

ON CENTRAL EXTENSIONS AND DEFINABLY COMPACT GROUPS IN O-MINIMAL STRUCTURES

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ABSTRACT. We prove several structural results on definable, definably compact groups G in o-minimal expansions of real closed fields such as (i) G is definably an almost direct product of a semisimple group and a commutative group, (ii) (G, \cdot) is elementarily equivalent to $(G/G^{00}, \cdot)$. We also prove results on the internality of finite covers of G in an o-minimal environment, as well as deducing the full compact domination conjecture for definably compact groups from the semisimple and commutative cases which were already settled.

These results depend on key theorems about the interpretability of central and finite extensions of definable groups, in the o-minimal context. These methods and others also yield interpretability results for universal covers of arbitrary definable real Lie groups.

1. INTRODUCTION AND PRELIMINARIES

This paper is motivated partly by questions coming out of our paper [16], especially whether, for a definably compact group G in an o-minimal structure (say expanding a real closed field), G and G/G^{00} are elementarily equivalent as groups. We solve this problem (see Theorem 7.1) and in the process manage to tie up several loose ends regarding definable groups in o-minimal structures. For now we will just say “o-minimal structure” \mathcal{M} but often there are additional assumptions on \mathcal{M} such as expanding a real closed field, or expanding an ordered group, which appear explicitly in the statements. One of the main results, Theorem 6.1, considers a definably connected central extension \tilde{G} of a semisimple group G by A , all definable in \mathcal{M} and says that the exact sequence $A \rightarrow \tilde{G} \rightarrow G$ of groups, is essentially bi-interpretable with the pair $\langle G, A \rangle$ of groups. Corollary 6.2 deduces that any such \tilde{G} (in particular any definably compact group) is elementarily equivalent, as a group, to a semialgebraic real Lie group. From this it is not hard to deduce (Corollary 6.4) that a definably compact definably connected group is definably an almost direct product of a semisimple group and a commutative group. Corollary 6.5 strengthens this to central extensions of definably compact semisimple groups. Section 8 contains interpretability and internality results for finite (not necessarily central) extensions of groups, again definable in an o-minimal structure \mathcal{M} . In section 9, definable groups which are not necessarily definably connected are considered, and Corollary

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6.4 (elementary equivalence to semialgebraic Lie groups) is generalized (see Theorem 9.4). In section 10 we point out how the compact domination conjecture (for definably compact groups in o-minimal expansions of real closed fields) follows from our results, together with earlier work.

The above results rely on the main technical theorem (Theorem 2.1) about the interpretability of central extensions in an o-minimal context, which appears in section 2. The theorem roughly says that under certain assumptions, a definable central extension \tilde{G} of a definable group G can be interpreted in the two-sorted structure $\langle G, Z(\tilde{G}) \rangle$ (possibly, after expanding G by definably connected components of some definable sets). In fact we also note that if the base o-minimal structure \mathcal{M} is an expansion of the field of real numbers (in which case we sometimes call a group definable in \mathcal{M} , a definable real Lie group), the assumption that \tilde{G} is definable can be omitted, obtaining interpretability results for central topological extensions of suitable definable Lie groups (see section 2.1 and Theorem 2.8). A version for finite extensions appears in section 8.1 (see Theorem 8.4). Also in section 8.1 a result with a similar flavour is proved for arbitrary connected definable real Lie groups G : for example, the universal cover $\pi : \tilde{G} \rightarrow G$ is interpretable in the two-sorted structure consisting of the given o-minimal expansion of \mathbb{R} together with $\langle \ker(\pi), + \rangle$.

Sections 3, 4 and 5 are devoted to checking that various hypotheses of Theorem 2.1 hold in the cases we are interested in.

Our notation is on the whole standard. However, as we are concerned with issues of interpretability in certain reducts, we will mention the relevant notation.

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1.1. On definability, interpretability and definable Skolem functions. In general, \mathcal{M} is an o-minimal structure, with M is its universe. We will always make a distinction between definable and interpretable objects, but in order to make the language less awkward, maps between two interpretable or definable structures will always be called *definable*, even when their graph is actually interpretable.

Because the foundational results from [29] about the topology of definable groups in o-minimal structures are not known to hold for groups which are only interpretable in \mathcal{M} , we need to insist in many cases on definability of the said groups. It is known (see 6.1.2 in [5]) that if \mathcal{M} expands an ordered group then \mathcal{M} has definable Skolem functions and in particular elimination of imaginaries. Thus, every interpretable group in such an \mathcal{M} is definably isomorphic to a definable one. Also, as is shown in [6, Theorem 7.2], if \mathcal{M} is any o-minimal and G is a definable group in \mathcal{M} then \mathcal{M} has definable choice for every family of subsets of G^n which is definable in \mathcal{M} . Hence, if G is a definable group in any o-minimal structure then every quotient of G by a definable subgroup is definably isomorphic to a definable group.

For a structure \mathcal{N} and a set $A \subseteq N^{eq}$, we use $dcl_{\mathcal{N}}(A)$ and $dcl_{\mathcal{N}}^{eq}(A)$ to denote the elements in N and in N^{eq} which are in the definable closure of a set A . We similarly use $acl_{\mathcal{N}}$ and $acl_{\mathcal{N}}^{eq}$ for the model theoretic algebraic closure in \mathcal{N} . Note that because \mathcal{M} is an ordered structure, we have $acl_{\mathcal{M}} = dcl_{\mathcal{M}}$. The following observation is motivated by a question of the referee, and helps simplifying some of our arguments:

Claim 1.1. *If \mathcal{M} is an arbitrary o-minimal structure then $\text{acl}_{\mathcal{M}}^{\text{eq}} = \text{dcl}_{\mathcal{M}}^{\text{eq}}$. More precisely, if $X \subseteq M^n$ is a \emptyset -definable set and E is a \emptyset -definable equivalence relation on X with finitely many classes, then every class is \emptyset -definable.*

Proof. We use induction on n . Assume that there are k -many E -classes.

If $n = 1$ then X is a subset of M and the E -classes partition X into k sets. By o-minimality, exactly one of the classes contains elements which is larger than all elements of the other classes, and we can choose this class \emptyset -definably. We can now omit the class we have chosen and proceed by induction on k to show that all classes are \emptyset -definable.

Assume now that $X \subseteq M^{n+1}$ and let $Y \subseteq M^n$ be the projection of X onto the first n coordinates. By the $n = 1$ case and compactness, there are finitely many formulas

$$\phi_{i,j}(w, x), \quad i \in I, j = 1, \dots, k, \quad \text{and } |w| = n, |x| = 1,$$

such that: for every $a \in Y$, there is an $i \in I$ such that the intersection of every E -class with $Y_a = \{a\} \times M$ is given by one of the formulas $\phi_{i,j}(a, x)$, for $j = 1, \dots, k$. By partitioning Y further (\emptyset -definably), we may assume that $|I| = 1$, namely that for each a , the E -classes on Y_a are given precisely by the formulas $\phi_1(a, x), \dots, \phi_k(a, x)$. Let us call the E -class of Y_a defined by $\phi_i(a, y)$ “the i -th class of E on Y_a ”.

For each $i = 1, \dots, k$, we define on Y the equivalence relation \sim_i , given by $a_1 \sim_i a_2$ iff there exist $b_1, b_2 \in M$ such that $\phi_i(a_1, b_1) \wedge \phi_i(a_2, b_2) \wedge (a_1, b_1)E(a_2, b_2)$ (namely, the elements of the i -th classes of E on Y_{a_1} and Y_{a_2} are E -equivalent). Each \sim_i is \emptyset -definable and has at most k equivalence classes. Therefore the intersection $\sim := \bigcap_i \sim_i$ is \emptyset -definable, with finitely many classes. By induction, every class of \sim is \emptyset -definable.

We now define the equivalence relation E' on X by

$$(a_1, b_1)E'(a_2, b_2) \Leftrightarrow a_1 \sim a_2 \wedge \bigvee_{i=1}^k (\phi_i(a_1, b_1) \wedge \phi_i(a_2, b_2)).$$

Each E' -class of an element (a, b) is determined by the \sim -class of a together with the formula ϕ_i for which we have $\phi_i(a, b)$. Thus each E' -class is \emptyset -definable, and there are finitely many such classes.

Finally, note that E' is a refinement of E so every E -class must be \emptyset -definable. \square

1.2. Preliminary results. We now review some earlier results, mainly about definably simple and definable semisimple groups (for a survey of definable groups in o-minimal structures, see Otero’s [20]).

A *definably simple* group is a definable, non-abelian group with no definable normal subgroup. A *semisimple* group is a definable group with no infinite definable normal abelian subgroup (because of DCC, the definability requirement is superfluous).

We summarize the main results which we will be using here:

Fact 1.2. *Let \mathcal{M} be an o-minimal structure and G a definable group in \mathcal{M} .*

- (1) *If G is definably simple then there is in \mathcal{M} a definable real closed field R and a real algebraic group H defined over $R_{\text{alg}} \subseteq R$ the subfield of real algebraic numbers, such that G is definably isomorphic in \mathcal{M} to $H(R)^0$, the definably connected component of $H(R)$ (see [24, 4.1] for the existence of an algebraic*

- group H and [26, 5.1] and its proof, for the fact that H can be defined over R_{alg}).
- (2) If G is definably simple then it is either bi-interpretable, over parameters, with a real closed field or with an algebraically closed field of characteristic zero, [25].
 - (3) If G is definably connected and semisimple then $Z(G)$ is finite and $G/Z(G)$ is definably isomorphic in \mathcal{M} to the direct product of finitely many definably simple groups (see [24, 4.1]).
 - (4) If G is definably compact and definably connected then either G is abelian or $G/Z(G)$ is semisimple, [27, 5.4].
 - (5) Let G be a definably simple group. If G is not definably compact then it is necessarily abstractly simple (see [26, 6.3]). If G is definably compact and \mathcal{M} is sufficiently saturated structure then G is not abstractly simple (its infinitesimal subgroup with respect to the prime model is a normal in G).

Note that the bi-interpretability of (2) above is necessarily over parameters (see [25, Remark 4.11]).

2. THE MAIN INTERPRETABILITY THEOREM

We recall that \mathcal{M} is an o-minimal structure.

Let G be a definable group. By a *definable (interpretable) central extension* of G we mean the following data: definable (interpretable) groups A, \tilde{G} , definable homomorphisms: $i : A \rightarrow \tilde{G}$, $\pi : \tilde{G} \rightarrow G$ with

$$1 \rightarrow A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \rightarrow 1$$

exact and $i(A)$ central in \tilde{G} . We let $\langle A, \tilde{G}, G, i, \pi \rangle$ denote the three group structures, together with the maps i and π . We say that the central sequence above is (definably) isomorphic to another definable exact sequence

$$1 \rightarrow A_1 \xrightarrow{i_1} \tilde{G}_1 \xrightarrow{\pi_1} G_1 \rightarrow 1$$

if there are (definable) group isomorphisms $h_A : A_1 \rightarrow A$, $h_{\tilde{G}} : \tilde{G}_1 \rightarrow \tilde{G}$ and $h_G : G_1 \rightarrow G$, which commute with the exact sequence maps.

When G is a definable group in \mathcal{M} then it has a canonical group topology (see [29]), with respect to which every definable subset of G in \mathcal{M} has finitely many definably connected components. Moreover, if g is a generic point of G then the basis of neighborhoods of g in the ambient o-minimal topology of \mathcal{M} (namely the one induced by the order topology of M), is also a basis of neighborhoods of g in the sense of the group topology. When G is defined in an o-minimal expansion of the real field then G , with its canonical group topology, is a Lie group. If H is a definable subgroup of a definable group G then the canonical group topology on H can also be obtained as the one induced from the group topology of G .

Let \mathbf{G} be an arbitrary expansion of the group G (not necessarily definable in \mathcal{M}). We say that \mathbf{G} has *property ρ* if for every definable $X \subseteq G^n$ in the structure $\langle G, \cdot \rangle$, every definably connected component of X (with respect to the group topology of G) is definable in \mathbf{G} , possibly over new parameters.

Given the abelian group A , we use $\langle \mathbf{G}, A \rangle$ to denote the two-sorted structure of the two groups, where G is equipped with its \mathbf{G} -structure and A with just its group structure.

Recall that for a definable group H , the commutator subgroup $[H, H]$ is a countable union of definable sets, which might not be definable itself. More precisely, if we denote by $[H, H]_n$ the definable set of all products of n commutators in H , then $[H, H] = \bigcup_{n \in \mathbb{N}} [H, H]_n$.

