (1) By Fact 4.33, we may assume that A is a model so by [26, Proposition 4.6] we may further assume that $\text{dp-rk}(Y) = n$ and that $Y \subseteq D^n$ is open. Since in SW-uniformities every definable function is generically continuous ([26, Proposition 3.7]), and continuity is preserved under composition, we can, as in Proposition 5.11, find $C \supseteq A$ with $\text{dp-rk}(c/C) = \text{dp-rk}(c/A)$ and a C-definable open subset $Y_0 \subseteq Y$ containing c such that $F(x, y, z) = xy^{-1}z$ takes values in Y and is continuous on $(Y_0)^3$. We have $\nu = \nu_Y(c)c^{-1}$, and thus the pullback, under the map $x \mapsto x \cdot c$, of the topology on Y endows ν with a topology. It is a group topology since $(x, y) \mapsto xc^{-1}yc^{-1}c = xc^{-1}y$ is continuous and so is $x \mapsto (xc^{-1})^{-1}c = cx^{-1}c$.

(2) The injection $f : Y \to D^n$ endows Y with a definable topology and by [8, Lemma 4.6], this topology at every A-generic point of Y does not depend on f, call it τ_Y. Thus, we may assume that $Y \subseteq D^n$.

Now, given an A-generic c in Y, the above construction endows $\nu(\widehat{\mathcal{M}})$ with a group topology, call it $\tau_{Y,c}$, for which, by definition, the map $x \mapsto xc$ is a homeomorphism between $(\nu(\widehat{\mathcal{M}}), \tau_{Y,c})$ and $(\nu_Y(c)(\widehat{\mathcal{M}}), \tau_Y)$.

Similarly, if X is any other D-critical set, with d an A-generic in X then the map $x \mapsto xd$ endows $\nu(\widehat{\mathcal{M}})$ with a group topology $\tau_{X,d}$ which is homeomorphic, via $x \mapsto xd$, to $(\nu_X(d)(\widehat{\mathcal{M}}), \tau_X)$. By replacing A by a small model $\mathcal{N}$ containing it (see [8, Lemma A.1]), and applying Fact 4.32, we may assume that $\text{dp-rk}(c, d/N) = \text{dp-rk}(X \times Y)$.

Thus, the map $x \mapsto x \cdot (c^{-1}d)$ is a bijection of $\nu_Y(d)$ and $\nu_X(c)$, defined over $Nc^{-1}d$. Since c is generic in X over $Nc^{-1}d$ (by Lemma 4.22), the map is a homeomorphism of $(\nu_Y(c)(\widehat{\mathcal{M}}), \tau_Y)$ and $(\nu_X(d)(\widehat{\mathcal{M}}), \tau_X)$. This shows that $\tau_{Y,c} = \tau_{X,d}$.

(3) Let Y be a D-critical set over A, witnessed by some $f : Y \to D^n$ and assume $e \in Y$. Recall that

$$\nu = \bigcap \{V : V \in \tau_{Y,f} \text{ with } e \in V\}.$$

By Proposition 5.8(3), ν is invariant under conjugation by $g \in G(\mathcal{M})$, thus by compactness, for every $U \in \tau_{Y,f}$ with $e \in U$ there exists $V \in \tau_{Y,f}$ with $e \in V$ such that $gVg^{-1} \subseteq U$. Since this statement is first order it also holds of any $g \in G(\widehat{M})$ and U an $\widehat{M}$-definable open basic open set containing 0. This gives (i) and (ii). ☐

Using these result we may endow G with a definable group topology. {C: G locally

Corollary 5.14. *Let G be a definable group locally strongly internal to D.*

1. *The group G has a definable basis for a topology, making G a non-discrete Hausdorff topological group.*
2. *If G is dp-minimal then it is an SW-uniformity.*

Proof. (1) Apply Proposition 5.13(3) with Lemma 2.8 to conclude that $G(\widehat{\mathcal{M}})$ has a uniformly definable basis of neighbourhoods of the identity $e \in G(\widehat{\mathcal{M}})$, for $\widehat{\mathcal{M}} \succ \mathcal{M}$ an $|M|^+$-saturated extension. Since this is first order and G is definable, the same holds for $G = G(M)$. Since D is an SW-uniformity it has no infinite definable discrete sets, so – in particular – the topology on ν, and therefore also on G is non-discrete.

(2) Since the topology is a non-discrete group topology it has no isolated points and is automatically uniform by construction. So we only have to check that every definable infinite subset of G contains a definable open set.

Let $S \subseteq G$ be a definable infinite set and let $Y \subseteq G$ be D-critical set. By Lemma 4.26, there exists $g \in G$ with $S \cap gY$ infinite. Since gY is strongly internal to D and D is an SW-uniformity, there exists a definable open subset of $S \cap gY \subseteq S$, as required. ☐

6. THE CASE OF A STRONGLY MINIMAL D

{S: lsi to k}

We consider in this section the case where some infinite definable subset of a definable group G is almost strongly internal to a strongly minimal set.

We fix a sufficiently saturated (possibly multi-sorted) structure $\mathcal{M}$ and a definable strongly minimal set F. We assume further that F is stably embedded and eliminates imaginaries.

We observe:

for asi to k}

Lemma 6.1. *Assume that G is a definable group in $\mathcal{M}$ of finite dp-rank. If $X_1, X_2 \subseteq G$ are (almost) F-critical then so is $X_1 \cdot X_2$.*

Proof. Since X_1 and X_2 are both (almost) strongly internal to F, so is $X_1 \times X_2$, and because F eliminates imaginaries, it follows from Lemma 4.3, that $X_1 \cdot X_2$ is also (almost) F-strongly internal. Since $\text{dp-rk}(X_1 \cdot X_2) \geq \text{dp-rk}(X_1)$, if X_1, X_2 are (almost) F-critical then equality of the ranks follows. □

Recall that in a any strongly minimal definable set, the dp-rank and Morley rank coincide. Furthermore, in the following we will repeatedly use the fact that if $f : X \to Y$ is a finite-to-one definable functions then $\text{RM}(f(X)) = \text{RM}(X)$.

