O-minimal structures-basics and some applications

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11th Panhellenic Logic Symposium at Delphi

Map of the universe (thanks to Gabriel Conant)

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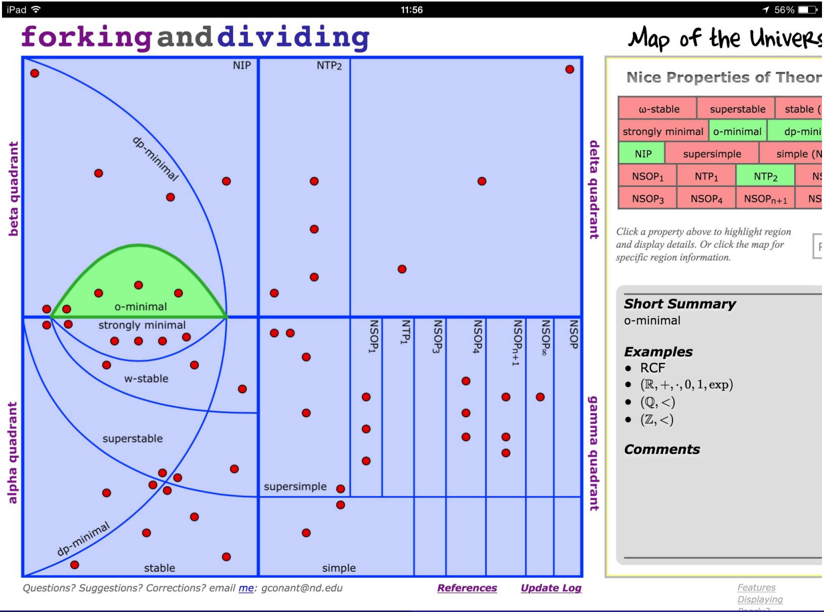


The genrel plan

Talk I

Definitions, examples and a variety of basic results about o-minimal structures. Including some proofs.

Talk II

O-minimality and Diophantine geometry.

O-minimal structures

Definition

A linearly ordered structure $\mathcal{M} = \langle M; <, \cdots \rangle$, is o-minimal if every definable subset of M (with parameters) is of the form

$$\bigcup_{i=1}^{k} (a_i, b_i) \cup \underbrace{F}_{\text{finite}}$$

with $a_i, b_i \in M \cup \{\pm\infty\}$.

Negative Examples

- $\langle \mathbb{Z}; <, + \rangle$ (why? $2\mathbb{Z}$)
- $\langle \mathbb{R}; <, +, \cdot, \sin x \rangle$ (why? the set $\{x : \sin x = 0\}$).
- $\langle \mathbb{Q}; <, \cdot \rangle$ (why? The set $\{x : x^2 < 2\}$)
- $\langle \gamma; < \rangle$, γ an ordinal $\geqslant \omega^2$ (**Exercise**).

Main examples of o-minimal structures

For simplicity we restrict to dense linear orderings.

- $\langle D; <\rangle$ a dense linear ordering (by quantifier elimination).
- $\langle G; <, +\rangle$ an ordered divisible abelian group (by quantifier elimination).
- $\langle R; <, +, \cdot\rangle$ a real closed field (for example, the field $\mathbb{R}$). **why?** By QE (Tarski) and the fact that every polynomial has finitely many roots.
- $\mathbb{R}_{an} = \langle \mathbb{R}; <, +, \cdot, f \restriction [-1, 1]^n\rangle$, for f real analytic (v.d.Dries, based on Gabrielov).
- $\mathbb{R}_{exp} = \langle \mathbb{R}; <, +, \cdot, e^x\rangle$ (Wilkie).
- $\mathbb{R}_{an,exp} = \langle \mathbb{R}_{an}, e^x\rangle$ (vdD-Miller). Most of recent applications of o-minimality take place in $\mathbb{R}_{an,exp}$.