Theorem 2.1. *Let \mathcal{M} be an o-minimal structure and assume $E = 1 \rightarrow A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \rightarrow 1$ is a \emptyset -definable central extension of G in the structure \mathcal{M} , \mathbf{G} an arbitrary expansion of G such that:*

- (1) \mathbf{G} has property ρ .
- (2) For every n , the set $i(A) \cap [\tilde{G}, \tilde{G}]_n$ is finite (we call this property Z).
- (3) There exists $r \in \mathbb{N}$ with $G = [G, G]_r$.

Then E (that is, the structure $\langle A, \tilde{G}, G, i, \pi \rangle$) can be interpreted in $\langle \mathbf{G}, A \rangle$ over A and G . More precisely, there is an exact sequence

$$E' = 1 \rightarrow A \rightarrow \tilde{G}' \rightarrow G \rightarrow 1,$$

interpretable in $\langle \mathbf{G}, A \rangle$ over an imaginary parameter \bar{c} , such that E' is definably isomorphic, in the structure $\langle A, \tilde{G}, \mathbf{G}, i, \pi \rangle$ (note that G is expanded), to the sequence E , with h_A, h_G the identity maps.

The imaginary parameter \bar{c} names a map from the (finite) set of definably connected components of a \emptyset -definable set in the group $\langle G, \cdot \rangle$, onto a finite subset of A . This map is \emptyset -definable in the structure $\langle A, \tilde{G}, G, i, \pi \rangle$.

Proof. Note that our assumption implies that \tilde{G} can be written as the group product of the subgroups $i(A)$ and $[\tilde{G}, \tilde{G}]_r$. Our goal is to produce in \mathcal{M} a definable surjective map $j : G^{2r} \times A \rightarrow \tilde{G}$, such that the pull-back under j of equality in \tilde{G} and of the group operation are both definable in $\langle \mathbf{G}, A \rangle$.

For $x, y \in G$, we let $[x, y] = xyx^{-1}y^{-1}$. For $n \geq 0$, we let $w_n(x_1, y_1, \dots, x_n, y_n)$ be the word in the free group given by the product $[x_1, y_1] \cdots [x_n, y_n]$ of n commutators. For any group K , we let $F_{n,K} : K^{2n} \rightarrow K$ be the associated function which evaluates the word w_n in K . The image of K^{2n} under $F_{n,K}$ is exactly $[K, K]_n$. We also have, for $\bar{h}_1 \in K^{2m}, \bar{h}_2 \in K^{2n}$,

$$F_{m,K}(\bar{h}_1) \cdot F_{n,K}(\bar{h}_2) = F_{m+n,K}(\bar{h}_1, \bar{h}_2),$$

and if for $\bar{h} = (x_1, y_1, \dots, x_n, y_n)$ we let $\text{inv}(\bar{h}) = (y_n, x_n, y_{n-1}, x_{n-1}, \dots, y_1, x_1)$ then

$$F_{n,K}(\bar{h})^{-1} = F_{n,K}(\text{inv}(\bar{h})).$$

We use π to denote the map from \tilde{G}^{2n} to G^{2n} which is induced by π in each coordinate.

Claim 2.2. *For every $\bar{g}_1, \bar{g}_2 \in \tilde{G}^{2n}$, if $\pi(\bar{g}_1) = \pi(\bar{g}_2)$ then $F_{n,\tilde{G}}(\bar{g}_1) = F_{n,\tilde{G}}(\bar{g}_2)$.*

Proof. This is immediate from the fact that each coordinate of \bar{g}_1 differs from the corresponding coordinate of \bar{g}_2 by a central element of $i(A)$, and on tuples from the center of \tilde{G} , the map $F_{n,\tilde{G}}$ is identically 1. \square

It follows from 2.2 that there is a definable surjective map $k_n : G^{2n} \rightarrow [\tilde{G}, \tilde{G}]_n$ in \mathcal{M} such that $F_{n,\tilde{G}}$ factors through π and k_n (see diagram below). Also, for $\bar{g}_1 \in G^{2m}, \bar{g}_2 \in G^{2n}$, we have

$$k_m(\bar{g}_1) \cdot k_n(\bar{g}_2) = k_{m+n}(\bar{g}_1, \bar{g}_2) \text{ and } k_n(\bar{g}_1)^{-1} = k_n(\text{inv}(\bar{g}_1)).$$

$$\begin{array}{ccc}
\tilde{G}^{2n} & \xrightarrow{F_{n,\tilde{G}}} & \tilde{G} \\
\pi \downarrow & \nearrow k_n & \downarrow \pi \\
G^{2n} & \xrightarrow{F_{n,G}} & G
\end{array}$$

Fact 2.3. (i) The function $k_n : G^{2n} \rightarrow \tilde{G}$ is continuous (here and below we always refer to the group topology) and $\pi k_n(\bar{g}) = F_{n,G}(\bar{g})$ for all $\bar{g} \in G^{2n}$.

(ii) For every $\bar{g} \in G^{2n}$, $F_{n,G}(\bar{g}) = 1$ if and only if $k_n(\bar{g}) \in i(A)$.

(iii) For every $\bar{g}_1, \bar{g}_2 \in G^{2n}$, the following are equivalent:

- (1) $F_{n,G}(\bar{g}_1) = F_{n,G}(\bar{g}_2)$.
- (2) $k_n(\bar{g}_1)$ and $k_n(\bar{g}_2)$ are in the same $i(A)$ -coset in \tilde{G} .
- (3) $F_{2n,G}(\bar{g}_1, \text{inv}(\bar{g}_2)) = 1$.

Proof. (i) Because the topology on G equals the quotient topology which is inherited from \tilde{G} , (see [1, Theorem 4.3]) and because $F_{n,\tilde{G}}$ is continuous, the map k_n is also continuous.

(ii) This follows from the fact that $\pi k_n = F_{n,G}$.

(iii) (1) \Leftrightarrow (2): Because π is a homomorphism, we have $\pi k_n(\bar{g}) = F_{n,G}(\bar{g})$, for every $\bar{g} \in G^{2n}$. Hence, $F_{n,G}(\bar{g}_1) = F_{n,G}(\bar{g}_2)$ if and only if $\pi k_n(\bar{g}_1) = \pi k_n(\bar{g}_2)$ if and only if $k_n(\bar{g}_1)$ and $k_n(\bar{g}_2)$ are in the same $i(A)$ -coset.

(2) \Leftrightarrow (3): $k_n(\bar{g}_1) \cdot k_n(\bar{g}_2)^{-1} \in i(A)$ if and only if $k_{2n}(\bar{g}_1, \text{inv}(\bar{g}_2)) \in i(A)$ if and only if $F_{2n,G}(\bar{g}_1, \text{inv}(\bar{g}_2)) = 1$. □

For $k \geq 0$, we let

$$G(k) = \{\bar{g} \in G^{2k} : F_{k,G}(\bar{g}) = 1\}.$$

By our assumption, the set $i(A) \cap [\tilde{G}, \tilde{G}]_n$ is finite for every n . Because k_n is continuous and surjective on $[\tilde{G}, \tilde{G}]_n$ (since $F_{n,\tilde{G}}$ is) we have:

Fact 2.4. The set $k_n(G(n))$ equals $i(A) \cap [\tilde{G}, \tilde{G}]_n$, and the function k_n is constant on every definably connected component of $G(n)$.

Given n , let $b_1, \dots, b_{\ell_n} \in A$ be such that

$$\{i(b_1), \dots, i(b_{\ell_n})\} = i(A) \cap [\tilde{G}, \tilde{G}]_n.$$

We have a corresponding partition of $G(n)$ into relatively clopen sets $W_n(b_1), \dots, W_n(b_{\ell_n})$, with $k_n(W_n(b_j)) = \{i(b_j)\}$. Each $W_n(b_j)$ is a finite union of definably connected components of the set $G(n)$ which is itself definable in $\langle G, \cdot \rangle$. Hence, by property ρ , each such $W_n(b_j)$ is definable in \mathbf{G} , possibly over some parameters.

The interpretation

Let r be given as in assumption (3). Hence, $\tilde{G} = i(A) \cdot [\tilde{G}, \tilde{G}]_r$.

The Universe: Consider the map $j : A \times G^{2r} \rightarrow \tilde{G}$ defined by

$$j(a, \bar{g}) = i(a) \cdot k_r(\bar{g}).$$

Because k_r is surjective on $[\tilde{G}, \tilde{G}]_r$, the map j is surjective on \tilde{G} , and we have $j(a_1, \bar{g}_1) = j(a_2, \bar{g}_2)$ if and only if $k_r(\bar{g}_1) \cdot k_r(\bar{g}_2)^{-1} = i(a_1)^{-1} \cdot i(a_2)$ if and only if

$$k_{2r}(\bar{g}_1, \text{inv}(\bar{g}_2)) = i(a_1^{-1} \cdot a_2).$$

Let

$$B_{2r} = \{b_1, \dots, b_{\ell_{2r}}\} \subseteq A$$

for b_i chosen as above, and let

$$\mathcal{W}_{2r} = \{W_{2r}(b_1), \dots, W_{2r}(b_{\ell_{2r}})\}$$

be the corresponding partition of $G(2r)$, as given above (namely, $k_{2r}(W_{2r}(b_j)) = \{i(b_j)\}$). Let

$$c_{2r} : B_{2r} \rightarrow \mathcal{W}_{2r}$$

be the bijection which sends b_j to $W_{2r}(b_j)$. Note that the map c_{2r} is \emptyset -definable in the structure $\langle A, \tilde{G}, G, i, \pi \rangle$ because $G(2r)$, B_{2r} and k_{2r} are \emptyset -definable there.

Consider now the equivalence relation \sim induced on $A \times G^{2r}$ by the map j . It is defined by

$$j(a_1, \bar{g}_1) = j(a_2, \bar{g}_2),$$

or equivalently by

$$(\bar{g}_1, \text{inv}(\bar{g}_2)) \in c_{2r}(a_1^{-1} \cdot a_2).$$

We let $\mathcal{U} = A \times G^{2r} / \sim$. Notice that the equivalence relation is \emptyset -definable in the structure $\langle \langle G, \cdot \rangle, \langle A, \cdot \rangle, c_{2r} \rangle$, which in particular names the finite set $B_{2r} \subseteq A$. By property ρ , the function c_{2r} itself is definable, over parameters, in $\langle \mathbf{G}, A \rangle$ (one way to obtain c_{2r} is by naming each element of B_{2r} and then naming an element in each definably connected component of $W(b_j) \in \mathcal{W}_{2r}$).

We denote by $[(a, \bar{g})]$ the \sim -class of (a, \bar{g}) .

The group operation: We now consider the pull-back on \mathcal{U} , via the map j , of the group operation from \tilde{G} : we get (because $i(A)$ is central) for every $(b, \bar{h}), (a_1, \bar{g}_1), (a_2, \bar{g}_2) \in A \times G^{2r}$,

$$[(b, \bar{h})] = [(a_1, \bar{g}_1)] \cdot [(a_2, \bar{g}_2)] \Leftrightarrow b \cdot k_r(\bar{h}) = a_1 \cdot a_2 \cdot k_r(\bar{g}_1) \cdot k_r(\bar{g}_2)$$

if and only if

$$(\bar{g}_1, \bar{g}_2, \text{inv}(\bar{h})) \in c_{3r}(a_2^{-1} \cdot a_1^{-1} \cdot b).$$

As before, this last expression can be defined using the pure group structure of A and G , and a function symbol for $c_{3r} : B_{3r} \rightarrow \mathcal{W}_{3r}$. The map c_{3r} itself is defined, over parameters, in $\langle \mathbf{G}, A \rangle$.

We thus proved the interpretation of the group \tilde{G} in $\langle \mathbf{G}, A \rangle$, over the imaginary parameter $\bar{c} = (c_{2r}, c_{3r})$. The map $i : A \rightarrow \tilde{G}$ is interpreted by $i(a) = [(a, (1, \dots, 1))]$ and the map $\pi : \tilde{G} \rightarrow G$ is interpreted via $\pi([(a, \bar{g})]) = F_{r,G}(\bar{g})$.