{P: si to sm}

Proposition 6.2. *Let G be a definable group of finite dp-rank in $\mathcal{M}$, locally almost strongly internal to F. Then there exist a definable normal subgroup $H \trianglelefteq G$ and a definable finite normal $H_0 \trianglelefteq H$, such that*

(1) *H is almost F-critical.*
(2) *H/H_0 is strongly internal to F.*

Proof. Let $Y \subseteq G$ be almost F-critical with $\text{dp-rk}(Y) = n$. Because F is strongly minimal this implies that $\text{RM}(Y) = n$ and by replacing Y with a subset, we may assume that its Morley degree is 1. Note that the Morley rank of subsets of Y is definable in parameters, since Y is almost strongly internal to a strongly minimal definable set and so the latter eliminates $\exists^\infty$ and Morley rank is given by acl-dimension.

Define a relation E on G:

$$g \; E \; h \iff \text{RM}(Yg \cap Yh) = n.$$

Notice that $\text{RM}(Yg \cap Yh) = \text{RM}(Y \cap Yhg^{-1})$, and hence it follows from the definability of RM that E is definable (one can also use the fact that the generic type of Y is definable). Since $\text{DM}(Y) = 1$, E is an equivalence relation on G, which is moreover right-invariant under G. Furthermore, the E-class of the identity element e, call it H, is closed under group inverse. Thus, H is a definable subgroup of G. Notice that if $Yg \cap Y \neq \emptyset$ then $g \in Y^{-1}Y$, hence $H \subseteq Y^{-1}Y$.

Claim 6.2.1. *H is almost strongly internal to F, $\text{RM}(H) = n$ and H is connected and normal in G.*

Proof. By Lemma 6.1, $Y^{-1}Y$ is almost strongly internal to F so has finite Morley rank. By the maximality assumption on $\text{RM}(Y)$, we have $\text{RM}(Y^{-1}Y) = n$ (it is clear that $n = \text{RM}(Y) \leq \text{RM}(Y^{-1}Y)$), hence $\text{RM}(H) \leq n$. We similarly have $\text{RM}(Y \cdot Y) = n$.

To see that $\text{RM}(H) = n$, we let $k = \text{DM}(Y \cdot Y)$. Notice first that $g_1 \; E \; g_2$ if and only if $Hg_1 = Hg_2$. Now, for $g_1, g_2 \in Y$, we have $\text{RM}(Yg_1) = \text{RM}(Yg_2) = n$. If $\neg(g_1 \; E \; g_2)$, then $\text{RM}(Yg_1 \cap Yg_2) < n$, but since the Morley degree of YY is k, there can be at most k different E-classes intersecting Y, so Y is covered by at most k-many right cosets of H. It follows that there is some $g_0 \in Y$ such that $\text{RM}(Hg_0 \cap Y) = n$. In particular, $\text{RM}(H) = n$. The group H is connected because $\mathrm{DM}(Y) = 1$. Indeed, if H_1 was a subgroup of finite index of H then one of H-cosets of H_1 will contain Yg_0^{-1} up to a set of smaller rank, contradicting the definition of H.

It is left to see that H is normal in G. Indeed, if $H^g \neq H$ for some $g \in G$ then, since H is connected, it follows that $[H : H^g \cap H]$ is infinite, and therefore the set $H^g H$, which by Lemma 6.1is almost strongly internal to F, contains infinitely many pairwise disjoint right cosets of H^g. It follows that $\mathrm{RM}(H^g H) > n$, contradicting the maximality of n. □ (claim)

It is left to find the desired finite H_0. We have so far shown that H is almost strongly internal to a stably embedded set F, thus by the work of Hrushovski-Rideau, [12, Lemma 2.25], there exists a finite normal subgroup $H_0 \leq H$ such that H/H_0 is internal to F. As F eliminates imaginaries, H/H_0 is strongly internal to F. □

7. The main theorems

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In this section we apply the results obtained in the previous sections to study groups interpretable in some dp-minimal valued fields. We start with a lemma on definable groups in a slightly more general context: we show that groups *definable* in a dp-minimal field, $\mathcal{K}$, of characteristic 0 have unbounded exponent, under the additional assumption that definable functions are generically differentiable (e.g., if $\mathcal{K}$ is 1-h-minimal).

The idea for the proof of the next lemma is of S. Starchenko:

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Lemma 7.1. *Let $\mathcal{K}$ be a sufficiently saturated SW-uniform structure expanding a field of characteristic 0 with generic differentiability.*

(1) *If G is an infinite interpretable group locally strongly internal to K then G is a D-group for $D = K$ and the associated type-definable subgroup $\nu_D(G)$ is torsion-free.*
(2) *If G is locally almost strongly internal to K then G has unbounded exponent.*

Proof. Since K is an SW-uniformity it is vicinic (Fact 4.6), and thus if G is almost locally strongly internal to $D = K$ it is an almost D-group (Lemma 2.6 and Corollary 4.22). By Proposition 4.34 there is a finite normal subgroup $H \leq G$ such that G/H is locally strongly internal to K. If G/H has unbounded exponent then so does G. So it suffices to prove the former. Hence, replacing G with, G/H Clause (2) follows from Clause (1).

For (1), since K is an SW-uniformity and G is locally strongly internal to K, it is a D-group (Fact 4.25(1)). Let $\nu := \nu_K(G)$ as provided by Proposition 5.8, $\widehat{\mathcal{K}} \succ \mathcal{K}$ some $|\mathcal{K}|^+$-saturated extension. As in [8, Proposition 4.19], we endow $\nu(\widehat{\mathcal{K}})$ with a differential group structure that we identify with a type-definable group in $K^m(\widehat{\mathcal{K}})$ for some integer m.

Let $\lambda(x, y)$ be the multiplication map on $\nu := \nu(\widehat{\mathcal{K}})$. It follows easily from the chain rule that the differential of λ at (e, e), as a map from the tangent space at (e, e) of $\nu \times \nu$ into $T_e(\nu)$, is $x + y$. Thus, for any $n \in \mathbb{N}$, the K-differential of the n-fold multiplication $\lambda_n(x)$ mapping x to $\lambda(x, \lambda(x, \lambda(\dots \lambda(x, x) \dots)$ equals to nId, where Id is the identity (matrix or differential). Since $\mathrm{char}(K) = 0$, nId is an invertible linear map. By the very definition of the derivative, this means that there exists an open neighbourhood U of e, such that $\lambda_n(x) = x^n \neq e$ for all $x \in U$. Since it is a first-order property, we can take U to be $\mathcal{K}$-definable. As $\nu(G)$ is the intersection of all $\mathcal{K}$-definable open neighbourhoods of e, $x^n \neq e$ for all n and any $x \in \nu(\widehat{\mathcal{K}})$, as claimed. □

7.1. **Power bounded T-convex valued fields.** Let $\mathcal{K} = (K, v, \dots)$ be a T-convex valued field expanding a power bounded o-minimal field which we assume to be one sorted. By naming one constant $c \notin \mathcal{O}$ we may assume that $\mathcal{K}$ has definable Skolem functions. We assume that $\mathcal{K}$ is sufficiently saturated.