Basic properties of o-minimal structures

We assume that $\mathcal{M} = \langle M, <, \cdots \rangle$ is o-minimal.

Observations

- An ordered reduct of an o-minimal structure is still o-minimal.
- **Definable completeness:** If $X \subseteq M$ is definable and bounded below then X has a maximal lower bound.
- Every definable **infinite** $X \subseteq M$ contains an open interval.
- Every definable **unbounded** $X \subseteq M$ contains a ray (a, ∞) or $(-\infty, a)$.

Namely, assume that

$$\mathcal{M} \models \forall x \exists y (y > x \wedge \phi(y, \bar{a})).$$

Then

$$\mathcal{M} \models \exists x \forall y (y > x \rightarrow \phi(y, \bar{a})).$$

Definable Choice (Skolem functions)

Assume that $\mathcal{M} = \langle M, <, +, 1, \cdots \rangle$ is an o-minimal expansion of an ordered (abelian) group, with 1 a non-zero element.

Theorem

Let $\{X_{\bar{a}} : \bar{a} \in T\}$ be a definable family of nonempty subsets of M^n. I.e. for some $\phi(\bar{x}, \bar{y})$ and definable $T \subseteq M^k$, $X_{\bar{a}} := \{\bar{b} \in M^n : \phi(\bar{b}, \bar{a})\}$. Then there is a definable $f : T \to M^n$ (i.e. a definable graph) such that (1) for all $\bar{a} \in T$, $f(\bar{a}) \in X_{\bar{a}}$ and (2) If $X_{\bar{a}} = X_{\bar{b}}$ then $f(\bar{a}) = f(\bar{b})$.

Proof By induction on n. For $n = 1$, write $X_{\bar{a}}$ as an $<$-increasing union of intervals $X_{\bar{a}} = J_{1,\bar{a}} \cup J_{2,\bar{a}} \cdots$ ......
If $J_{1,\bar{a}}$ is a bounded interval, let $f(\bar{a})$ be its midpoint (use $+$). Otherwise, let $f(\bar{a}) = Inf(J_{1,\bar{a}}) + 1$ or $sup(J_{\bar{a}}) - 1$.

Exercise

- Show that f is definable.
- Finish by induction.

Structural properties of definable sets I

The Monotonicity-Continuity Theorem

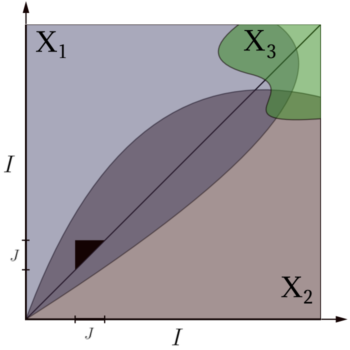
Let $f : I = (a, b) \to M$ be a definable function (i.e. its graph is a definable set).Then there are $a = a_0 < a_1 \cdots < a_n = b$ such that on each interval (a_i, a_{i+1}) the function is continuous and either strictly monotone or constant.

Idea of Proof

We prove a much weaker statement: There is an open (nonempty) interval $J \subseteq I$ such that f is either constant or strictly monotone on J.

Lemma-an ordered Ramsey Theorem (no proof here)

Let $I \subseteq M$ be an open interval, and assume $I^2 \subseteq X_1 \cup \cdots \cup X_n$, for definable X_i's. Then there exists an open interval $J \subseteq I$ and a fixed $1 \leqslant j \leqslant n$, such that for every $a < b$ in J, $(a, b) \in X_j$.



$\{(x, y) \in J^2 : x < y\} \subseteq X_2$

End of the proof of CMT

Back to our $f : I \to M$.
Let

$$X_< = \{(x,y) \in I^2 : f(x) < f(y)\}.$$

$$X_> = \{(x,y) \in I^2 : f(x) > f(y)\}.$$

$$X_= = \{(x,y) \in I^2 : f(x) = f(y)\}.$$

Clearly $I^2 = X_< \cup X_> \cup X_=$.
By Ordered Ramsey, there is an open interval $J \subseteq I$ and $\diamondsuit \in \{>, <, =\}$ such that for all $x < y$ in J, $f(x) \diamondsuit f(y)$.