We therefore interpreted in $\langle \mathbf{G}, A \rangle$ a central extension of G which is isomorphic to the original one, as required. \square

Corollary 2.5. *Consider the assumptions of Theorem 2.1 and assume further that \mathbf{G} is just the group structure G . Then $\langle A, \tilde{G}, G, i, \pi \rangle$ and $\langle G, A \rangle$ are bi-interpretable over parameters. The parameters come from $G \cup A$ and they are in $\text{dcl}^{\text{eq}}(\emptyset)$ of the structure $\langle A, \tilde{G}, G, i, \pi \rangle$.*

Proof. This is immediate from Theorem 2.1 together with the fact that B_{2r}, B_{3r}, c_{2r} and c_{3r} are \emptyset -definable in $\langle A, \tilde{G}, G, i, \pi \rangle$ (and using the fact that the isomorphism between the two exact sequences is the identity when restricted to A and to G , in the notation of that theorem). \square

Remark 2.6. *Let us return to Theorem 2.1 and to the imaginary parameters c_{2r}, c_{3r} used there: The maps c_{2r}, c_{3r} define bijections between finite subsets $B_{2r}, B_{3r} \subseteq A$, respectively, and finite unions of definably connected components of \emptyset -definable sets in $\langle G, \cdot \rangle$. Even though the group G is definable in the o-minimal \mathcal{M} , its canonical group topology is not necessarily definable in the pure group $\langle G, \cdot \rangle$. It is however \emptyset -definable in the o-minimal reduct $\langle M, <, \langle G, \cdot \rangle \rangle$. Furthermore, in an o-minimal structure the finitely many definably connected components of a \emptyset -definable set are themselves \emptyset -definable (this is known but also follows from 1.1). Therefore, c_{2r} and c_{3r} are \emptyset -definable in the structure*

$$\langle M, <, \langle G, \cdot \rangle, \langle A, \cdot, B_{2r}, B_{3r} \rangle \rangle$$

(by that we mean that we add predicates for B_{2r} and B_{3r}). In the special case that A itself is finite then, because it is a subset of M^k for some k , every subset of A is \emptyset -definable in $\langle M, <, \langle A, \cdot \rangle \rangle$ and therefore the predicates B_{2r}, B_{3r} can be omitted. We will later make use of this fact.

Finally let us mention an easy general result on definable splitting, which we do not really use, but nevertheless is in the spirit of the other results.

Fact 2.7. *Suppose that $E : A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G$ is a central extension (of abstract) groups, and that $G = [G, G]_k$ for some k . Suppose that E splits abstractly, then, in the structure $\langle A, \tilde{G}, G, i, \pi \rangle$, the group $[\tilde{G}, \tilde{G}]$ is definable and the sequence definably splits.*

Proof. By the splitting assumption, \tilde{G} can be written abstractly as a direct product of $i(A)$ and a subgroup $H \subseteq \tilde{G}$, with $\pi|_H : H \rightarrow G$ an isomorphism. It follows that $[\tilde{G}, \tilde{G}] \subseteq H$ and because $[\tilde{G}, \tilde{G}]_k$ projects onto G we have $[\tilde{G}, \tilde{G}]_k = [H, H] = H$. In particular $H = [\tilde{G}, \tilde{G}]$ is definable hence \tilde{G} split definably. \square

2.1. The real case. There was actually not much use of o-minimality in the proof of Theorem 2.1. Mainly, it was used in order to obtain the canonical partition of $G(2n)$ into finitely many definably connected components, on each of which the map k_{2n} is constant. Because of o-minimality this partition could be read off just using G (and the definably connected components of sets definable in $\langle G, \cdot \rangle$), independently of \tilde{G} and π . In particular, if we work over the ordered real numbers then this assumption can be partially omitted:

We say that

$$E : 1 \rightarrow A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \rightarrow 1$$

is a central extension of topological groups if A, \tilde{G} and G are topological groups, and the maps i and π are homomorphisms of topological groups which in addition are quotient maps (a set in the image is closed iff its pre-image is closed). When G is definable in an o-minimal structure (but possibly not \tilde{G} and A), we always consider G with its canonical o-minimal group topology which we just call “the group topology”.

Theorem 2.8. *Let \mathcal{M} be an o-minimal expansion of $\langle \mathbb{R}, < \rangle$, G a definable group in \mathcal{M} . Let $E = 1 \rightarrow A \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \rightarrow 1$ be a topological central extension of G (so A, \tilde{G}, π and i are not assumed to be definable), \mathbf{G} an arbitrary expansion of G such that:*

- (1) \mathbf{G} has property ρ .
- (2) For every n , the set $i(A) \cap [\tilde{G}, \tilde{G}]_n$ is finite.

(3) There exists $r \in \mathbb{N}$ with $G = [G, G]_r$.

Then there is an exact sequence $E' : 1 \rightarrow A \rightarrow \tilde{G}' \rightarrow G \rightarrow 1$, interpretable in the structure $\langle \mathbf{G}, A \rangle$ (over parameters) such that E' is isomorphic (in the group language) to the sequence E , with h_A, h_G the identity maps.

If, moreover, we assume that A is definable in \mathcal{M} , then \tilde{G}' may be taken to be definable in \mathcal{M} and then the isomorphism between \tilde{G} and \tilde{G}' (with its group topology) is also a topological one.

Later on, in section 8, we will show how, for finite central extensions, assumption (3) can be omitted. For now, let us note that the above already implies that every finite topological cover of $SL(2, \mathbb{R})$ is topologically isomorphic to a semialgebraic cover.

Proof of the theorem All one has to do is go back to the proof of Theorem 2.1 and see where definability and o-minimality were used. While the existence of the map $k_n : G^{2n} \rightarrow \tilde{G}$ is just a group theoretic fact (of course now k_n is not definable in \mathcal{M}), something should be said about the continuity of k_n , mentioned in 2.3. Indeed, this just follows from the fact that the topology on G is the quotient topology inherited from \tilde{G} . The rest of 2.3 is just group theoretic.

Proceeding to 2.4, we still have $k_n(G(n)) = i(A) \cap [\tilde{G}, \tilde{G}]_n$ and because of our assumption, this set is finite, which implies by continuity, that k_n is locally constant. We now wish to use the fact (see p.59, Ex. 7 in [5]) that over the real numbers the definably connected components of definable subsets of \mathbb{R}^n are just the usual connected components and then conclude then that k_n is constant on every (definably) connected component of the set $G(n)$. However, some care is needed since the topology on G which we consider here is the canonical group topology. We digress and claim that for a definable set $X \subseteq G$, the usual connected components of X with respect to the group topology are definable (and obviously definably connected). Indeed, by the definition of the group topology, G is covered by finitely many definable sets U_i , each definably identified with an open subset of \mathbb{R}^n , such that on U_i the group topology is identical to the \mathbb{R}^n -topology. If $X \subseteq G$ is definable then each $X_i = X \cap U_i$ is definable and we can use the above fact to conclude that each of the (finitely many) connected components of X_i are definable. Each connected component of X with respect to the group topology is a finite union of connected components of the X_i 's, so definable. With that we end the digression and conclude that k_n is constant on every (definably) connected component of $G(n)$.

Since the rest of the argument takes place fully in $\langle \mathbf{G}, A \rangle$, we obtain in this last structure an interpretable central exact sequence

$$E : 1 \rightarrow A \xrightarrow{i'} \tilde{G}' \xrightarrow{\pi'} G \rightarrow 1,$$

together with a group isomorphism $h_{\tilde{G}} : \tilde{G}' \rightarrow \tilde{G}$, such that all maps commute (with the identity maps on A and G).

Assume now that A itself is definable in the o-minimal structure \mathcal{M} . Let us see why $h_{\tilde{G}}$ is a topological homeomorphism as well.

The group \tilde{G}' is obtained as a quotient of $A \times G^{2r}$ by a definable equivalence relation \sim in \mathcal{M} , which is itself the pre-image of equality under the continuous map j . The isomorphism $h_{\tilde{G}} : \tilde{G}' \rightarrow \tilde{G}$ is just the map induced by j . Note that by [6], the structure \mathcal{M} has definable choice functions for subsets of $A \times G^{2r}$, hence there exists a definable set of representatives $X \subseteq A \times G^{2r}$ and a definable bijection

$\alpha : \tilde{G}' \rightarrow X$. By the definition of the group topology on \tilde{G}' , the map α is continuous on some open subset $U \subseteq \tilde{G}'$ and therefore the composition $j \circ \alpha$, which is just $h_{\tilde{G}}$, is continuous on U as well. Since $h_{\tilde{G}}$ is a group isomorphism it must be continuous everywhere. Because \tilde{G}' is locally compact (and \tilde{G} is Hausdorff) the inverse map is continuous as well. \square

In the next section, we investigate each of the three assumptions of Theorem 2.1.

3. PERFECT GROUPS

In this section \mathcal{M} can be taken to be an arbitrary o-minimal structure. Recall that G is perfect if $[G, G] = G$.

Claim 3.1. *Let G be a definable group in \mathcal{M} .*

- (i) *If G is perfect then every homomorphic image of G is perfect.*
- (ii) *The direct product of finitely many perfect groups is perfect.*
- (iii) *(Assume that $\langle G, \cdot \rangle$ is sufficiently saturated). If \tilde{G} is definably connected, G is perfect and $\pi : \tilde{G} \rightarrow G$ is a finite extension then \tilde{G} is perfect as well.*
- (iv) *If G is definably simple then it is perfect and moreover there is an r such that $G = [G, G]_r$.*
- (v) *If G is semisimple and definably connected then it is perfect and moreover there is an r such that $[G, G]_r$.*

Proof. (i) and (ii) are easy. For (iii), the assumption implies that $\pi[\tilde{G}, \tilde{G}] = G$, and hence $\tilde{G} = F \cdot [\tilde{G}, \tilde{G}]$ for some finite group $F \subseteq \tilde{G}$. It follows that $[\tilde{G}, \tilde{G}]$ has finite index in \tilde{G} . Since it is a \forall -definable group (i.e. a countable union of definable sets) of finite index, its complement is also \forall -definable. This implies, using saturation, that $[\tilde{G}, \tilde{G}]$ is a definable proper subgroup of finite index, contradicting connectedness.

For the next two clauses, note that definable simplicity and semi-simplicity are preserved under elementary extensions hence we may assume that \mathcal{M} is sufficiently saturated. Indeed, this is immediate for definably simple groups. As for semi-simple groups, we use the fact that groups with no *definable* normal abelian subgroups are semi-simple.

(iv) If G is definably compact and definably simple then G is elementarily equivalent to a compact simple real Lie group H , by 1.2(1). By topological compactness, there exists an r such that $[H, H]_r = H$. This is now true for G as well.

If G is not definably compact then by 1.2(5) it is abstractly simple and therefore $[G, G] = G$. Since \mathcal{M} is sufficiently saturated we can conclude that there exists r such that $[G, G]_r = G$.

(v) Assume that G is semisimple and definably connected. Then $G/Z(G)$ is centerless, definably connected and semisimple. By 1.2(3), the group $G/Z(G)$ can be written as a direct product of finitely many definably simple groups that are definable in \mathcal{M} . The result now follows from (iv), (ii) and (iii). \square

4. PROPERTY ρ AND SEMISIMPLE GROUPS

In this section we assume that \mathcal{M} is an arbitrary o-minimal structure. Recall that an expansion \mathbf{G} of a group G definable in \mathcal{M} is said to have property ρ if the definably connected components of every $\langle G, \cdot \rangle$ -definable subset of G^n are definable in \mathbf{G} (possibly over new parameters).

Claim 4.1. *Assume that G_1, \dots, G_k are definable groups, such that the theory of each pure group $\langle G_i, \cdot \rangle$ satisfies property ρ and let $G = G_1 \times \dots \times G_k$. Expand the pure group G by a predicate for every*

$$G_i = \{(x_1, \dots, x_n) \in G : x_i \in G_i \text{ \& for all } j \neq i, x_j = 1\}.$$

Then the expanded group satisfies property ρ .

Proof. Here and subsequently (section 7) we will make use of the rather obvious facts that

- (i) The expanded group G (equipped with predicates for the G_i) is naturally interpretable in (in fact bi-interpretable with) the many sorted structure (G_1, \dots, G_k) , consisting of the k “pure groups” with no additional relations between the sorts.
- (ii) Any subset of $G_1^{n_1} \times \dots \times G_k^{n_k}$ definable in the structure (G_1, \dots, G_k) is a finite union of “rectangular boxes” $X_1 \times \dots \times X_k$, where X_i is a definable (in G_i) subset of $G_i^{n_i}$.

Regarding (ii) this can be easily proved by passing to a saturated model and showing that if a_i is a tuple from G_i , then the type of (a_1, \dots, a_k) in the many sorted structure is determined by the types of the a_i in G_i for $i = 1, \dots, k$ (and using compactness). This is all of course related to Feferman-Vaught.

Below (section 7) we may denote many sorted structures (G_1, \dots, G_k) with no relations between the sorts as “disjoint unions” $G_1 \sqcup \dots \sqcup G_k$. In any case we will use freely facts (i), (ii) above, as well as appropriate variants without giving further details.

Returning to the main thread, we see by (i), (ii) above that every definable set $X \subseteq G^n$ in $\langle G, \cdot \rangle$ is a finite union of sets of the form $X_1 \times \dots \times X_k$, where X_i is a definable subset of $G_i^{n_i}$ in the pure group G_i . By our assumptions on each G_i , we may assume that each X_i is definably connected. Each definably connected component of X is a finite union of such cartesian products and therefore definable in the expansion of $\langle G, \cdot \rangle$ by the predicates for every G_i . \square

Lemma 4.2. *If G is a definably simple group then the pure group $\langle G, \cdot \rangle$ has property ρ .*

Proof. By 1.2(2), there are two cases to consider: If G is unstable then, since property ρ is preserved under definable isomorphism, we may assume that it is a semialgebraic group which is bi-interpretable (over parameters) with a real closed field. It follows that every definable set $X \subseteq G^n$ in $\langle G, \cdot \rangle$ is semialgebraic and every definably connected component of X is again semialgebraic, and therefore definable in $\langle G, \cdot \rangle$ (possibly over parameters).