We first need the following results.

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Fact 7.2. *Assume that $\mathcal{M}$ is a structure such that every dp-minimal group interpretable in $\mathcal{M}$ has unbounded exponent then every such group is abelian by finite.*

Proof. By [23, Proposition 3.1] a dp-minimal group G has a normal abelian subgroup H such that G/H has bounded exponent. Our assumption implies that G/H is finite, with the desired conclusion. □

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Lemma 7.3. *Every infinite set interpretable in $\mathcal{K}$ is locally strongly internal to one of the distinguished sorts.*

Proof. Let $G = X/E$ be an infinite interpretable set, where $X \subseteq K^n$. By [8, Lemma 5.10 and Proposition 5.5] there exists an infinite definable subset $S \subseteq G$ in definable bijection with an infinite definable subset of K/E', and a definable finite-to-finite correspondence between S and one of K, $K/\mathcal{O}$, $\mathbf{k}$ or Γ. By weak o-minimality each E'-equivalence class is a finite union of convex sets. By replacing E' with the equivalence relation E'', choosing the first component in each E'-class, we may assume that E' is a a convex equivalence relation. Thus S is linearly ordered (and weakly o-minimal), in particular it eliminates finite imaginaries in the sense of [8, Section 4.7]. As each of the sorts K, $K/\mathcal{O}$, $\mathbf{k}$ or Γ is an SW-uniformity, the result follows by [8, Lemma 4.28]. □

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Theorem 7.4. *Let G be an infinite group interpretable in $\mathcal{K}$ and $\widehat{\mathcal{K}}$ a $|\mathcal{K}|^+$-saturated elementary extension. Then G has unbounded exponent and a type-definable infinite subgroup ν satisfying one of the following:*

1. *ν is definably isomorphic to an infinite type-definable group in $\mathcal{K}$ and $\nu(\widehat{\mathcal{K}})$ carries the structure of a differential group with respect to K.*
2. *ν is definably isomorphic to an infinite type-definable group in $\mathbf{k}$ and $\nu(\widehat{\mathcal{K}})$ carries the structure of a differential group with respect to $\mathbf{k}$.*
3. *ν is definably isomorphic to a type-definable subgroup of $(\Gamma^r, +)$ for some integer r.*
4. *There exists an infinite definable subgroup $H \leq G$ definably isomorphic to a subgroup of $((K/\mathcal{O})^r, +)$ for some integer r.*

Moreover, if G is dp-minimal then G is abelian-by-finite.

Proof. Let G be in an infinite interpretable group. By Lemma 7.3, G is locally strongly internal to D where D is one of the distinguished sorts. By Fact 4.6, every such D is vicinic and by Fact 4.25, G is a D-group.

If $D = K$ or $D = \mathbf{k}$ then let $\nu := \nu_K(G)$ (or $\nu_{\mathbf{k}}(G)$). As in [8, Proposition 4.19], we may endow $\nu(\widehat{\mathcal{K}})$ with a structure of a differential group. By Lemma 7.1, G has unbounded exponent. This shows (1) and (2).

If $D = \Gamma$ then D is locally linear by Fact 5.10 and [31, Theorem B] so (3) follows from Proposition 5.11. In this case G has unbounded exponent since every subgroup of Γ is torsion-free.

If $D = K/\mathcal{O}$ then (4) follows by Proposition 5.12. In this case G has unbounded exponent since obviously every subgroup of $K/\mathcal{O}$ is torsion-free.

Finally, since as we have just shown, every infinite group interpretable in $\mathcal{K}$ has unbounded exponent, if G is dp-minimal then G is abelian-by-finite by Fact 7.2. □

As we shall see later (Corollary 7.16), if G is dp-minimal then exactly one of $(1),(2),(3),(4)$ holds.

7.2. **V-minimal valued fields.** Let $\mathcal{K} = (K, v, \dots)$ be a sufficiently saturated V-minimal valued field. We need the following result, appearing implicitly in [8]:

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Lemma 7.5. *Assume that X and Y are infinite sets definable in some $|T|^+$-saturated (multi-sorted) structure $\mathcal{M}$ and $C \subseteq X \times Y$ a definable finite-to-finite definable correspondence. If Y is either a field or supports an SW-uniform structure then X is locally almost strongly internal to Y.*

Proof. It is not hard to show (see the first part of the proof of [8, Lemma 4.28] for the details) that if Y eliminates finite imaginaries (in the sense of [8, Section 4.7]) then any finite-to-finite correspondence $C \subseteq X \times Y$ gives rise to a definable finite-to-one function $f : X' \to Y$ for some infinite definable $X' \subseteq X$. This gives the desired conclusion if Y is a field.

Assume now that Y supports an SW-uniform structure. For $x \in X$ and $y \in Y$ denote

$$C_x = \{y \in Y : (x, y) \in C\},\ C^y = \{x \in X : (x, y) \in C\}$$

and note that by $\aleph_0$-saturation, $|C_x|$ is uniformly bounded.

Assume everything is definable over some parameters set A and let $d \in Y$ with $\text{dp-rk}(d/A) = \text{dp-rk}(Y) = 1$. Since C^d is finite so is $(C^d)^{-1}$, where $(C^y)^{-1} := \{y \in Y : y \in C_x,\ x \in C^y\}$.

Since the topology on the SW-uniformity is Hausdorff, we can find a relatively open subset of Y, $U \ni d$ with $(C^d)^{-1} \cap U = \{d\}$. By [8, Proposition 3.12] there is some $B \supseteq A$ and a B-definable open neighbourhood $U_0 \subseteq U$ with $d \in U_0$ and $\text{dp-rk}(d/B) = \text{dp-rk}(d/A) = 1$.

Let $Y' = \{y \in U_0 : |(C^y)^{-1} \cap U_0| = \{y\}\}$; it is C-definable and as $d \in Y'$, it has dp-rank 1.