So, f is either constant, of strictly increasing, or strictly decreasing on J. Piecewise montonicity follows. □

Topological properties of definable sets

• The order topology on M and box-topology on M^n have definable basis.

• **Caution:** In general, the space M is not locally connected, or locally compact (unless the underlying ordered set is $\mathbb{R}$).

BUT

Exercise

- M is **definably connected**, i.e. it cannot be partitioned into two definable open sets.
- M^n is also definably connected.

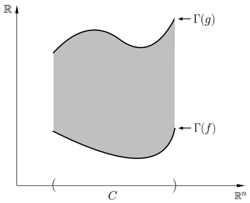
Cell decomposition

Definition of a k-cell

• In M: A 0-cell is a point, a 1-cell is an open interval.

• In M^{n+1}: Assume $C \subseteq M^n$ is a k-cell. *A cell C' in M^{n+1}* is either
(i) the graph $\Gamma(f)$ of a definable continuous function $f : C \to M$, in which case C' is a k-cell. OR
(ii) $C' = \{(\bar{x}, y) \in C \times M : f(\bar{x}) < y < g(\bar{x})\}$, for $f, g : C \to M$ definable continuous, with $f < g$. In which case $\dim C'$ is a $k + 1$-cell.

Exercise: Each cell is definably connected.



The fundamental theorem of o-minimality-Knight Pillay Steinhorn

Every definable set in M^n can be partitioned into finitely many cells. Hence, every definable set has finitely many definably connected components.
Moreover, if $\{X_t : t \in T\}$ is a definable family of sets then there is a uniform bound on the number of these components.

Many important corollaries

- Finer stratification results yield strong geometric tameness.
- **Dimension theory**: $\dim X :=$ the maximal dimension of a cell in X.
- **Important!**: For every definable set $X \subseteq M^n$,

$$\dim(Cl(X) \setminus X) < \dim X.$$

(easy to prove for cells).

A model theoretic corollary

Theorem

If $\mathcal{M}$ is o-minimal and $\mathcal{N} \equiv \mathcal{M}$, then $\mathcal{N}$ is also o-minimal. I.e. o-minimality is a property of complete theories.

Exercise: The above is equivalent to:

Uniform o-minimality

Let $\{X_{\bar{a}} : \bar{a} \in T\}$ be a *definable family of subsets of* M. I.e. for some formula $\phi(x, \bar{t})$ and a definable $T \subseteq M^n$, for every $\bar{a} \in T$,

$$X_{\bar{a}} := \{b \in M : \phi(b, a)\}.$$

Then there is a fixed $K \in \mathbb{N}$ such that every $X_{\bar{a}}$ can be written as a finite union of at most K-many open intervals and K-many points. And this follows from cell decomposition.

O-minimality and algebra

Theorem

Let $\langle G; <, \cdot \rangle$ be an o-minimal ordered group. Then G is abelian and divisible.

Proof

1. **There are no proper nontrivial definable subgroups of G**: Indeed, G must be torsion-free (ordered), so every nontrivial subgroup is infinite. So, if $H < G$ is definable then $Int(H) \neq \varnothing$. Hence, H is open in G. Hence, H is closed (its complement is open). G definably connected, so $H = G$.

2. G **ia abelian**: For every $g \in G$, its centralizer $\{h \in G : gh = hg\}$ is definable, nontrivial subgroup so equals G.

3. G **is divisible**: For every $n \in \mathbb{N}$, nG is a definable nontrivial subgroup, so $nG = G$. □

O-minimality and algebra-definable groups

Definition

A group G is called **definable in** $\mathcal{M}$ if the set G and the group operation are definable in $\mathcal{M}$.