If G is stable then we may assume that it is a linear algebraic group over a definable algebraically closed field K . Because K is a definable algebraically closed field in the o-minimal structure \mathcal{M} , then, by [28], a maximal real closed subfield $R \subseteq K$ is definable in \mathcal{M} and we have $K = R(\sqrt{-1})$. Since G is a linear algebraic group over K , we may assume that $G \subseteq K^\ell$ for some ℓ and that its group topology agrees with that of K^ℓ , when identified with $R^{2\ell}$. In particular, the definably connected components of every definable subset of G^n in the sense of the group topology are the same as those in the sense of the field R .

By 1.2, G is bi-interpretable (again over parameters) with K and hence the $\langle G, \cdot \rangle$ -definable subsets of G^n are exactly the K -constructible sets. It is therefore sufficient to prove:

Claim 4.3. *If $K = R(\sqrt{-1})$ is an algebraically closed field definable in an o-minimal \mathcal{M} expanding R and $X \subseteq K^n$ is a K -constructible set then every definably connected component of X (in the sense of R) is K -constructible.*

Proof. The set X is of the form

$$X = \bigcup_{i=1}^r (X_i \setminus Y_i),$$

with each X_i an irreducible algebraic variety and $Y_i \subseteq X_i$ an algebraic variety of smaller algebraic dimension. We claim that each $X_i \setminus Y_i$ is definably connected.

Indeed, it is known that if $V \subseteq \mathbb{C}^n$ is an irreducible complex variety then $\text{Reg}(V)$ (the set of complex regular points of V) is a connected set, dense in V . If we now work in the structure $\langle \mathbb{R}, <, +, \cdot, \rangle$ then, by quantifying over parameters, this fact carries over to $\langle R, <, +, \cdot, K \rangle$. Thus, every $\text{Reg}(X_i)$ is definably connected in the sense of R and dense in X_i .

Thus, the set $\text{Reg}(X_i)$ is a definably connected R -manifold of even R -dimension $2k$, and we have $\dim_R(Y_i) \leq 2k - 2$ (we let $\dim_R(Y_i)$ denote the o-minimal dimension of Y_i with respect to R , which is twice the algebraic dimension of Y_i). The set $\text{Reg}(X_i) \setminus Y_i$ is therefore still definably connected, dense in $\text{Reg}(X_i)$ and so, also dense in $X_i \setminus Y_i$. It follows that $X_i \setminus Y_i$ is definably connected.

Finally, each definably connected component of X must be a finite union of sets of the form $X_i \setminus Y_i$, so constructible.

With this ends the proof of the claim and of Lemma 4.2. \square

Part (ii) of the theorem below is closely related to a result by Edmundo, Jones and Peatfield, (see [9, Theorem 1.1]), proved differently.

Theorem 4.4. *If \tilde{G} is semisimple and definably connected then*

(i) $\langle \tilde{G}, \cdot \rangle$ is bi-interpretable with $\langle \tilde{G}/Z(\tilde{G}), \cdot \rangle$, after naming an imaginary parameter \bar{b} from $G = \tilde{G}/Z(\tilde{G})$. The parameter can be chosen in $\text{dcl}_{\mathcal{N}}(\emptyset)$, for $\mathcal{N} = \langle M, <, \langle G, \cdot \rangle \rangle$ and also in $\text{dcl}_{\tilde{G}}^{\text{eq}}(\emptyset)$, where $\tilde{G} = \langle \tilde{G}, \cdot \rangle$.

(ii) There are finitely many real closed fields R_1, \dots, R_k in \mathcal{M} , such that the group \tilde{G} is definably isomorphic in \mathcal{M} , over parameters, to a group \tilde{G}' which is \emptyset -definable in the multi-sorted structure of the k fields. If $\tilde{G}/Z(\tilde{G})$ is simple, or if \mathcal{M} expands an ordered group then only one such field is required.

(iii) \tilde{G} has property ρ .

Proof. Consider the extension

$$Z(\tilde{G}) \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G.$$

Case I G is definably simple.

(i) By 4.2, the pure group G has property ρ and by 3.1, it is a perfect group. Clearly \tilde{G} has finite center and hence has property Z (see Theorem 2.1(2)). Therefore, by Corollary 2.5, the structure $\langle Z(\tilde{G}), \tilde{G}, G, i, \pi \rangle$ is bi-interpretable with the structure $\langle \langle G, \cdot \rangle, \langle Z(\tilde{G}), \cdot \rangle \rangle$, after naming finitely many elements in G and in $Z(\tilde{G})$. Since $\text{dcl}_{\tilde{G}}^{\text{eq}}(\emptyset)$ is infinite, any finite structure can be interpreted in G over \emptyset . It

follows that the finite group $Z(\tilde{G})$ itself can be interpreted in $\langle G, \cdot \rangle$ over \emptyset . By 1.1 every subset of $Z(\tilde{G})$ is \emptyset -interpretable in $\langle G, \cdot \rangle$.

Thus, $\langle Z(\tilde{G}), \tilde{G}, G, i, \pi \rangle$ is bi-interpretable, over parameters in G , with $\langle G, \cdot \rangle$. Finally, we note that $\langle Z(\tilde{G}), \tilde{G}, G, i, \pi \rangle$ is bi-interpretable with $\langle \tilde{G}, \cdot \rangle$, hence $\langle \tilde{G}, \cdot \rangle$ is bi-interpretable over parameters, with $\langle G, \cdot \rangle$. As was observed in Remark 2.6, the parameters which we use can be taken in $dcl_{\mathcal{N}}(\emptyset)$, where $\mathcal{N} = \langle M, <, \langle G, \cdot \rangle, \langle A, \cdot \rangle \rangle$, where A is a finite \emptyset -interpretable set in $\langle G, \cdot \rangle$.

(ii) By 1.2(1), we may assume that G is an R -semialgebraic group, defined over $R_{alg} \subseteq R$, for some \mathcal{M} -definable real closed field R . We now consider the field $\langle R, <, +, \cdot \rangle$ and then, since $dcl_R(\emptyset)$ is infinite it follows that the finite group $\langle A, \cdot \rangle$ is \emptyset -definable in R . Since G is \emptyset -definable in R it follows from (i) that $\langle \tilde{G}, \cdot \rangle$ is interpretable over the empty set in R . In particular, the group \tilde{G} is definably isomorphic in \mathcal{M} to a group H which is interpretable in the field R , over R_{alg} . By elimination of imaginaries in real closed fields, H can be chosen to be definable.

(iii) We now want to show that \tilde{G} has property ρ . If G is stable then, by 1.2(2), it is bi-interpretable, over parameters, with an algebraically closed field K . In this case, because \tilde{G} and G are bi-interpretable, \tilde{G} is definably isomorphic, over parameters, with an algebraic group H over K . By Claim 4.3, if $X \subseteq H^n$ is constructible over K then its definably connected components are constructible over K as well. Because of the bi-interpretability of G (so also of H) with K , these components are definable in $\langle H, \cdot \rangle$, possibly over parameters, so H (hence \tilde{G}) has property ρ .

If G is unstable then it is bi-interpretable with a real closed field R and therefore, by (i), \tilde{G} is also bi-interpretable with R . By (ii), we may assume that \tilde{G} is semialgebraic over R_{alg} . This implies that every semialgebraic subset of \tilde{G}^n is definable in the pure group $\langle \tilde{G}, \cdot \rangle$, possibly over parameters. In particular, every definably connected component of every definable set in the pure group \tilde{G} is semialgebraic, so definable in \tilde{G} , hence \tilde{G} has property ρ .

Case II G is semisimple.

(i) We first claim that \tilde{G} is bi-interpretable with $G = \tilde{G}/Z(\tilde{G})$ after possibly naming in G finitely many constants. For that, we need to establish the three assumptions of Theorem 2.1 (with $A = Z(\tilde{G})$):

By 1.2, G is definably isomorphic in \mathcal{M} to a product $H_1 \times \cdots \times H_k$, of definably simple groups. Each of the H_i 's is the centralizer of the other $k-1$ groups hence, by DCC, it is definable in the pure group language of G (after naming parameters). It follows from 4.2 and 4.1 that G has property ρ . By 3.1, \tilde{G} is perfect. Because $Z(\tilde{G})$ is finite, we clearly have property Z . We can now apply Theorem 2.1 exactly as in Case I.

(ii) By 1.2(1), we may assume that each H_i above is a \emptyset -definable group in a definable real closed field R_i . Let $\pi : \tilde{G} \rightarrow G$ be the quotient map and let $\tilde{H}_i = \pi^{-1}(H_i)$.

Each \tilde{H}_i is a finite central extension of H_i , and if we let $\tilde{H} = \tilde{H}_1 \times \cdots \times \tilde{H}_k$ and $\tilde{\pi} : \tilde{H} \rightarrow G$ be the natural projection, then $\tilde{\pi}$ factors through the finite extensions $\pi' : \tilde{H} \rightarrow \tilde{G}$ and $\pi : \tilde{G} \rightarrow G$.

$$\begin{array}{ccc}
\tilde{H} & \xrightarrow{\pi'} & \tilde{G} \\
& \searrow \tilde{\pi} & \swarrow \pi \\
& & G
\end{array}$$

By Case I, each \tilde{H}_i is definably isomorphic in \mathcal{M} to a \emptyset -definable group in the field R_i , hence the group \tilde{H} is definably isomorphic to a \emptyset -definable group in the multi-sorted structure, call it \mathcal{N}_{mult} , of the fields R_1, \dots, R_k . We assume that \tilde{H} is actually \emptyset -definable there. Note that the kernel of $\tilde{\pi}$ is a finite subgroup of $Z(\tilde{H}_1) \times \dots \times Z(\tilde{H}_k)$. Because $Z(\tilde{H}_i)$ is a \emptyset -definable finite set in the ordered field R_i , every subset of $Z(\tilde{H}_i)$ is \emptyset -definable in R_i and therefore $\ker(\tilde{\pi})$ is \emptyset -definable in \mathcal{N}_{mult} . It follows that \tilde{G} is \emptyset -definable in \mathcal{N}_{mult} .

In case that \mathcal{M} expands an ordered group, all the R_i 's are definably isomorphic to each other (see [21]), so we may assume that \mathcal{N}_{mult} contains actually a single sort.

(iii) We claim that \tilde{G} has property ρ and use the above setting. First we prove:

Claim \tilde{H} has property ρ .

Proof. Each \tilde{H}_i is a finite central extension of a definably simple group so by Case I, it has property ρ . Therefore, by 4.1, in order to see that \tilde{H} itself has property ρ it is sufficient to see that each \tilde{H}_i , $i = 1, \dots, k$, is definable in the pure group $\langle \tilde{H}, \cdot \rangle$, possibly with parameters. Let us see why \tilde{H}_1 is definable. First note that the centralizer of $\tilde{H}_2 \cup \dots \cup \tilde{H}_k$, call it Z_1 , is $\tilde{H}_1 \cdot Z(\tilde{H})$ (where $Z(\tilde{H})$ is finite). By DCC, Z_1 is definable in $\langle \tilde{H}, \dots \rangle$. By 3.1(v), \tilde{H}_1 is perfect, and it is easy to see that $[Z_1, Z_1] \subseteq \tilde{H}_1$. Hence, $\tilde{H}_1 = [Z_1, Z_1]_k$ for some k and this last group is clearly definable. End of Claim.

Because G is perfect and has property ρ , we can apply Theorem 2.1 to the finite central extension $\tilde{\pi} : \tilde{H} \rightarrow G$ and conclude that the pure groups G and \tilde{H} are bi-interpretable, after naming constants from G .

Let $X \subseteq \tilde{G}^n$ be a \tilde{G} -definable set and let X_1, \dots, X_k be its definably connected components, with respect to the group topology of \tilde{G} . Because \tilde{G} and $\pi' : \tilde{H} \rightarrow \tilde{G}$ are definable in $\langle \tilde{H}, \cdot \rangle$ and continuous, the set $Y = \pi'^{-1}(X)$ is definable in \tilde{H} and each $\pi'^{-1}(X_i)$ is a finite union of definably connected components of Y , hence definable in \tilde{H} (because \tilde{H} has property ρ). It follows that each X_i is definable in \tilde{H} . However, as we already saw, \tilde{H} and G are bi-interpretable and G and \tilde{G} are bi-interpretable as well, and therefore each X_i is definable in \tilde{G} (after possibly naming finite many parameters). Hence, \tilde{G} has property ρ . \square

5. PROPERTY Z

Assume now that \mathcal{M} expands a real closed field R and some neighborhood of the identity of a definable group G is contained in R^n for some n . We denote by \mathcal{G} its Lie algebra whose underlying R -vector space is the tangent space of G at 1, $T_1(G)$. We recall some facts about groups and Lie algebras, as presented in [24].