Consider $C' = \{(x, y) \in X \times Y' : (x, y) \in C\}$. It is still a finite-to-finite correspondence and we claim that for any $x \in X$, $|C'_x| = 1$. Indeed, if $y \in C'_x$ then by definition $y \in (C^y)^{-1}$ so it follows that $|C'_x| = 1$ by the definition of Y'.

It follows that C' gives a finite-to-one definable map between X and Y'. □

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Corollary 7.6. *Every infinite set interpretable in $\mathcal{K}$ is locally almost strongly internal to one of the distinguished sorts.*

Proof. Let $G = X/E$ be an infinite intrepretable set, where $X \subseteq K^n$. By [8, Proposition 5.6 and Proposition 5.5] there exists an infinite definable subset $S \subseteq G$, in definable bijection with an infinite definable subset of K/E', and a definable finite-to-finite correspondence between S and and either K, $K/\mathcal{O}$, **k** or Γ. Since each of these sorts is either a field or an SW-uniformity we can conclude by Lemma 7.5. □

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Theorem 7.7. *Let G be an infinite group interpretable in $\mathcal{K}$, and $\widehat{\mathcal{K}}$ a $|\mathcal{K}|^+$-saturated elementary extension. Then G has unbounded exponent and there exists a finite normal subgroup $H \trianglelefteq G$ such that G/H has a type-definable infinite subgroup ν satisfying one of the following:*

(1) *ν is definably isomorphic to a type-definable subgroup in $\mathcal{K}$ and $\nu(\widehat{\mathcal{K}})$ carries the structure of a differential group with respect to K.*
(2) *There exists an infinite definable subgroup $H' \leq G/H$ definably isomorphic to a **k**-definable. In particular, H' is a **k**-algebraic group.*
(3) *ν is definably isomorphic to a type-definable subgroup of $(\Gamma^r, +)$ for some integer r.*
(4) *There exists an infinite definable subgroup $H' \leq G/H$ definably isomorphic to a subgroup of $((K/\mathcal{O})^r, +)$ for some integer r.*

Moreover, if G is dp-minimal then G is abelian-by-finite.

Proof. Let G be in an infinite interpretable group. By Corollary 7.6, G is locally almost strongly internal to D where D is one of the distinguished sorts.

Assume first that $D = \mathbf{k}$. By Proposition 6.2, there exists definable subgroups $H \trianglelefteq H_1 \trianglelefteq G$, where H is finite and H_1/H is strongly internal to **k**. Since **k** is a stably embedded algebraically closed field, H_1/H is **k**-algebraic ([21]) and we take $H' = H_1/H$.

It is well known that every algebraic group over an algebraically closed field of characteristic 0 has unbounded exponent. This finishes the proof in case $D = \mathbf{k}$.

We assume that G is locally strongly internal to one of Γ, K or $K/\mathcal{O}$. By Fact 4.6, each of those is vicinic and by Fact 4.25, G is an almost D-group (for a suitable D). Thus, there are $H \trianglelefteq G$, a finite normal subgroup as provided by Proposition 4.34 and ν the type definable subgroup of G/H as in Proposition 5.8.

Since Γ is a pure ordered vector space it is locally linear. From this point the proof of Theorem 7.4 goes through verbatim for the above three sorts.

Finally, if G is dp-minimal then by what we have just shown and Fact 7.2, G is abelian-by-finite. □

As we shall see later (Corollary 7.16), if G is dp-minimal exactly one of $(1), (2), (3), (4)$ holds.

Remark 7.8. It may be worth pointing out that Theorem 7.7 is wrong in $\mathrm{ACVF}_{p,p}$: interpretable groups (and even definable ones) need not have unbounded exponent, and as shown by Simonetta [27] dp-minimal such groups need not be abelian-by-finite. Our methods do carry us a long way in $\mathrm{ACVF}_{p,p}$ (as well as in $\mathrm{ACVF}_{0,p}$), and the failure of our main results in this case seems local. More explicitly, Lemma 7.5 holds in any C-minimal valued field, and so do [8, Proposition 5.5, Proposition 5.6]. Since all the distinguished sorts in the C-minimal case are either fields or SW-uniformities, Corollary 7.6 goes through to assure that and group interpretable in $\mathrm{ACVF}_{p,p}$ or $\mathrm{ACVF}_{0,p}$ is almost locally strongly internal to one of the distinguished sorts. Thus, interpretable groups are almost D-groups for D a vicinic sort – and our construction of the infinitesimal group $\nu_D(G)$ goes through unaltered. Thus, it seems that clauses (1)-(4) of Theorem 7.7 could still be true, if – in case $D = K/\mathcal{O}$ – we require only that ν be type-definable in $K/\mathcal{O}$ (not necessarily a subgroup). The reason the for this change is that we do not know that $K/\mathcal{O}$ is locally linear in this setting.

An interesting question we leave open is, therefore:

Question 7.9. Let $\mathcal{K}$ be a saturated enough algebraically closed valued field or, more generally, a pure dp-minimal valued field. Let G be a dp-minimal group interpretable in $\mathcal{K}$. If G is locally almost strongly internal to K, is G abelian-by-finite?

7.3. **p-adically closed valued fields.** Let $\mathcal{K} = (K, v, \dots)$ be a sufficiently saturated p-adically closed valued field (or a model of $\mathbb{Q}_p^{an}$). As we noted before, it has definable Skolem functions.

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Lemma 7.10. *Every infinite set interpretable in $\mathcal{K}$ is locally almost strongly internal to one of the sorts $K, K/\mathcal{O}$ and Γ.*

Proof. Assume that X/E is a definable quotient in $\mathcal{K}$. We first apply [8, Proposition 5.5] (noting that $\mathcal{K}$ satisfies the necessary assumptions, see [8, Proposition 5.8]), and conclude that there are an infinite $T \subseteq X/E$, a distinguished sort D, an infinite $D' \subseteq D$ and a definable finite-to-finite correspondence $C \subseteq T \times D'$.

When $D = K$, it is an SW uniformity and the result follows from Lemma 7.5.

When $D = \Gamma$, then, since Γ is linearly ordered, the correspondence gives rise to a finite-to-one function from T into Γ, as needed.