Examples of definable groups in o-minimal structures

1. Semi-linear groups (groups definable in ordered vector spaces).
2. All real algebraic groups $H(\mathbb{R})$, all semi-algebraic groups (e.g. $\langle \mathbb{R}^{>0}, \cdot \rangle$), are definable in the real field.
3. Groups definable in $\mathbb{R}_{exp}$: E.g.

$$\left\{ \begin{pmatrix} e^t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix} : t, u, v \in \mathbb{R} \right\}$$

Solvable, not isomorphic to a semi-algebraic group.

Definable groups, rich theory, many authors

Some general principles

Definable groups resemble:

- Lie groups (e.g. admit a manifold-like, definable topology),
- Algebraic groups (e.g. A descending Chain Condition for definable subgroups. Definable simple groups are isomorphic to algebraic groups).
- Compact groups (e.g. finite number of connected components).

Note: The complex field $(\mathbb{C},+,\cdot)$ is definable in $(\mathbb{R};+,\cdot)$: The field operations on $\mathbb{C}\sim\mathbb{R}^2$ are polynomial in the real and imaginary coordinates.

Question

Which holomorphic functions are definable in o-minimal structures?

Fact

If z_0 is an isolated singularity of a definable holomorphic function f then z_0 is not an essential singularity.

Proof Consider $F=\{(z,f(z))\in\mathbb{C}^2 : z\neq z_0\}$.
If z_0 is an essential singularity then $z_0\times\mathbb{C}\subseteq Cl(F)\setminus F$. But then $\dim(Cl(F)\setminus F)=\dim F=2$. Contradicting topological fact. □

O-minimality and complex analysis II

Based on this simple observation, there is a rich theory of complex analytic sets definable in o-minimal structures.

Positive (and negative) results

1. (**positive**) If $f : \mathbb{C}^n \to \mathbb{C}$ is definable and holomorphic then f is necessarily a polynomial map.
2. Every definable complex analytic subset of $\mathbb{C}^n$ is algebraic.
3. (**Negative**) As a result, all classical periodic, modular and automorphic functions are not definable on their entire domain.
4. Still, as we shall see next talk, some of these are definable on partial domain, which is enough for applications.

Some references

1. **Book**: *Tame topology and o-minimal structures*, L. van den Dries, London Lecture Notes 248, 1998.
2. *Notes on o-minimality and variations*, D. Macpherson, in Model Theory, Algebra and geometry, 2000.
3. *Introduction to o-minimal geometry*, M. Coste, 1998.
4. *A self-guide to o-minimality*, Y. Peterzil, 2005, on http://math.haifa.ac.il/kobi
5. *A survey on groups definable in o-minimal structures*, M. Otero, 2006.

O-minimal structures II

Definition

A linearly ordered structure $\mathcal{M} = \langle M; <, \cdots \rangle$, is o-minimal if every definable subset of M (with parameters) is of the form

$$\bigcup_{i=1}^{k} (a_i, b_i) \cup \underbrace{F}_{\text{finite}}$$

with $a_i, b_i \in M \cup \{\pm\infty\}$.

Talk II-Diophantine applications of o-minimality

We fix an o-minimal expansion $\bar{\mathbb{R}}$ of the real field $\mathbb{R}$.

Let $X \subseteq \mathbb{R}^n$ be a definable set. What can we say about the number of rational points in X?, namely about $X \cap \mathbb{Q}^n$?

A general principle

A definable subset of $\mathbb{R}^n$ should not have "many" integer or rational points unless for a "a good reason".

Examples:

- The set $\{(x, x^2) : x \in \mathbb{R}\}$, has infinitely many points in $\mathbb{Q}^2$ (because of algebraicity).
- But also the set $\{(x, 2^x) : x \in R\}$ has infinitely many points in $\mathbb{Q}^2$ (in fact in $\mathbb{Z}^2$). Not algebraic.
- And the (non definable) set $\{(x, sin(x/\pi))\}$ has infinitely many points in $\mathbb{Z}^2$ (because of "periodicity").