Assume that G is definably connected. To every definable subgroup $H \subseteq G$ there is an associated Lie subalgebra $\mathfrak{h} \subseteq \mathcal{G}$. The subgroup H is normal in G if and only if \mathfrak{h} is an ideal in \mathcal{G} (see [24, Claim 2.32]). For every $g \in G$, we denote by $Ad_g : T_1(G) \rightarrow T_1(G)$ the differential of the inner automorphism $a_g : x \mapsto gxg^{-1}$. If

\mathcal{G}_1 is a linear subspace of \mathcal{G} then \mathcal{G}_1 is an ideal if and only if it is invariant under Ad_g for all $g \in G$. (See Claim 2.31 there).

Assume now that H is a \vee -definable subgroup of G . By that we mean (see [25, Definition 2.1]) a subgroup of G given as a bounded directed union $\bigcup_{i \in I} U_i$ of definable subsets such that for every $i, j \in I$, $U_i \cdot U_j$ and U_i^{-1} are contained in some U_k (e.g. $H = [G, G]$). We would like to associate to H a Lie subalgebra of \mathcal{G} . For that it is convenient to first move to a sufficiently saturated extension \mathcal{M}^* of \mathcal{M} (although the associated Lie subalgebra will be defined over \mathcal{M}).

By [25, Proposition 2.3(1)], there is an $i \in I$ such that for some small definable open neighborhood $W \subseteq G$ of 1 we have $W \cap H = W \cap U_i$. Since the Lie algebra of a definable subgroup is determined by its intersection with any open neighborhood of the identity one can associate to H a Lie subalgebra $L(H) \subseteq \mathcal{G}$ using the definable $W \cap H$ instead of H (see the discussion in section 2.4 of [24]). If H is normal in G then $L(H)$ is an ideal in \mathcal{G} (indeed, because the whole analysis is local in nature, it is enough to consider H at a neighborhood of 1 and in this case the arguments work as in the definable case). Notice that because $L(H)$ is invariant under automorphisms of \mathcal{M}^* fixing \mathcal{M} , it is defined over \mathcal{M} , so we can talk about it in \mathcal{M} as well.

Recall that a subalgebra $\mathcal{A} \subseteq \mathcal{G}$ is called central if for every every $\xi \in \mathcal{A}$ and $\eta \in \mathcal{G}$, we have $[\xi, \eta] = 0$. Equivalently (see [24, Claim 2.31(1)]), for every $g \in G$, $Ad_g|_{\mathcal{A}} = id$.

Theorem 5.1. *Let G be a definably connected group. Assume that $\mathcal{G} = \mathcal{A} + \mathcal{I}$ for a central subalgebra \mathcal{A} , and an ideal $\mathcal{I} \subseteq \mathcal{G}$. Then $L([G, G]) \subseteq \mathcal{I}$.*

Proof. Since, as we pointed out above, $L([G, G])$ is defined over \mathcal{M} , we may prove the statement in an elementary extension of \mathcal{M} so we may assume that \mathcal{M} is sufficiently saturated.

We first introduce some notation. Let $\mathfrak{h} = L([G, G])$ and for $g \in G$, let $\ell_g : G \rightarrow G$ be left-multiplication by g and $r_g : G \rightarrow G$ be right-multiplication by g . For every $g \in G$ we have

$$T_g(G) = d_1(\ell_g)(T_1(G)) = d_1(r_g)(T_1(G)),$$

and similarly, for every $h \in [G, G]$ we have

$$T_h([G, G]) = d_1(\ell_h)(\mathfrak{h}) = d_1(r_h)(\mathfrak{h}).$$

It is therefore sufficient to prove, for some $h \in [G, G]$, that $d_h(r_h^{-1})(T_h([G, G])) \subseteq \mathcal{I}$.

As pointed out above, because $[G, G]$ is \vee -definable and \mathcal{M} is sufficiently saturated, there exists an n and an open neighborhood $U \subseteq G$ of 1 such that $U \cap [G, G] = U \cap [G, G]_n$. Consider the function $F_n = F_{n,G} : G^{2n} \rightarrow G$ as given earlier, by the product of n many group commutators. It is not hard to see that for sufficiently generic $\bar{g} = (g_1, \dots, g_{2n})$ in $F_n^{-1}(U \cap [G, G])$, we have

$$d_{\bar{g}}(F_n)(T_{g_1}(G) \times \cdots \times T_{g_{2n}}(G)) = T_{F_n(\bar{g})}([G, G]).$$

Using the chain rule, it is sufficient to prove that for every $\bar{g} \in G^{2n}$, we have

$$d_{\bar{g}}(r_{F_n(\bar{g})}^{-1} \circ F_n)(T_{g_1}(G) \times \cdots \times T_{g_{2n}}(G)) \subseteq \mathcal{I}.$$

We are going to prove this by induction on n . For that purpose, let us call a definable function $\alpha : G^k \rightarrow G$ good at $\bar{g} \in G^k$ if it satisfies

$$d_{\bar{g}}(r_{\alpha(\bar{g})}^{-1} \circ \alpha)(T_{g_1}(G) \times \cdots \times T_{g_k}(G)) \subseteq \mathcal{I}.$$

Claim (i) If $\alpha : G^k \rightarrow G$ is good at \bar{g} and $\alpha(\bar{g}) = 1$ then for every $h \in G$, the function $a_h \circ \alpha$ is good (recall $a_h(x) = h x h^{-1}$).

(ii) If $\alpha, \beta : G^k \rightarrow G$ are good at $\bar{g} \in G^k$ then so is $\alpha \cdot \beta$, the group product of the two.

Proof (i) This is immediate from the fact that \mathcal{I} is invariant under $d_1(a_g) = Ad_g$.

(ii) If $\mu : G \times G \rightarrow G$ is the group product then $d_{(1,1)}(\mu) = (id, id)$. Now, in the special case that $\alpha(\bar{g}) = \beta(\bar{g}) = 1$ we have, by the chain rule, $d_{\bar{g}}(\mu(\alpha, \beta)) = d_{\bar{g}}(\alpha) + d_{\bar{g}}(\beta)$, and therefore $\mu(\alpha, \beta)$ is good at \bar{g} .

More generally, if $\alpha(\bar{g}) = h_1, \beta(\bar{g}) = h_2$ then

$$\alpha(\bar{x})\beta(\bar{x})h_2^{-1}h_1^{-1} = \alpha(\bar{x})h_1^{-1}(h_1(\beta(\bar{x})h_2^{-1})h_1^{-1}),$$

and hence

$$r_{(h_1 h_2)}^{-1} \circ \mu(\alpha, \beta) = \mu(r_{h_1^{-1}} \circ \alpha, a_{h_1} \circ r_{h_2^{-1}} \circ \beta).$$

By definition, $r_{h_1^{-1}} \circ \alpha$ and $r_{h_2^{-1}} \circ \beta$ are good at \bar{g} and the two functions send \bar{g} to 1. By (i) and the special case we just did, $\mu(\alpha, \beta)$ is good at \bar{g} as well. End of Claim.

Because every F_n is a product of commutators, it is sufficient, using Claim (ii) above, to prove that $F_1(x, y) = x y x^{-1} y^{-1}$ is good at every $(g, h) \in G^2$. Because $F_1(g, h) = g h g^{-1} h^{-1}$, we need to show that $r_{h g h^{-1} g^{-1}} \circ F_1$ is good at (g, h) , or equivalently, that $\sigma(x, y) = r_{h g h^{-1} g^{-1}} \circ F_1(g x, h y)$ is good at $(1, 1)$. Rewriting $\sigma(x, y)$ we get:

$$\begin{aligned} g x h y x^{-1} g^{-1} y^{-1} h^{-1} h g h^{-1} g^{-1} &= g x g^{-1} g h (y x^{-1} g^{-1} y^{-1} g) (g h)^{-1} \\ &= a_g(x) \cdot a_{gh}(y \cdot x^{-1} \cdot a_{g^{-1}}(y^{-1})). \end{aligned}$$

The right-most expression can be re-written as

$$a_g(x) \cdot a_{gh}(y) \cdot a_{gh}(x)^{-1} \cdot a_{g^{-1}hg}(y)^{-1}.$$

We have a product of four functions, each sending 1 to 1. Taking the differential and applying the chain rule we obtain, for every $u, v \in T_1(G)$:

$$d_{(1,1)}\sigma(u, v) = Ad_g(u) + Ad_{gh}(v) - Ad_{gh}(u) - Ad_{ghg^{-1}}(v).$$

We now return to our assumptions. Every $u \in \mathcal{G}$ can be written as $u = u_1 + u_2$, where $u_1 \in \mathcal{A}$ and $u_2 \in \mathcal{I}$. Because \mathcal{A} is central we have $Ad_g(u_1) = u_1$ for every $g \in G$. Hence, $d_{(1,1)}\sigma(u_1 + u_2, v_1 + v_2)$ equals:

$$\begin{aligned} u_1 + Ad_g(u_2) + v_1 + Ad_{gh}(v_2) - u_1 - Ad_{gh}(u_2) - v_1 - Ad_{ghg^{-1}}(v_2) &= \\ = Ad_g(u_2) + Ad_{gh}(v_2) - Ad_{gh}(u_2) - Ad_{ghg^{-1}}(v_2). \end{aligned}$$

Because \mathcal{I} is invariant under every Ad_g and under $+$, the sum on the right belongs to \mathcal{I} . \square

Corollary 5.2. *Let G be a definably connected group, $A \subseteq G$ a definable central subgroup and let \mathcal{A} be the Lie algebra of A . Assume that \mathcal{G} can be written as a direct sum $\mathcal{G} = \mathcal{A} \oplus \mathcal{I}$, for some ideal \mathcal{I} . Then for every n , $A \cap [G, G]_n$ is finite.*

Proof. The Lie algebra \mathcal{A} is central in \mathcal{G} (this follows from [24, Claim 2.26 and Claim 2.31](1)). By our assumption, and Theorem 5.1 we have $L([G, G]) \subseteq \mathcal{I}$ and therefore $\mathcal{A} \cap L([G, G]) = \{0\}$. Because $A \cap [G, G]$ is a locally definable group it has a Lie Algebra of the same dimension, which equals $\mathcal{A} \cap L([G, G])$. Hence, $\dim(A \cap [G, G]) = 0$ and therefore $A \cap [G, G]_n$ is finite for every n . \square

Corollary 5.3. *Let \tilde{G} be a definably connected central extension of a semisimple group G , with $L(\tilde{G}) = \tilde{\mathcal{G}}$. Then*

(i) *For every n , the set $Z(\tilde{G}) \cap [\tilde{G}, \tilde{G}]_n$ is finite.*

(ii) *The Lie algebra of the locally definably group $[\tilde{G}, \tilde{G}]$ equals to $[\tilde{\mathcal{G}}, \tilde{\mathcal{G}}]$ and we have $\tilde{\mathcal{G}} = \mathcal{Z} \oplus L([\tilde{G}, \tilde{G}])$, where $\mathcal{Z} = L(Z(\tilde{G}))$. Moreover,*

$$L([\tilde{G}, \tilde{G}]) \simeq L(G).$$

Proof. (i) First note that $\mathcal{Z} = L(Z(\tilde{G}))$ is the center of $\tilde{\mathcal{G}}$ (see [24, Claim 2.26 and Claim 2.31](1)) and the Lie algebra of $\tilde{G}/Z(\tilde{G})$ equals $\tilde{\mathcal{G}}/\mathcal{Z}$. It follows that $\tilde{\mathcal{G}}/\mathcal{Z}$ is a semisimple Lie algebra (see Theorem 2.34 there).

By the Levi decomposition theorem for Lie algebras, $\tilde{\mathcal{G}}$ is the semi-direct product of its solvable radical and a semisimple Lie sub-algebra \mathfrak{h} . Because $\tilde{\mathcal{G}}/\mathcal{Z}$ is semisimple it follows that \mathcal{Z} is the solvable radical of $\tilde{\mathcal{G}}$. We claim that $\mathfrak{h} = [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}]$.