When $D = K/\mathcal{O}$, the second projection map $\pi_2 : C \to K/\mathcal{O}$ proves that C is almost strongly internal to $K/\mathcal{O}$. We now consider the first projection $\pi_1 : C \to T \subseteq X/E$. By Lemma 3.9 (2), there exists an infinite subset of T which is almost strongly internal to $K/\mathcal{O}$. □

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Theorem 7.11. *Let G be an infinite group interpretable in $\mathcal{K}$ and $\widehat{\mathcal{K}}$ a $|\mathcal{K}|^+$-saturated elementary extension. Then G has unbounded exponent and there exists a finite normal subgroup $H \trianglelefteq G$ such that G/H has a type-definable infinite subgroup ν satisfying one of the following:*

1. *ν is definably isomorphic to a type-definable subgroup in $\mathcal{K}$ and $\nu(\widehat{\mathcal{K}})$ carries the structure of a differential group with resepct to K.*
2. *ν is definably isomorphic to a type-definable subgroup of $(\Gamma^r, +)$ for some integer r.*
3. *There exists an infinite definable subgroup $H' \leq G/H$ definably isomorphic to a subgroup of $((K/\mathcal{O})^r, +)$ for some integer r.*

Moreover, if G is dp-minimal then G is abelian-by-finite.

Proof. Let G be in an infinite interpretable group. By Lemma 7.10, G is locally almost strongly internal one of the distinguished sorts, D. By Fact 4.6 every such D is vicinic and by Fact4.25 G is an almost D-group. Let $H \trianglelefteq G$ be a finite normal subgroup as provided by Proposition 4.34 and let ν be the type definable subgroup of G/H as in Proposition 5.8.

If $D = K$ then the proof is as in Theorem 7.4(1). If $D = \Gamma$ then the result follows by Proposition 5.11 once we recall that Γ is a model of Presburger arithmetic and therefore by Fact 5.10 is locally linear. Therefore G has unbounded exponent since it has a type-definable subgroup isomorphic to a type-definable subgroup of Γ^r (some r) and every subgroup of Γ^r has unbounded exponent. This covers Clause (2).

If $D = K/\mathcal{O}$ then the result follows by Proposition 5.12 since $K/\mathcal{O}$ is locally linear by Fact 5.10. To show that G has unbounded exponent it will suffice to note that any infinite subgroup of $(K/\mathcal{O})^r$ has unbounded exponent: This follows from Fact 3.1(3). This covers Clause (3).

Finally, if G is dp-minimal then as in previous cases, G is abelian-by-finite by Fact 7.2. □

As we shall see later (Corollary 7.16), if G is dp-minimal then exactly one of $(1), (2), (3)$ holds.

7.4. **Summary of proof.** The combination of Theorem 7.4, Theorem 7.7 and Theorem 7.11 imply Theorem 1.1 and Theorem 1.2, as stated in the introduction. We summarize briefly, without repeating the references, the strategy implemented in the proof of these results.

Given an interpretable group G, we first find an an infinite definable $X \subseteq G$ that is almost strongly internal to one of the distinguished sorts, D. Namely, G is locally almost strongly internal to D, and show that G is an almost D-group.

We then find a finite normal $H \leq G$ such that G/H is a D-group, and fix $X \subseteq G/H$ strongly internal to D of maximal dp-rank, which is a D-set. Using the fact that G/H is a D-group, we show that for $d \in X$ generic, the partial type $\nu_X(d) \vdash G/H$ is a right coset of the desired group $\nu_D(G/H)$. Thus, the type definable subgroup we obtain lives in a quotient of our original G by a finite normal subgroup.

It is possible that from the start, G contains a definable infinite $X \subseteq G$ which is *strongly internal* to D. In this case, we can directly replace X with a critical D-set, which we still call X.

However, in order to apply our machinery we need to show that G is D-group (see Fact 4.25). When $\mathcal{K}$ is either V-minimal or power bounded T-convex, then indeed G is a D-group for all D, and thus the type definable ν is a subgroup of G itself. The same is true if $\mathcal{K}$ is P-minimal and D is either K or Γ. However, when $\mathcal{K}$ is P-minimal and $D = K/\mathcal{O}$ then Fact 4.25 implies only that $K/\mathcal{O}$ is locally *almost* strongly internal to $K/\mathcal{O}$ and we are still forced to mod G by a finite normal H, and thus ν is a subgroup of G/H.

7.5. **The distinguished sorts are foreign.** Using our main theorems we can prove a certain orthogonality result for the distinguished sorts. More precisely:

Definition 7.12. Let $\mathcal{M}$ be any structure. Two $\mathcal{M}$-definable sets D_1, D_2 are *foreign* if there is no definable finite-to-finite correspondence $C \subseteq X \times Y$, where $X \subseteq D^n$ and $Y \subseteq D_2^m$ are definable infinite subsets.

We leave to the reader the proof of the following easy observation:

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Lemma 7.13. *If there exists a definable finite-to-finite correspondence between infinite definable subsets of D_1^n and D_2^m then there also exists one between infinite definable subsets of D_1 and D_2.*

We assume now that $\mathcal{K}$ is either power bounded T-convex, V-minimal or P-minimal.

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Proposition 7.14. *Any two distinct distinguished sorts in $\mathcal{K}$ are foreign.*

Proof. Most of the cases could have been proved earlier using more elementary methods, but we find it to be a good application of our results here.

First, assume that D_1, D_2 are sets that are not foreign, namely (applying Lemma 7.13) there exists with a definable finite-to-finite correspondence between respective infinite subsets thereof. We repeatedly use [8, Lemma 4.28], stating that if D_1 eliminates finite imaginaries (EfI for short) and D_2 either has (EfI) or supports an SW-uniformity then D_1 is locally strongly internal to D_2. We shall use the fact that expansions of fields have (EfI).

K and **k** are foreign: Note first that if $\mathcal{K}$ is P-minimal then there is nothing to prove (**k** is finite), so we assume we are not in this case.

By the above, K is locally strongly internal to **k**. By [8, Theorem 4.21], in the power-bounded T-convex case, or [8, Theorem 4.24], in the V-minimal case, K is definably isomorphic to **k**. But K is a valued field and **k** is not: it is strongly minimal in the V-minimal case and o-minimal in the T-convex case.

K and **k** are foreign to Γ: Since Γ has definable choice in all settings, it eliminates imaginaries so by the above, if the sorts were not foreign we would get K (respectively, **k**) locally strongly internal to Γ, contradicting [8, Proposition 6.29].

K and **k** are foreign to $K/\mathcal{O}$: In the P-minimal case, we only need to check that K and $K/\mathcal{O}$ are foreign, which follows from [8, Proposition 6.29].