What is “many”? The height function

Definition

The height of a reduced fraction $m/k \in \mathbb{Q}$ is $\max\{|m|, |k|\}$.
For $\bar{q} = (q_1, \ldots, q_n) \in \mathbb{Q}^n$, $ht(\bar{q}) = \max\{ht(q_i) : i = 1, \ldots, n\}$.

• There are between H and H^2 points in $\mathbb{Q}$ of height $\leqslant H$ (to be precise $(H+1) \cdot \phi(H)$, where $\phi(H)$ is Euler’s totient function).

Definition

For $X \subseteq \mathbb{R}^n$ and $H \in \mathbb{N}$, we let $\#X(\mathbb{Q}, H) = |\{\bar{q} \in X \cap \mathbb{Q}^n : ht(\bar{q}) \leqslant H\}|$ (so always $\#X(\mathbb{Q}, H) \leqslant H^{2n}$).

• For $X = \{(x, x^2) : x \in \mathbb{R}\}$, $\#X(\mathbb{Q}, H) \geqslant H^{1/2}$ (definable-algebraic).
• For $X = \{(x, 2^x) : x \in \mathbb{R}\}$, $\#X(\mathbb{Q}, H) \leqslant \log(H^2)$ (definable in $\mathbb{R}_{exp}$-nonalgebraic).
• For $X = \{(x, \sin(\pi x)) : x \in \mathbb{R}\}$, $\#X(\mathbb{Q}, H) \geqslant H$. (non-definable, non-algebraic).

The Pila-Wilkie Theorem

Let $X \subseteq \mathbb{R}^n$ be definable in an o-minimal structure. Assume that X contains "polynomially many" rational points. Namely, there exists $\epsilon > 0$ such that,

$$\limsup_H \frac{\#X(\mathbb{Q}, H)}{H^\epsilon} = +\infty.$$

Then X contains a semialgebraic set (defined over $\mathbb{Q}$), of positive dimension.

Moreover, if we let
$X^{trans} =$
$X \setminus$ all positive dimensional connnected semialgebraic susets of X,
then for every $\epsilon > 0$,

$$\limsup_H \frac{\#X^{trans}(\mathbb{Q}, H)}{H^\epsilon} < \infty.$$

Many Diophantine applications

Definition

A subset $X \subseteq \mathbb{C}^n$ is called **algebraic variety** if it is the solution sets to finitely many polynomial equations in n variables. It is **irreducible** if cannot be written as union of two proper algebraic subvarieties.

(For later use)
A closed $X \subseteq \mathbb{C}^n$ is called **an analytic subvariety of** $\mathbb{C}^n$ if it is given *locally* by finitely many **complex analytic** equations.
E.g. the graph of $exp(z)$ is an analytic subvariety of $\mathbb{C}^2$ (but not algebraic).

General "principle"

Given an abelian algebraic group G, an algebraic subvariety $V \subseteq G$ should not contain "many" torsion points, unless for a "good reason".

Concrete conjectures and theorems

The Manin-Mumford Conjecture-Raynaud's theorem

Let $A \subseteq \mathbb{P}^n(\mathbb{C})$ be an abelian variety defined over $\mathbb{Q}$ (i.e. a projective algebraic group) and let $V \subseteq A$ be an irreducible algebraic subvariety. If the torsion points of A are Zariski dense in V then V **is a coset of an algebraic subgroup of** A.

An analogous simpler result (Laurent 1984)

Let $\mathbb{G}_m^n = ((\mathbb{C}^*)^n, \cdot)$. Its torsion subgroup $\mathbb{U}^n = (u_1, \ldots, u_n)$ (all tuples of roots of unity).
If $V \subseteq \mathbb{G}_m^n$ is an irreducible algebraic variety and $V \cap \mathbb{U}^n$ is Zariski dense in V then there exists an algebraic subgroup $A \leqslant \mathbb{G}_m^n$ and $p \in \mathbb{U}^n$ such that $V = A \cdot p$.