Indeed, for $\zeta_i = \xi_i + \eta_i \in \tilde{\mathcal{G}}$, $i = 1, 2$, and $\xi_i \in \mathcal{Z}$ and $\eta_i \in \mathfrak{h}$, we have

$$[\zeta_1, \zeta_2] = [\xi_1, \xi_2 + \eta_2] + [\eta_1, \xi_2] + [\eta_1, \eta_2] = [\eta_1, \eta_2] \in \mathfrak{h}.$$

It follows that $[\tilde{\mathcal{G}}, \tilde{\mathcal{G}}] \subseteq \mathfrak{h}$ and because \mathfrak{h} is semisimple we also have $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \subseteq [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}]$, and hence $\mathfrak{h} = [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}]$. Therefore, \mathfrak{h} is an ideal in $\tilde{\mathcal{G}}$ and we have $\tilde{\mathcal{G}} = \mathcal{Z} \oplus [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}]$.

We can now apply Corollary 5.2 and conclude that $Z(\tilde{G}) \cap [\tilde{G}, \tilde{G}]_n$ is finite for every n .

(ii) By dimension considerations, the above implies that

$$\dim L([\tilde{G}, \tilde{G}]) = \dim([\tilde{G}, \tilde{G}]) = \dim(\tilde{\mathcal{G}}) - \dim Z(\tilde{G}) = \dim[\tilde{\mathcal{G}}, \tilde{\mathcal{G}}].$$

By Theorem 5.1, $L([\mathcal{G}, \mathcal{G}]) \subseteq [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}]$, and therefore $L([\mathcal{G}, \mathcal{G}]) = [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}]$.

Again, by dimension considerations, $\dim([\tilde{G}, \tilde{G}]) = \dim G$ and hence $d\pi$ is an isomorphism of $L([\tilde{G}, \tilde{G}])$ and $L(G)$. \square

Because every definably compact definably connected group is a central extension of a semisimple group (1.2) we immediately conclude the result below. As we will later see (Corollary 6.4), this is only a first approximation to the stronger theorem about the commutator subgroup of a definably compact group.

Corollary 5.4. *Let \tilde{G} be a definably connected definably compact group. Then for every n , $Z(\tilde{G}) \cap [\tilde{G}, \tilde{G}]_n$ is finite.*

Question Is it true that for any definable group G , the Lie algebra of the derived group $[G, G]$ equals the derived Lie algebra $[\mathcal{G}, \mathcal{G}]$?

5.1. Omitting the real closed field assumption. The real closed field assumption was of course necessary for the discussion in the last section, because it involved the tangent space of \tilde{G} at 1. However, once used, this assumption can be weakened, at least in the definably compact case.

We first recall some notions: An o-minimal expansion of an ordered group is called *semi-bounded* if there is no definable bijection between bounded and unbounded intervals. There are three different possibilities for an o-minimal expansion \mathcal{M} of an ordered group (see discussion in [22, section 2.1]):

1. $Th(\mathcal{M})$ is linear, i.e. \mathcal{M} is elementarily equivalent to an ordered reduct of an ordered vector space over an ordered division ring.
2. $Th(\mathcal{M})$ is not linear and not semi-bounded, in which case there exists a definable real closed field whose domain is M .

3. \mathcal{M} is semi-bounded and $Th(\mathcal{M})$ is not linear.

We can now state the following generalization of Corollary 5.3:

Corollary 5.5. *Let \mathcal{M} be an o-minimal expansion of an ordered group and let G be an \mathcal{M} -definable, definably compact, definably connected group. Then for every $n \in \mathbb{N}$, $Z(G) \cap [G, G]_n$ is finite.*

Proof. We need to examine the three cases above. In case 1, when \mathcal{M} is elementarily equivalent to a reduct of an ordered vector space then every definable group is abelian-by-finite (indeed, by [27, Corollary 5.1], if not then a field is definable in \mathcal{M} . It is easy to see that this is impossible), so G is abelian.

In case 2, when \mathcal{M} is not linear and not semi-bounded, the structure \mathcal{M} expands a real closed field and therefore Corollary 5.3 applies. We are thus left with the semi-bounded nonlinear case. Recall the following from [22]:

If \mathcal{M} is semi-bounded and $Th(\mathcal{M})$ is not linear then we use [22, Theorem 6.1]: There exists $\mathcal{N} \succ \mathcal{M}$ and an o-minimal expansion $\widehat{\mathcal{N}}$ of \mathcal{N} (by “expansion” we mean here that every definable subset of \mathcal{N} is definable in $\widehat{\mathcal{N}}$, possibly with additional parameters) and an elementary substructure $\widehat{D} \prec \widehat{\mathcal{N}}$ such that every bounded interval in \widehat{D} is the domain of a definable real closed field. Given G , an \mathcal{M} -definable, definably connected and definably compact group, we can view G as an \mathcal{N} -definable, and therefore also $\widehat{\mathcal{N}}$ -definable, group. Hence, there exists in $\widehat{\mathcal{N}}$ a \emptyset -definable family $\mathcal{F} = \{G_s : s \in S\}$ of groups such that $G = G_{s_0}$ for some $s_0 \in S$. By [22, Lemma 7.4], we may assume that all groups G_s in the family are definably compact and definably connected and furthermore (Lemma 7.1 there) that the domain of each G_s is a bounded subset of $\widehat{\mathcal{N}}$. Given a fixed $m \in \mathbb{N}$, we now argue in \widehat{D} :

For every $s \in S(\widehat{D})$, because G_s is bounded it is contained in the I^n for some bounded interval $I \subseteq M$, and hence by assumption on \widehat{D} , its underlying set is contained in R^n , for some definable real closed field R in \widehat{D} . The field R , with all its \widehat{D} -induced structure is o-minimal, and therefore, Corollary 5.3 applies and hence $Z(G_s) \cap [G_s, G_s]_m$ is finite. Since this is true for every $s \in S(\widehat{D})$, there exists, by o-minimality, a bound $k = k(n)$ such that $\widehat{D} \models \forall s \in S |Z(G_s) \cap [G_s, G_s]_m| \leq k$. This is a first-order statement which carries over to $\widehat{\mathcal{N}}$ and therefore to \mathcal{N} as well. It follows that $Z(G) \cap [G, G]_n$ is finite. \square

Question. Is there a direct proof, avoiding the Lie algebra argument for the following result: If \tilde{G} is a definably connected central extension of a semisimple group in an arbitrary o-minimal structure then the set $Z(\tilde{G}) \cap [\tilde{G}, \tilde{G}]_n$ is finite for every n ?

6. THE MAIN RESULTS

6.1. Interpretability results.

Theorem 6.1. *Let \mathcal{M} be an o-minimal expansion of a real closed field and let $A \rightarrow \tilde{G} \rightarrow G$ be a \emptyset -definable central extension of a semisimple group G , with \tilde{G} definably connected. Then*

- (1) $\langle A, \tilde{G}, i, \pi \rangle$ is bi-interpretable, over an imaginary parameter \bar{c} , with the structure $\langle \langle G, \cdot \rangle, A \rangle$. The parameter \bar{c} names a map from a family of definably connected components of a \emptyset -definable set in $\langle G, \cdot \rangle$ onto a finite subset of A .

- (2) The exact sequence $A \rightarrow \tilde{G} \rightarrow G$ is elementarily equivalent, after naming parameters on both sides, to a semialgebraic central extension $A' \rightarrow \tilde{G}' \rightarrow G'$, defined over the real algebraic numbers, with $\dim(\tilde{G}') = \dim(\tilde{G})$.

If \tilde{G} is definably compact then, in both (1) and (2), it is sufficient to assume that \mathcal{M} expands an ordered group. In this case \tilde{G}' of (2) can be chosen definably compact as well.

Proof of Theorem 6.1:

- (1) We need only to establish the three assumptions of Theorem 2.1.

By Theorem 4.4, $\langle G, \cdot \rangle$ has property ρ with respect to the pure group structure. By Corollary 5.3, \tilde{G} has the property Z . By Claim 3.1, G is perfect.

(2) By (1), $\langle A, \tilde{G}, G, i, \pi \rangle$ is bi-interpretable with $\langle G, A \rangle$, after naming the necessary map. By 4.4(ii), G itself is definably isomorphic in \mathcal{M} to a semialgebraic group G' over the real algebraic numbers, which is clearly definably compact if G is. Therefore, in order to prove (2), it is sufficient to show: Given a finite set $C \subseteq A$, the structure $\langle A, +, C \rangle$ is elementarily equivalent to $\langle A', +, C' \rangle$ for some semialgebraic group A' over the real algebraic numbers, and a finite subset $C' \subseteq A'$. Moreover, if A is definably compact then so is A' . This last statement is exactly the content of Lemma 11.2 in the Appendix.

Assume now that \mathcal{M} expands an ordered group and that \tilde{G} is definably compact. As above, G as property ρ , and by Corollary 5.5, \tilde{G} has property Z . We now proceed as before. □

Remark As the proof of (2) above shows, the only obstacle for a definable central definably connected extension \tilde{G} of a definable semisimple group to be definably isomorphic to a semialgebraic group is the group $Z(\tilde{G})$. Hence, if $Z(\tilde{G})$ is definably isomorphic in \mathcal{M} to a semialgebraic group then so is G . We also prove below an analogue for algebraic groups:

Corollary 6.2. *If \mathcal{M} is an o-minimal expansion of an ordered group then every definably compact, definably connected group is elementarily equivalent, in the pure group language, to a compact semialgebraic (in particular real Lie) group over the real algebraic numbers.*

Proof. If G is abelian and definably connected then this follows from Lemma 11.2. Otherwise, by 1.2(4), $G/Z(G)$ is semi-simple and therefore Theorem 6.1(2) applies. □

Corollary 6.3. *Assume that \mathcal{M} is an o-minimal expansion of a real closed field R . Let $A \rightarrow \tilde{G} \rightarrow G$ be a definable central extension of a definably connected semisimple group G .*

If G is a stable group and A is definably isomorphic in \mathcal{M} to an algebraic group over $K = R(\sqrt{-1})$, then \tilde{G} is definably isomorphic in \mathcal{M} to an algebraic group over K .

Proof. Since G is stable, $G/Z(G)$ is a direct product of definably simple stable groups, which we may assume are all algebraic groups over K (see 1.2). Hence, $G/Z(G)$ is definably isomorphic to an algebraic group over K . By Theorem 4.4(1), G is definable, possibly over parameters, in the group $G/Z(G)$ and therefore it is

definably isomorphic in \mathcal{M} to an algebraic group over K . We continue as in the proof of Theorem 6.1. \square

We end this discussion with an example showing that not every definably connected group in an o-minimal structure is elementarily equivalent to a *real* Lie group which is definable in an o-minimal structure. This is a small variation of an example from [26], so we will be brief:

Example Let $\mathcal{M} = \langle R, <, +, \cdot, \exp \rangle$ be a nonstandard model theory of the real exponential field, and let $\alpha \in R$ be an element greater than all natural numbers. We define

$$G = \left\{ \begin{pmatrix} t & 0 & u \\ 0 & t^\alpha & v \\ 0 & 0 & 1 \end{pmatrix} : u, v \in M, t > 0 \right\}.$$

The group G is a solvable centerless group, and as is shown in [26, p.4], the structure $\mathcal{M}_\alpha = \langle R, <, +, \cdot, t \mapsto t^\alpha \rangle$ is interpretable in the pure group G . If G were elementarily equivalent to a definable real Lie group H in some o-minimal structure over the field of real numbers then H would interpret a structure $\mathcal{N}_\alpha \equiv \mathcal{M}_\alpha$ so the underlying field of \mathcal{N}_α is non-archimedean. However, every real closed field which would be interpretable in an o-minimal structure over the field of real numbers must be archimedean (its ordering is Dedekind complete). Contradiction.

6.2. Structural results. We can now deduce a structural result about definably compact groups in o-minimal structures.

Corollary 6.4. *Let \mathcal{M} be an o-minimal expansion of an ordered group. Let \tilde{G} be a definably compact, definably connected group. Then*

- (i) $[\tilde{G}, \tilde{G}]$ is definable, definably connected and semisimple.
- (ii) \tilde{G} equals the almost direct product of $[\tilde{G}, \tilde{G}]$ and $Z(\tilde{G})^0$. Namely, \tilde{G} is the group product of $[\tilde{G}, \tilde{G}]$ and $Z(\tilde{G})^0$ and the intersection of the two groups is finite. In particular, $\tilde{G} \simeq (Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}]) / F$, for a finite central subgroup $F \subseteq Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}]$.

Proof. (i) By Corollary 6.2, \tilde{G} is elementarily equivalent to a compact real Lie group H . By classical Lie group theory, $[H, H]$ is a closed connected semisimple subgroup of H (indeed, this can be found, for example, in [19, Chapter 5.2, Theorem 4]).

By topological compactness, there exists a k such that $[H, H]_k = H$, i.e. the set $[H, H]_k$ is already a subgroup of H . It follows that the same is true for \tilde{G} hence $[\tilde{G}, \tilde{G}] = [\tilde{G}, \tilde{G}]_k$ is definable. The group $[\tilde{G}, \tilde{G}]$ is definably connected as the continuous image of the definably connected group \tilde{G} .