In the remaining cases, $K/\mathcal{O}$ is an SW-uniformity and K, **k** are fields, so satisfy (Efi), hence by [8, Proposition 4.28] any correspondence between $K/\mathcal{O}$ and K or **k** such two implies that K, **k** are locally strongly internal to $K/\mathcal{O}$. This contradicts [8, Proposition 6.24].

Γ and $K/\mathcal{O}$ are foreign: Assume towards contradiction that this is not the case. We first claim that Γ is locally strongly internal to $K/\mathcal{O}$.

Indeed, Γ eliminates (finite) imaginaries and in the V-minimal and T-convex cases, $K/\mathcal{O}$ is an SW-uniformity, so by [8, Lemma 4.28], Γ is locally strongly internal to $K/\mathcal{O}$ in these cases. In the P-minimal case, we apply Lemma 3.9 (with $X = K/\mathcal{O}$ and $T = \Gamma$), and conclude that Γ is locally almost strongly internal to $K/\mathcal{O}$. However, since Γ is ordered it follows that Γ is in fact locally strongly internal to $K/\mathcal{O}$.

Since Γ is an interpretable group, we may apply Proposition 5.12 and conclude that there exists a definable infinite subgroup $G_1 \leq \Gamma$, which is definably isomorphic to a definable subgroup of $(K/\mathcal{O})^r$, for some integer r.

In $(K/\mathcal{O})^r$, every infinite definable subgroup has many infinite definable proper subgroups (intersection with balls). However, in the V-minimal and T-convex power bounded cases, Γ is an ordered vector space thus has no infinite definable subgroups other than itself, contradiction.

In the P-minimal case,[3] Γ is torsion-free while every definable subgroup of $(K/\mathcal{O})^r$ has torsion (Lemma 3.10(2)), leading also to a contradiction. □

Question 7.15. Note that a definable quotient of a distinguished sort D by a definable equivalence relation with infinitely many infinite classes can be foreign to D itself, and thus it is locally (almost) strongly internal to one of the other sorts. In fact, the sorts $K/\mathcal{O}$, Γ, and **k** are all quotients of K, or some subset of K, by such an equivalence relation.

By repeatedly taking appropriate quotients one can alternate between local strong internality to two different sorts: Consider $G = K/\mathbf{m}$ (locally strongly internal to **k**), and the definable subgroups $r\mathcal{O}$ and $s\mathcal{O}$ for $r, s \in K$ such that $v(r) < v(s) < 0$. Then $(r\mathcal{O}/\mathbf{m})/(s\mathcal{O}/\mathbf{m}) \cong r\mathcal{O}/s\mathcal{O} \cong (r/s)\mathcal{O}/\mathcal{O}$ (all isomorphisms definable), with the latter definably isomorphic to a ball in $K/\mathcal{O}$ (so obviously locally strongly internal to $K/\mathcal{O}$). Every ball in $K/\mathcal{O}$ has a quotient definably isomorphic to a subgroup of $K/\mathbf{m}$ thus this quotient is locally strongly internal to **k**, and we can choose the subgroups along the way so this process will go on indefinitely.

It is interesting to ask which of the distinguished sorts may appear in such a sequence of quotients.

Proposition 7.14 allows us to show that dp-minimal groups interpretable in $\mathcal{K}$ are pure in the following sense:

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[3] A direct proof, which does not make use of our work here, and is based on [3, Proposition 3.1], was suggested to us by P. Cubides Kovacsics.

Corollary 7.16. *If G is interpretable in $\mathcal{K}$ and $X_1, X_2 \subseteq G$ are almost strongly internal to foreign sorts D_1, D_2, respectively, then* $\text{dp-rk}(G) \geq \text{dp-rk}(X_1) + \text{dp-rk}(X_2)$. *In particular, if* $\text{dp-rk}(G) = 1$ *then G can be locally almost strongly internal to at most one distinguished sort.*

Proof. Assume toward contradiction that $\text{dp-rk}(G) < \text{dp-rk}(X_1) + \text{dp-rk}(X_2)$, and consider the function $f : X_1 \times X_2 \to G$, defined by $f(x_1, x_2) = x_1 \cdot x_2$.

We have $\text{dp-rk}(X_1 \times X_2) = \text{dp-rk}(X_1) + \text{dp-rk}(X_2)$, thus by the dp-rank assumption, f cannot be everywhere finite-to-one. Hence there is some $g \in G$ such that $f^{-1}(g)$ is infinite. But, by its definition, $f^{-1}(g) \subseteq X_1 \times X_2$ is the graph of a bijection between (infinite) subsets of X_1 and X_2. This gives rise to a finite-to-finite correspondence between infinite subsets of D_1 and D_2, contradiction. □

Fact 7.14 and Corollary 7.16 raise interesting questions about the possible dp-rank of subsets of G that are almost strongly internal to the distinguished sorts. For example:

Question 7.17. Are there always definable $X_1, \dots, X_4 \subseteq G$ (some possibly empty), with X_i almost strongly internal to $K, \mathbf{k}, \Gamma, K/\mathcal{O}$, respectively, such that $\text{dp-rk}(G) = \sum_{i=1}^{4} \text{dp-rk}(X_i)$?

We end with an example of a dp-minimal valued field where the distinguished sorts are not foreign.

Example 7.18. Let $\mathcal{R}$ a sufficiently saturated extension of $\mathbb{R}_{\exp}$ and let $\mathcal{K}$ be $\mathcal{R}$ expanded by a predicate for the convex hull of $\mathbb{Z}$, which we denote by $\mathcal{O}$. So $\mathcal{K}$ an exponential T-convex valued field and, therefore, dp-minimal ($\mathcal{R}$ is o-minimal and $\mathcal{O}$ is externally definable). We claim that $K/\mathcal{O}$ is strongly internal to Γ.

We first note that $\exp(\mathcal{O}) = \mathcal{O}_{>0} \setminus \mathbf{m}$. Indeed, for the right-to-left, since $\log$ is a $\emptyset$-definable continuous function, if $x \in \mathcal{O}_{>0}$ then $\log(x) \in \mathcal{O}$. For the other direction, assume for a contradiction that $a = \exp(b) \in \mathbf{m}$ for some $b \in \mathcal{O}$. Then $a^{-1} = \exp(-b) \notin \mathcal{O}$, contradicting T-convexity.

Thus (and as $\mathcal{O}^\times = \mathcal{O} \setminus \mathbf{m}$), $\exp$ induces map $E : K/\mathcal{O} \to K^\times/\mathcal{O}^\times$ given by $E(x + \mathcal{O}) := \exp(x) + \mathcal{O}^\times$. It is easy to check that E is a homomorphism of (ordered) groups. It is injective because $\exp$ is.