For the rest of the talk I will present the Pila-Zannier proof of this test case.

Translating to the analytic context

The exponential covering map

Consider $exp : (\mathbb{C}^n, +) \to ((\mathbb{G}_m)^n, \cdot)$, given by $exp(\bar{z}) = (e^{z_1}, \ldots, e^{z_n})$.

So exp is a holomorphic group homomorphism.

$\Gamma := Ker(exp) = (2\pi i\mathbb{Z})^n$, is an infinite discrete subgroup.

From torsion to rational

Because exp is a homomorphism, $exp(\bar{z})$ is a torsion point of order k in $\mathbb{G}_m^n$ if and only if $k\bar{z} \in (2\pi i\mathbb{Z})^n$.
So, $exp(\bar{z})$ is a torsion of $\mathbb{G}_m^n$ if and only if there exists k such that $\bar{z} \in 1/k(2\pi i\mathbb{Z})^n$ iff $\bar{z} \in (2\pi i\mathbb{Q})^n$.

Subgroups to linear subspaces

If $L \subseteq \mathbb{C}^n$ is a $\mathbb{C}$-subspace with a basis in $\mathbb{Q}^n$ then $exp(L)$ is a connected algebraic subgroup of $\mathbb{G}_m^n$.

A rough description of the strategy

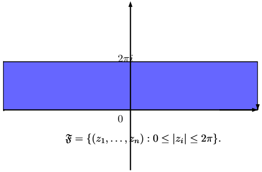
1. We start with an irreducible algebraic subvariety $V \subseteq \mathbb{G}_m^n$ such that $V \cap \mathbb{U}^n$ is Zariski dense in V. Using Galois Theory we shall see that V contains "many" torsion points in the sense of Pila-Wilkie.
2. We let $X = exp^{-1}(V)$. As we shall see, a subset of X (but not all of X!) is definable in an o-minimal structure. Thus the set X contains "many" points in $2\pi i\mathbb{Q}^n$.
3. By Pila-Wilkie, X contains a semialgebraic set of dimension one. Taking Zariski closure, X contains also a complex algebraic curve $S \subseteq X$.
4. By o-minimality, the Zariski closure of $exp(S)$ is a coset of an algebraic subgroup of V. Thus V contains a coset of a subgroup.

Definability in an o-minimal structure

• Because *exp* : $\mathbb{C}^n \to \mathbb{C}^n$ has an infinite discrete kernel Γ, it cannot be definable in any o-minimal structure.

• Instead, we consider a restriction of *exp* to a fundamental set $\mathfrak{F} \subseteq \mathbb{C}^n$, namely $\mathfrak{F}$ such that $\mathfrak{F} + \Gamma = \mathbb{C}^n$.

$$\mathfrak{F} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : 0 \leqslant Im(z_i) \leqslant \pi\}.$$



The restricted function $exp \restriction \mathfrak{F}$ **is definable in** $\mathbb{R}_{an,exp}$:
$e^{x+iy} = e^x(cosy + isiny)$. The function $e^x : \mathbb{R} \to \mathbb{R}$ is definable in $\mathbb{R}_{exp}$ and $\sin x$, $\cos x \restriction [0, \pi]$ is definable in $\mathbb{R}_{an}$.

From “infinity” to “polynomially many” torsion pts

- Assume that $V \subseteq \mathbb{G}_m^n$ is algebraic, say defined over $\mathbb{Q}$, and $V \cap \mathbb{U}^n$ is infinite (weaker than Zariski dense).
- Hence, there are $m_1 < m_2 < \cdots < \cdots$ in $\mathbb{N}$ and g_i's in V such that the order of g_i in $\mathbb{G}_m^n$ equals m_i.
- By elementary considerations, $[\mathbb{Q}(g_i) : \mathbb{Q}] = \phi(m_i)$, where ϕ is the Euler totient function.
- Since V is $\mathbb{Q}$, all conjugates of g_i are in V, hence V contains $\phi(m_i)$-many conjugates of g_i, all elements of order m_i.