(ii) Because the intersection of $Z(\tilde{G})$ and $[\tilde{G}, \tilde{G}]$ is 0-dimensional (Corollary 5.3), it must be finite. The final clause is immediate from the fact that $[\tilde{G}, \tilde{G}]$ acts trivially on $Z(\tilde{G})^0$, and hence the map $(g, h) \mapsto gh$ from $Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}]$ to \tilde{G} is a homomorphism with finite kernel. \square

Remark Actually, by a theorem of Goto, [15, Theorem 6.55], every element of $[H, H]$ is a commutator (i.e, $[H, H] = [H, H]_1$) hence the same is true in every definably compact group.

Corollary 6.5. *Let \mathcal{M} be an o-minimal expansion of an ordered group and assume that \tilde{G} is definably connected with $\tilde{G}/Z(\tilde{G})$ semisimple and definably compact. Then*

(i) $[\tilde{G}, \tilde{G}]$ is definable.

(ii) $\tilde{G} \simeq (Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}])/F$, for a finite central subgroup $F \subseteq Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}]$

Proof. By 6.1 (2), the assumption implies that $Z(G)^0 \rightarrow \tilde{G} \rightarrow G$ is elementarily equivalent to a semialgebraic extension $Z_0 \rightarrow \tilde{G}_0 \rightarrow G_0$ over the real numbers, with G_0 a compact connected semisimple real Lie group. By Corollary 5.3 (ii), we have $L(\tilde{G}_0) \simeq L(Z_0) \oplus [L(\tilde{G}_0), L(\tilde{G}_0)]$, and the Lie algebra of the locally definable group $[\tilde{G}_0, \tilde{G}_0]$ equals $[L(\tilde{G}_0), L(\tilde{G}_0)]$. Moreover, $[L(\tilde{G}_0), L(\tilde{G}_0)]$ is isomorphic to $L(G_0)$.

We first claim that $[\tilde{G}_0, \tilde{G}_0]$ is a compact subgroup of \tilde{G}_0 .

We recall the following definition: A Lie algebra over \mathbb{R} is called *compact* if it admits an invariant positive definite scalar product. Clearly, a subalgebra of a compact Lie algebra is also compact and if a Lie algebra is commutative then it is compact (any positive definite scalar product will do). Furthermore, the direct sum of two compact Lie algebras is compact as well. Finally, the Lie algebra of a compact real Lie group is compact, see [19, p. 228, 12].

Because G_0 is compact, its Lie algebra is compact. Hence $L(G_0) = [L(\tilde{G}_0), L(\tilde{G}_0)]$ is a compact Lie algebra as well. Because $L(Z_0)$ is abelian it is also compact. It follows that $L(\tilde{G}_0)$ is a compact Lie algebra as well. We now apply a theorem about connected Lie groups with compact Lie algebras (see [19, p. 242, Theorem 5]) and conclude that, as a Lie group, $\tilde{G}_0 = B \times C \cdot [\tilde{G}_0, \tilde{G}_0]$, for Lie subgroups $B, C \subseteq Z(\tilde{G})^0$ (B torsion-free and C compact), and with $C \cdot [\tilde{G}_0, \tilde{G}_0]$ a compact Lie subgroup, which we denote by H (these groups are not claimed to be definable).

Since $H/[\tilde{G}_0, \tilde{G}_0]$ is abelian and its Lie algebra equals the abelian algebra $L(H)/L([\tilde{G}_0, \tilde{G}_0])$, the ideal $L([\tilde{G}_0, \tilde{G}_0])$ is a maximal semisimple subalgebra of $L(H)$. It follows, using Levi decomposition theorem, that $L(H) = L(C) \oplus L([\tilde{G}_0, \tilde{G}_0])$. As we saw before, since H is compact, the group $[H, H]$ is a compact subgroup of H , which in this case must equal $[\tilde{G}_0, \tilde{G}_0]$. Thus, $[\tilde{G}_0, \tilde{G}_0]$ is a compact subgroup of \tilde{G}_0 .

It follows that there is a k (actually, as remarked above, $k = 1$), such that $[\tilde{G}_0, \tilde{G}_0]_k = [\tilde{G}_0, \tilde{G}_0]$. By elementary equivalence, $[\tilde{G}, \tilde{G}]_k = [\tilde{G}, \tilde{G}]$, hence this group is definable.

As we already saw, see 5.3, the intersection of $[\tilde{G}, \tilde{G}]$ and $Z(\tilde{G})$ has zero dimension and therefore is finite. Hence, $\tilde{G} \simeq (Z(\tilde{G})^0 \times [\tilde{G}, \tilde{G}])/F$ for a finite central subgroup F . \square

As is shown in [3, Example 3.1.7], the compactness assumption in the above result is necessary. Namely, there are semi-algebraic central extensions of definably simple groups in which the derived group intersects the center in an infinite discrete set, and therefore cannot be definable in any o-minimal structure.

7. THE CONNECTION TO PILLAY'S CONJECTURE

Corollary 6.2 gives a strong connection between definably compact, definably connected groups in o-minimal structures and compact real Lie groups.

Pillay's Conjecture (now a theorem, for expansions of ordered groups, see [16] for the main case) suggests another such connection to compact real Lie groups:

Let G be a definable group in a sufficiently saturated o-minimal structure. Then the intersection of all type-definable (possibly with parameters) subgroups of bounded index, call it G^{00} , is again type-definable of bounded index, and G/G^{00} , equipped with the logic topology, is isomorphic to a compact real Lie group. Finally, if G is definably compact then the topological dimension of G/G^{00} equals the o-minimal dimension of G .

Our goal is to prove, in the definably connected case, that the pure groups G and G/G^{00} are elementarily equivalent. More precisely, we will prove:

Theorem 7.1. *Let \mathcal{M} be a sufficiently saturated o-minimal expansion of an ordered group and let G be a definably compact, definably connected definable group. Then*

$$\langle G, \cdot \rangle \equiv \langle G/G^{00}, \cdot \rangle.$$

Moreover, the map $\pi : G \rightarrow G/G^{00}$ “splits elementarily”, namely there exists an elementary embedding (with respect to the group structure) $\sigma : G/G^{00} \rightarrow G$ which is also a section for π .

By Corollary 6.4, definably compact groups can be analyzed using abelian and semisimple subgroups. We first handle the abelian case. Because we are going later on to treat definable groups which are not necessarily connected, we will need to work in a more general setting of abelian groups together with finitely many automorphisms.

7.1. Definable abelian groups with an additional abelian structure. Let A be an abelian group definable in an o-minimal structure \mathcal{M} , which we assume to expand an ordered group. We denote by \mathbf{A}_{ab} the expansion of the group A by all \mathcal{M} -definable subgroups of A^n , $n \in \mathbb{N}$, and let L_{ab} be the associated language (note that if A is definably compact and abelian then, by [27, Cor. 5.2], every \mathcal{M} -definable subgroup of A is actually definable in \mathcal{M} over the same parameters defining A). For B a subgroup of A we let \mathbf{B}_{ab} be the associated L_{ab} -substructure of \mathbf{A}_{ab} .

In the appendix we treat the general (not necessarily o-minimal) such situation and observe, using known results:

Fact 7.2. *Let A be an abelian definable group in an o-minimal structure, then*

- (1) *The structure \mathbf{A}_{ab} eliminates quantifiers.*
- (2) *Assume that $B \leq A$ is an arbitrary subgroup of A .
Then $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$ if and only if the following hold:*
 - (i) *For every \emptyset -definable (in \mathbf{A}_{ab}) subgroup $S \leq A^{n+k}$ and $b \in B^k$,*

$$S(B^n, b) \neq \emptyset \Leftrightarrow S(A^n, b) \neq \emptyset$$

and (ii) For every \emptyset -definable (in \mathbf{A}_{ab}) subgroups $S_1 \leq S_2 \leq A^n$,

$$[S_2 : S_1] = [S_2 \cap B^n : S_1 \cap B^n],$$

with the meaning that if this index is infinite on one side then it is infinite on the other.

- (3) *If $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$, for a subgroup B of A , then there exists a surjective group homomorphism $\phi : A \rightarrow B$ which is the identity map on B and in addition sends every \emptyset -definable group $S \subseteq A^n$ onto $S \cap B^n$. (We call such a map ϕ a homomorphic retract).*

Lemma 7.3. *Let A be a definably compact, definably connected abelian group, in an \mathcal{O} -minimal expansion \mathcal{M} of an ordered group and let \mathbf{A}_{ab} be as above.*

- (1) *Assume that $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$ for a subgroup B of A , $\phi : A \rightarrow B$ a homomorphic retract and let $A_1 = \ker \phi$. Then $(\mathbf{Tor}(\mathbf{A}) + \mathbf{A}_1)_{ab} \prec \mathbf{A}_{ab}$.*
- (2) *If \mathbf{A}_{ab} is sufficiently saturated then $(\mathbf{Tor}(\mathbf{A}) + \mathbf{A}^{00})_{ab} \prec \mathbf{A}_{ab}$.*

Note that if we take, in (1) above, $B = A$ and ϕ the identity map then the lemma implies in particular that $\mathbf{Tor}(\mathbf{A})_{ab} \prec \mathbf{A}_{ab}$.

Proof. (1) We need to see that $Tor(A) + A_1$ satisfies the two requirements of 7.2(2). Note that A has finitely many torsion elements of every order (by Strzebonski's work, see [20, Theorem 5.1]), so since it is definably connected it is also divisible. It follows that B is divisible and contains all torsion elements of A . Therefore, since $A = B \oplus A_1$, the group A_1 is divisible as well.

Clause (i): Let S be an \mathcal{M} -definable subgroup of A^{n+k} and assume that $(b + a_1, c) \in S$ for some $b \in B^n, a_1 \in A_1^n$ and $c \in (Tor(A) + A_1)^k$. We want to show that there is $a \in (Tor(A) + A_1)^n$ such that $(a, c) \in S$.

We write $c = c_1 + c_2$, for $c_1 \in Tor(A)^k$ and $c_2 \in A_1^k$. If $mc_1 = 0$ then

$$m(b + a_1, c) = m(b + a_1, c_1 + c_2) = (mb + ma_1, mc_2) \in S,$$

with $(ma_1, mc_2) \in A_1^{n+k}$. Because ϕ is a retract, we have $(mb, 0) \in S$. Now, if k is the index of $S(A^n, 0)^0$ in $S(A^n, 0)$ then $(kmb, 0) \in S(A^n, 0)^0$. Because this last group is divisible, there exists $(a, 0) \in S(A^n, 0)^0$ such that $kma = kmb$ and therefore $b - a \in Tor(A)^n$. Finally, we have $(b + a_1, c) - (a, 0) = (b - a + a_1, c) \in S$, with $b - a + a_1 \in (Tor(A) + A_1)^n$, as needed.

Clause (ii): We will actually prove a stronger statement than needed here:

(*) *If $C \subseteq A^n$ is a divisible subgroup containing $Tor(A)^n$ then for every $S_1 \subseteq S_2 \subseteq A^n$ definable groups,*

$$[S_2 : S_1] = [S_2 \cap C : S_1 \cap C]$$

(and both are infinite if one of them is).

We clearly have $[S_2 \cap C : S_1 \cap C] \leq [S_2 : S_1]$, so we only need the opposite inequality.

Assume first that S_1 is definably connected, hence divisible. It follows that every torsion element of the group S_2/S_1 (i.e. a coset of S_1) contains a torsion element of S_2 and therefore an element of $S_2 \cap C$. Hence, we have an injective map from $Tor(S_2/S_1)$ into $S_2 \cap C/S_1 \cap C$ and hence $|Tor(S_2/S_1)| \leq [S_2 \cap C : S_1 \cap C]$.

If $[S_2 : S_1]$ is infinite then $\dim S_1 < \dim S_2$ and therefore S_2/S_1 is a definably compact group of positive dimension. It follows from [10] that $Tor(S_2/S_1)$ is infinite and therefore, by the inequality above, so is $[S_2 \cap C : S_1 \cap C]$. If S_2/S_1 is finite then all its elements are torsion and therefore, by the same inequality we have

$$S_2/S_1 = Tor(S_2/S_1) = [S_2 \cap C : S_1 \cap C].$$

If S_1 is not definably connected then we apply the above argument first to $[S_2 : S_1^0]$ and $[S_1 : S_1^0]$ and then conclude the result for $[S_2 : S_1]$.

(2) Since A^{00} is divisible we only need, by (*) above, to see that Clause (i) holds for $Tor(A) + A^{00}$.