8. Examples

{S: examples}

We end by studying some examples of interpretable groups in the valued fields we considered and see how our results are reflected in those examples. The examples are, mostly, common to all contexts, but their nature may vary between the different settings.

Let $\mathcal{K} = (K, v, \dots)$ be some expansion of a valued field.

Example 8.1 ($K/\mathbf{m}$)**.** If $\mathcal{K}$ is V-minimal or power bounded T-convex then as $\mathbf{m}$ is an additive subgroup of K, $K/\mathbf{m}$ is an infinite interpretable group. Since $\mathcal{O} \subseteq K$, $(\mathbf{k}, +)$ is a subgroup of $K/\mathbf{m}$. I.e. $K/\mathbf{m}$ is locally strongly internal to $\mathbf{k}$. If $\mathcal{K}$ is P-minimal then $K/\mathbf{m} \cong K/\mathcal{O}$.

Example 8.2 ($K/\mathcal{O} \rtimes \mathcal{O}^\times$)**.** Multiplication defines an action of $\mathcal{O}^\times$ on $K/\mathcal{O}$ by automorphisms and $K/\mathcal{O} \rtimes \mathcal{O}^\times$ is a solvable group of class 2 since $G' \leq K/\mathcal{O}$ (so, in particular, it is not abelian-by-finite). Its dp-rank is 2 (the universe of the group being $K/\mathcal{O} \times \mathcal{O}^\times$). It is locally strongly internal to both K and $K/\mathcal{O}$: the subgroup $0 \times \mathcal{O}^\times$ witnesses the former and $(K/\mathcal{O}, 1)$ the latter. The admissible dimension (in the sense of [13]) is 1 (since it is the admissible dimension of $K/\mathcal{O}\times\mathcal{O}^\times$). This shows that Theorem 1.1 does not extend to definable groups of admissible dimension 1.

Example 8.3 (RV_γ)**.** (1) Assume first that (K, v) is p-adically closed and identify the standard part of Γ with $\mathbb{Z}$. Since in p-adically closed fields there are definable angular component maps $\mathrm{ac}_n : K^\times \to \mathcal{O}/\mathbf{m}_{n-1}$, for any (standard) natural n, we can identify Γ with the definable subset of RV_{n-1} defined by $\mathrm{ac}_n(x) = 1$. Thus, RV_n is locally strongly internal to Γ.

For $\gamma \in \Gamma$ non-standard (i.e., $\gamma > n$ for all $n \in \mathbb{N}$) the picture is different. Since $\gamma > \mathbb{Z}$ we have $|\mathcal{O}^\times/\mathbf{m}_{n-1}| \leq |\mathcal{O}^\times/\mathbf{m}_\gamma|$ for all standard $n \in \mathbb{N}$ and as the left-hand side is unbounded with n it follows that $\mathcal{O}^\times/\mathbf{m}_\gamma$ is infinite. The map $a + \mathbf{m}_\gamma \mapsto a(1 + \mathbf{m}_\gamma)$ is a definable injection from $\mathcal{O}^\times/\mathbf{m}_\gamma$ into RV_γ. Since Γ is discrete $\mathbf{m}_\gamma$ is a closed ball (of valuative radius $\gamma + 1$), so $\mathcal{O}^\times/\mathbf{m}_\gamma$ is in definable bijection with a subset of $K/\mathcal{O}$ and thus locally strongly internal to $K/\mathcal{O}$.

In fact, if γ is non-standard, and $\delta \in \Gamma$ is such that $2\delta > \gamma$ and such that $\gamma - \delta > \mathbb{Z}$. Then $(1 + \mathbf{m}_\delta)/(1 + \mathbf{m}_\gamma)$ is an infinite subgroup of RV_γ definably isomorphic to the additive group $\mathbf{m}_\delta/\mathbf{m}_\gamma$. Indeed, the definable map $a + \mathbf{m}_\gamma \mapsto (1 + a)(1 + \mathbf{m}_\gamma)$ for $a \in \mathbf{m}_\delta$ from $\mathbf{m}_\delta/\mathbf{m}_\gamma$ to $(1 + \mathbf{m}_\delta)/(1 + \mathbf{m}_\gamma)$ is a bijective group homomorphism.

(2) Now assume that (K, v) is power-bounded T-convex or V-minimal (actually Γ dense and **k** infinite is sufficient).

Consider the definable subset $\mathcal{O}^\times/(1 + \mathbf{m}_\gamma)$ of RV_γ. We claim that there is a definable injection from **k** into $\mathcal{O}^\times/(1 + \mathbf{m}_\gamma)$; thus RV_γ is locally strongly internal to **k**. Indeed, pick some $t \in K$ with $v(t) = \gamma$, thus $a + \mathbf{m}_\gamma \mapsto at^{-1}(1 + \mathbf{m}_\gamma)$ (with $v(a) = \gamma$) is an injection of $\mathcal{O}_\gamma/\mathbf{m}_\gamma$ (where $\mathcal{O}_\gamma = \{x : v(x) \geq \gamma\}$) into $\mathcal{O}^\times/(1 + \mathbf{m}_\gamma)$; finally note that $\mathcal{O}_\gamma/\mathbf{m}_\gamma \cong \mathbf{k}$.

Example 8.4. The next example shows, in the power bounded T-convex case, the necessity for the group $\nu \subseteq G$ to be type-definable (rather than definable). It is similar to examples from [28].

We consider the two o-minimal structures, Γ and **k** and fix a positive $\gamma_0 \in \Gamma$. We start with $H = \Gamma \times \mathbf{k}$ and in it consider the group Λ generated by $\langle\gamma_0, 1\rangle$.

We let $S = \Gamma \times [0, 1) \subseteq H$, and on S define the operation

$$(x_1, y_1) \oplus (x_2, y_2) = \begin{cases} (x_1 + x_2, y_1 + y_2) & y_1 + y_2 < 1 \\ (x_1 + x_1 - \gamma_0, y_1 + y_2 - 1) & y_1 + y_2 \geq 1 \end{cases}$$

This is a group operation which makes $G = (S, \oplus)$ an interpretable group isomorphic to the quotient of the group $\langle S\rangle \subseteq H$ by the subgroup Λ.