Move to the analytic side: Recall $X = \exp^{-1}(V)$. It follows that $X_0 := X \cap \mathfrak{F}$ contains at least $\phi(m_i)$-many “rational” points of height m_i (actually points in $2\pi i\mathbb{Q}^n$).

It is known that $\phi(n)/n^{1/2} \to \infty$.

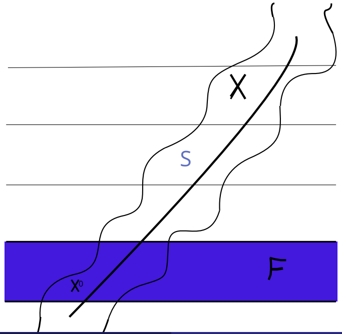
Applying Pila-Wilkie

- The set $X = exp^{-1}(V)$ is a analytic subset of $\mathbb{C}^n$ (in general not definable in an o-minimal structure). We have $\Gamma + X = X$.
- The set $X_0 = X \cap \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.
- We saw: $\frac{\# X_0(2\pi i\mathbb{Q}, m_i)}{m_i^{1/2}} \to \infty$.
- By Pila-Wilkie, there exists a semialgebraic set $S_0 \subseteq X_0$ of dimension 1.
- Let S be the Zariski closure of S_0 (the smallest algebraic subvariety of $\mathbb{C}^n$ containing S_0).
- It is a complex algebraic curve that is contained in X.

Summary thus far

We thus have

- A Γ-invariant analytic set $X \subseteq \mathbb{C}^n$ (not definable!).
- The image $exp(X) = V$ is an algebraic variety in $\mathbb{G}_m^n$.
- The set $X_0 = X \cap \mathfrak{F}$ is definable.
- In X we have a complex algebraic curve (so definable!) S, such that $\dim(S \cap \mathfrak{F}) = 1$.



Finding the coset inside V

Without loss of generality, $exp(S)$ is Zariski dense in V.
Let

$$\Sigma = \{g \in \mathbb{C}^n : (g+S) \cap \mathfrak{F} \neq \varnothing \;\&\; \dim(g+S) \cap X_0 = 1\}.$$

In fact, by the analyticity of X and the irreducibility of S,

$$\Sigma = \{g \in \mathbb{C}^n : (g+S) \cap \mathfrak{F} \neq \varnothing \;\&\; g+S \subseteq X\}.$$

For $g \in \Sigma$, $exp(S) \subseteq exp(X-g) = V \cdot exp(g)^{-1}$.

Because $V = Zarcl(expS)$, we have

$$V \cdot exp(g)^{-1} = V.$$

It follows that $exp(\Sigma)$ is contained in the group-stabilizer of V, in the group $\mathbb{G}_m^n$ (=$\{h \in \mathbb{G}_m^n : h \cdot V = V\}$).

Ending the proof

So,we have

$$\Sigma = \{g \in \mathbb{C}^n : (g+S) \cap \mathfrak{F} \neq \varnothing \,\&\, \dim(g+S) \cap X_0 = 1\},$$

and we saw that $exp(\Sigma)$ is contained in $Stab_{\mathbb{G}_m^n}(V)$.

Note: Since S and X_0 are definable, then Σ is definable.

And because Σ contains all $\gamma \in \Gamma$ such that $S \cap (-\gamma + \mathfrak{F}) \neq \varnothing$, Σ is also infinite.

By o-minimality, $|\Sigma| = 2^{\aleph_0}$, so $exp(\Sigma)$ is still infinite.