The group G has one definable subgroup $\Gamma \times 0$ isomorphic to Γ. However, since Γ and **k** are foreign, the only definable subsets of G which are (almost) strongly internal to **k** are subsets of $\{\gamma\} \times [0, 1)$, $\gamma \in \Gamma$, or finite unions of such. It is not hard to see that no finite union of such sets gives rise to a **definable** subgroup of G. On the other hand, G has a type definable subgroup ν which is definably isomorphic to the infinitesimals in $(\mathbf{k}, +)$.

Appendix A. Endowing an infinitesimal group in RCVF with a linear order

Here we show that if $\mathcal{K}$ is power bounded T-convex, G is a dp-minimal group interpretable in $\mathcal{K}$ then $\nu(G)$, the associated infinitesimal subgroup, is an ordered group with respect to the induced ordering:

Proposition A.1. *Let G be a dp-minimal interpretable group in a power bounded T-convex structure, $\mathcal{K}$ and assume that it is locally strongly internal to one of the distinguished sorts D. Then the group ν provided by Theorem 7.11 is ordered with respect to the order induced from its embedding into D.*

Proof. Assume that $\mathcal{K}$ is sufficiently saturated. We shall be brief. Since $K/\mathcal{O}$ and Γ are ordered groups and ν is a subgroup, then we have nothing to prove. We prove the result in case D is K. The proof translates verbatim to the case where $D = \mathbf{k}$ since it uses only weak o-minimality of K and [31, Corollary 2.8] asserting that any definable function $f : K \to K$ in a T-convex structure is piece-wise monotone. By o-minimality, this is also true in $\mathbf{k}$.

The following is a simple corollary of the piece-wise monotonicity of definable functions.

Claim A.1.1. *Assume that $f : K \to K$ is a definable, continuous, open and injective partial function with open domain. Then f is locally strictly monotone at every point.*

Our goal is to prove: If $\nu \subseteq K$ is the type-definable infinitesimal neighbourhood of $e \in K$, endowed with the K-ordering and a $\mathcal{K}$-definable topological group operation then left multiplication is order preserving (by symmetry, the same is true for right multiplication).

Let $e \in \nu$ be the identity element and $\lambda(x, y)$ a $\emptyset$-definable function whose restriction to ν is the group multiplication. We may assume, by compactness, that λ is defined and continuous on $U \times U$ for some K-definable open U, $\nu \vdash U$, which satisfies:

- For all $x \in U$ the function $\lambda_x(y) := \lambda(x, y)$ is an injective open map. In addition, $\lambda(x, e) = \lambda(e, x) = x$ for all $x \in U$.
  We may further find a definable open $V \subseteq U$, $\nu \vdash V$, such that:
- For all $x \in V$ there exists (a unique) $y \in U$ such that $\lambda(x, y) = \lambda(y, x) = e$. By abuse of notation we let x^{-1} denote this y.
- $V = V^{-1}$, $V \cdot V \cdot V \subseteq U$ and λ is associative on V.

Absorbing parameters into the language, assume that U and V above are $\emptyset$-definable.

By the above claim, for every $g \in U$ the function λ_g is locally (strictly) monotone at every point. Let $\widehat{\mathcal{K}} \succ \mathcal{K}$ be an $|K|^+$-saturated extension. We first show that for every $g \in \nu(\widehat{\mathcal{K}})$, the function λ_g is *locally* strictly increasing at e.

Let W^+ (resp. W^-) be the set of $g \in U(\widehat{\mathcal{K}})$ such that λ_g is locally strictly increasing (reps. decreasing) at e. Both sets are $\emptyset$-definable, hence $\nu^+ := \nu \cap (e, \infty)$, which – by weak o-minimality – is a complete type over $\emptyset$, is concentrated on one of W^+ and W^-, and the same for $\nu^- := \nu \cap (-\infty, e)$. We claim that both are concentrated on W^+, and hence $\nu \vdash W^+$ (clearly, $e \in W^+$).

Indeed, assume towards a contradiction that, say, $\nu^- \vdash W^-$, and fix any $a \in \nu^-(\widehat{\mathcal{K}})$ (in particular, $\text{dp-rk}(a/M) = 1$). Since $a \in W^-$, the function λ_a is strictly decreasing on some open interval $J \ni e$. As $ae = a$ belongs to the open set ν^-, it follows by continuity that there exists $b \models \nu^- \cap J$

sufficiently close to e, such that $ab \in \nu^-$. But then, $\lambda_{ab} = \lambda_a \circ \lambda_b$, λ_b is decreasing at e and λ_a is decreasing at b so λ_{ab} is increasing at e, contradicting the fact that $ab \in \nu^-(\widehat{\mathcal{K}}) \subseteq W^-$.

The cases where $\nu^+ \vdash W^-$ leads to a contradiction in the same way. We may therefore conclude that for every $g \in \nu(\widehat{\mathcal{K}})$, λ_g is locally increasing at e. By compactness, we may assume that $U \subseteq W^+$.

Fix $g \in V(\mathcal{K})$ (where associativity holds) such that $\text{dp-rk}(g) = 1$. The function λ_g is continuous at e, $\lambda_g(e) = g$, hence $\lambda_g(\nu) = \nu(g)$. Furthermore, since $g \in K$, by our assumptions, λ_g is strictly increasing on some $\mathcal{K}$-definable open interval $I \ni e$. By 2.5 we may choose I to be A-definable, $A \subseteq M$, such that $\text{dp-rk}(g/A) = 1$. It follows that for all $c \models \nu(g)$, λ_c is strictly increasing on I, so in particular on $\nu(\widehat{\mathcal{K}})$.

Fix any $h \in \nu$, we need to see that $x \mapsto hx$ is increasing on ν. We write $h = \lambda_g^{-1}(c)$, for some $c \in \nu(g)$. Now, for $x \in \nu(\widehat{\mathcal{K}})$, we have

$$h \cdot x = \lambda_g^{-1}(c) \cdot x = \lambda_{g^{-1}}(c) \cdot x = \lambda_{g^{-1}}(\lambda_c(x)),$$

where the right equality follows from the associativity on V. Now, if $x < y \in \nu(\widehat{\mathcal{K}})$ then $\lambda_c(x) < \lambda_c(y)$ and hence (because λ_g and its inverse are increasing on ν), we have

$$h \cdot x = \lambda_g^{-1}(\lambda_c(x)) < \lambda_g^{-1}(\lambda_c(y)) = h \cdot y.$$

It follows that multiplication is order preserving. □

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