Thus, the algebraic group $A = Stab_{\mathbb{G}_m^n}(V)$ has positive dimension. It follows that $A \cdot p \subseteq V$ for every $p \in V$.

Using a supped-up version of the above argument one can show $V = A \cdot p$. □

Conclusions and references

Similar strategy was used by Pila (2008) to prove unconditionally unknown cases of the André-Oort Conjecture.
Since then, there were many different applications of this strategy in Diophantine Geometry.

Surveys

T. Scanlon *A proof of the André-Oort Conjecture via mathematical logic [after Pila, Wilkie and Zannier]*, Sèminaire Bourbaki avril 2011 63èm anée. 2010-2011, no 1037.

T. Scanlon *Counting special points: Logic, Diophantine geometry and transcendence theory,* BAMS (N.S) 49 (2012), no 1 51-71.

K. Peterzil Video recording of MSRI talks, https://www.msri.org/people/8959, 2014.

Recent work on "Diophantine approximation"

Let Λ be a lattice in $\mathbb{R}^n$, namely $\Lambda = \oplus_{i=1}^{n} \mathbb{Z}\omega_i$, for $\omega_1, \ldots, \omega_n$ a basis for $\mathbb{R}^n$.
Let $T = \mathbb{R}^n/\Lambda$ be a compact real torus.
Let $\pi : \mathbb{R}^n \to T$ be the covering map (a transcendental map), and let $X \subseteq \mathbb{R}^n$ be some definable set in an o-minimal structure (e.g. a real algebraic variety).

A question of Ullmo and Yafaev

What is the topological closure of $\pi(X)$ in T?
I.e. Which points in $\mathbb{R}^n$ can be approximated by elements in Λ and X?

A simple observation

The topological closure of $\pi(X)$ is precisely $\pi(Cl(\Lambda + X))$.
So we need to answer: What is the topological closure of $\Lambda + X$ in $\mathbb{R}^n$?
Namely, which points in $\mathbb{R}^n$ can be approximated by elements of $\Lambda + X$

Some examples

Assume that $\Lambda = \mathbb{Z}^2 \subseteq \mathbb{R}^2$ and $X \subseteq \mathbb{R}^2$ closed and definable.

- If $X \subseteq \mathbb{R}^n$ is bounded the $\Lambda + X$ is closed.
- If $X \subseteq \mathbb{R}^n$ is an $\mathbb{R}$-subspace then $Cl(\Lambda + X) = \Lambda + X^{\Lambda}$,where X^{Λ} is the smallest $\mathbb{R}$-subspace with a basis in Λ containing X.
- If $X = \{(x, y) \in \mathbb{R}^2 : x \cdot y = 1\}$ then

$$Cl(\Lambda + X) = (\Lambda + X) \cup \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\}.$$

The main theorem

Theorem (Pe-Starchenko)

Let $X \subseteq \mathbb{R}^n$ be a closed definable set in an o-minimal structure. Then there are $\mathbb{R}$-subspaces $V_1, \ldots, V_r \subseteq \mathbb{R}^n$ of positive dimension, and definable closed sets $C_1, \ldots, C_r \subseteq \mathbb{R}^n$ such that for every lattice $\Lambda \subseteq \mathbb{R}^n$,

1. $Cl(\Lambda + X) = (\Lambda + X) \cup \bigcup_{i=1}^r \Lambda + V_i^\Lambda + C_i$. I.e. $Cl(\pi(X)) = \pi(X) \cup \bigcup_{i=1}^r T_i + \pi(C_i)$, for T_i real subtori of T.
2. For each $i = 1, \ldots, r$, $\dim C_i < \dim X$.
3. If V_i is maximal among $V_1, \ldots, V_r$ then C_i is bounded.

In particular, if $\dim X = 1$ (e.g. X is a real algebraic curve) then each C_i is finite, so

$$Cl(\pi(X)) = \pi(X) \cup \bigcup_{i=1}^r T_i + p_i$$

, for $p_i \in T$.