Let $S \subseteq A^{n+k}$ be a \mathcal{M} -definable group, $\pi_2 : S \rightarrow A^k$ the projection map onto the second set of coordinates and let $S_1 = \pi_2(S)$. As we saw in [16, see claim and preceding discussion on p. 587], we have

$$S^{00} = (A^{n+k})^{00} \cap S = (A^{00})^{n+k} \cap S$$

and

$$\pi_2(S^{00}) = S_1^{00} = (A^{00})^k \cap S_1.$$

Assume now that $(a, c_1 + c_2) \in S$ for $a \in A$, $c_1 \in \text{Tor}(A)$, $c_2 \in A^{00}$. If $mc_1 = 0$ then $(ma, mc_2) \in S$, with $mc_2 \in S_1^{00}$. Because $S_1^{00} = \pi_2(S^{00})$ there exists $e_1 \in (A^{00})^n$ such that $(e_1, mc_2) \in S$. Moreover, because A^{00} is divisible, we have $e_1 = me$ for some $e \in (A^{00})^n$. It follows that $(ma, mc_2) - (me, mc_2) = (ma - me, 0) \in S$.

If we let k be the index of $S(A^n, 0)^0$ in $S(A^n, 0)$ then $(kma - kme, 0) \in S(A^n, 0)^0$ and there exists $(d, 0) \in S$ such that $kmd = kma - kme$. In particular, $(a - e) - d \in \text{Tor}(A)^n$. We now have $(a, c_1 + c_2) - (d, 0) = (a - d, c_1 + c_2) \in S$, with $a - d \in (\text{Tor}(A)^n + A^{00})^n$, as needed. \square

By considering the special case of a compact real Lie group definable in the o-minimal structure \mathbb{R}_{an} (or by a direct modified version of the above proof) we also have:

Lemma 7.4. *Let B be a connected, compact abelian real Lie group and let \mathbf{B}_{an} be the expansion of $(B, +)$ by adding a predicate for every compact Lie subgroup of B^n . Then*

$$\mathbf{Tor}(\mathbf{B})_{an} \prec \mathbf{B}_{an}.$$

We can now state the main result in the abelian case:

Theorem 7.5. *Let A be a \emptyset -definable, definably compact, definably connected abelian group in a sufficiently saturated o-minimal expansion \mathcal{M} of an ordered group. Endow $B = A/A^{00}$ with an L_{ab} -structure by interpreting R_S , for every \emptyset -definable subgroup $S \subseteq A^n$, as the group $\pi(S) \subseteq B^n$ (where $\pi : A \rightarrow A/A^{00}$ is the projection map). Let \mathbf{B}_{ab} be the induced structure on B . Then,*

- (1) $\mathbf{B}_{ab} \equiv \mathbf{A}_{ab}$. Moreover, there is an L_{ab} -elementary embedding $\sigma : B \rightarrow A$ which is a section for π .
- (2) The structure \mathbf{B}_{ab} is a reduct of \mathbf{B}_{an} above.

Proof. (1) We start with the structure $T = (\mathbf{Tor}(\mathbf{A}) + \mathbf{A}^{00})_{ab}$, which, by 7.3 (2), is an elementary substructure of \mathbf{A}_{ab} . By 7.2(3), there exists a homomorphic retract $\phi : A \rightarrow T$ and if we let $A_1 = \ker \phi$ then, by 7.3(1), the structure $\mathbf{C}_{ab} = (\text{Tor}(A) + A_1)_{ab}$ is also elementary in \mathbf{A}_{ab} .

It is left to see that the restriction of π to \mathbf{C}_{ab} induces an isomorphism of \mathbf{C}_{ab} and \mathbf{B}_{ab} .

Let $S \subseteq A^n$ be an \mathcal{M} -definable group and $c \in C^n$. We need to see that $\pi(c) \in \pi(S)$ if and only if $c \in S$. Write $c = a + a_1$ for $a \in \text{Tor}(A)^n$, $a_1 \in A_1^n$, and assume that $\pi(a + a_1) \in \pi(S)$. It follows ($A^{00} \subseteq \ker \pi$) that for some $b \in (A^{00})^n$ we have $a + b + a_1 \in S$. Because A_1 is the kernel of the retract ϕ , we have $a + b \in S$. Because a is a torsion element, there exists an m such that

$$ma + mb = mb \in S \cap (A^{00})^n = S^{00}.$$

Because S^{00} is divisible, there exists $b_1 \in S^{00}$ such that $mb_1 = mb$ and therefore $b - b_1$ is a torsion element. However, $b - b_1$ belongs to the torsion-free group $(A^{00})^n$,

hence $b = b_1$ and $b \in S$. We can therefore conclude that $c = a + a_1 \in S$, and therefore $\phi|_C$ is an isomorphism of \mathbf{C}_{ab} and \mathbf{B}_{ab} . The inverse map $\sigma : B \rightarrow C$ is an elementary embedding of \mathbf{B}_{ab} into \mathbf{A}_{ab} .

(2) The image under the projection map of every definable $X \subseteq A^n$ is closed in the Euclidean (equivalently, Logic) topology on $(A/A^{00})^n$ (because $X + A^{00}$ is type-definable). Since every closed subgroup of Lie group is itself a Lie subgroup (see [18, 2.3.6]), it follows that for every definable $S \leq A^n$, $\pi(S)$ is a Lie subgroup of B^n . Therefore, \mathbf{B}_{ab} is a reduct of \mathbf{B}_{an} . \square

Remark. Note that for all the results above we did not require the full theorem of Edmundo-Otero about the structure of $Tor(A)$ for a definably compact abelian group A . We only needed the weaker statement that every definably compact infinite group has infinitely many torsion elements, together with the fact that for every n , the group of n -torsion is finite ([31]). However, without the stronger result we will not be able to conclude that $\dim(A/A^{00}) = \dim A$.

Remark. It is not hard to see that \mathbf{B}_{ab} is ω -saturated, hence it and \mathbf{A}_{ab} will actually be $L_{\infty, \omega}$ -equivalent, improving 7.5(1).

7.2. The general case. We prove Theorem 7.1 in several steps.

For any definably compact, definably connected K , we denote by \widehat{K} the group K/K^{00} . For $a \in K$, we let $\hat{a} = \pi(a) \in \widehat{K}$. Note that the restriction of π_K to every finite subgroup of K is an isomorphism, because K^{00} is torsion-free (see [2, Theorem 4.6]).

Claim I If A is abelian, definably connected and definably compact then for every finite subgroup $A_1 \subseteq A$,

$$\langle A, \cdot, \{a : a \in A_1\} \rangle \equiv \langle \widehat{A}, \cdot, \{\hat{a} : a \in A_1\} \rangle.$$

Moreover, there exists an elementary section $\sigma_A : \widehat{A} \rightarrow A$ with $\sigma_A(\hat{a}) = a$ for every $a \in A_1$.

Proof. Because A_1 is \mathcal{M} -definable and A^{00} is torsion free, this is almost immediate from 7.5. We only need to notice that since all elements of A_1 are torsion elements, the elementary embedding of \widehat{A} into A necessarily sends every $\hat{a} \in \widehat{A}$, with $a \in A_1$, to the element a .

Claim II Given H definably connected, definably compact and semisimple, for every finite central subgroup $H_1 \subseteq H$,

$$\langle H, \cdot, \{a : a \in H_1\} \rangle \equiv \langle \widehat{H}, \cdot, \{\hat{a} : a \in H_1\} \rangle.$$

Moreover, there exists an elementary embedding $\sigma_H : \widehat{H} \rightarrow H$ which is a section for π , such that $\sigma_H(\hat{h}) = h$ for every $h \in H_1$.

Proof. By 4.4(ii), using the fact that \mathcal{M} expands an ordered group, we may assume that H is a semialgebraic group definable over the real algebraic numbers (note the

parameters we used for the identification of H with a semialgebraic group do not affect H^{00}). Hence, every element of H_1 is a tuple of real algebraic numbers.

In this case (which was sometimes called “good reduction” (see [23, fact 4.1]) H^{00} is the set of elements which are infinitesimally close to the identity, in the sense of the field of reals, and $H(\mathbb{R})$ can be identified with \hat{H} . This identification gives an elementary embedding $\sigma_H : \hat{H} \rightarrow H$ which is also a section for $\pi : H \rightarrow \hat{H}$. In particular, $H = H^{00} \rtimes \sigma_H(\hat{H})$. Because σ_H is elementary, for every $\hat{h} \in \hat{H}_1$, the element $\sigma_H(\hat{h})$ is a central torsion element of H and we have $\sigma_H(\hat{h})h^{-1} \in H^{00}$ (since $\pi\sigma_H = id$). However H^{00} is torsion-free (see [2, Theorem 4.6]) and therefore $\sigma_H(\hat{h}) = h$.

Claim III If $G = (A \times H)/F$, for A definably connected, definably compact abelian, H definably connected, definably compact semisimple and F a finite central subgroup of $A \times H$, then $\langle G, \cdot \rangle \equiv \langle \hat{G}, \cdot \rangle$. Moreover, there exists an elementary embedding $\sigma_G : \hat{G} \rightarrow G$ which is a section for π .

Proof. Since F is finite it is contained in $A_1 \times H_1$ for some finite groups $A_1 \subseteq A$ and $H_1 \subseteq H$. It is easy to see that $\widehat{A \times H} \simeq \hat{A} \times \hat{H}$. By step I and step II, we have an elementary section:

$$\sigma : \langle \hat{A}, \cdot, \{\hat{a} : \hat{a} \in \hat{A}_1\} \rangle \sqcup \langle \hat{H}, \cdot, \{\hat{h} \in \hat{H}_1\} \rangle \rightarrow \langle A, \cdot, \{a \in A_1\} \rangle \sqcup \langle H, \cdot, \{h \in H_1\} \rangle$$

sending each \hat{a} and \hat{h} to a and h respectively. It follows (see the discussion in the proof of Claim 4.1) that we have elementary section

$$\sigma_1 : \langle \hat{A} \times \hat{H}, \cdot, \{(\hat{a}, \hat{h}) \in \hat{A}_1 \times \hat{H}_1\} \rangle \rightarrow \langle A \times H, \cdot, \{(a, h) \in A_1 \times H_1\} \rangle,$$

and hence also

$$\sigma_2 : \langle \widehat{A \times H}, \cdot, \{\hat{g} \in \hat{F}\} \rangle \simeq \langle \hat{A} \times \hat{H}, \cdot, \{\hat{g} \in \hat{F}\} \rangle \xrightarrow{\sigma_1} \langle A \times H, \cdot, \{g \in F\} \rangle.$$

This last section sends each $\hat{g} \in \hat{F}$ to $g \in F$.

In order to complete the proof of Claim III, it is therefore sufficient to prove the following general fact (with K now playing the role of $A \times H$):

Fact 7.6. *Let K be a definably connected, definably compact group and $F \subseteq K$ a finite central subgroup. Assume that $\sigma_K : \hat{K} \rightarrow K$ is an elementary section of $\pi_K : K \rightarrow \hat{K}$ which, for every $g \in F$ sends $\hat{g} \in \hat{F}$ to $g \in F$. Then the map $\sigma_{K/F}$ which sends the element $(gF)(K/F)^{00}$ of $\widehat{K/F}$ to $(\sigma_K(gK^{00}))F \in K/F$ is an elementary section for $\pi_{K/F} : K/F \rightarrow \widehat{K/F}$.*

Proof. By our assumption, if we expand the group language of K to include a constant for every element of F , the map $\sigma_K : \hat{K} \rightarrow K$ is still an elementary section. Since K/F , \widehat{K}/\hat{F} , are interpretable in $\langle K, \cdot \rangle$ and $\langle \hat{K}, \cdot \rangle$ in this expanded language using the same formula, it follows that the induced map $\sigma : \widehat{K}/\hat{F} \rightarrow K/F$, defined by

$$(gK^{00})\hat{F} \mapsto (\sigma_K(gK^{00}))F$$

is elementary as well.

Next, note that the map

$$\sigma' : (gF)(K/F)^{00} \mapsto (gK^{00})\hat{F}$$

gives an isomorphism of $\widehat{K/F}$ and \widehat{K}/\widehat{F} (indeed, this follows from the exactness of the functor $K \mapsto \widehat{K}$, see [2, Theorem 5.2]).

The composition of σ and σ' gives the desired $\sigma_{K/F}$. \square

Proof of Theorem 7.1: By 6.4 and our earlier analysis, the group $H = [G, G]$ is definable, semisimple and $Z(G) \cap H$ is finite. We have $G \simeq (Z(G) \times H)/F_1$, for a finite central subgroup $F_1 \subseteq Z(G) \times H$. We write $A = Z(G)^0$. Because G is definably connected, H is definably connected as well. Hence, we also have $G \simeq (A \times H)/F$, for a finite central subgroup $F \subseteq A \times H$. Let $A_1 \subseteq A$, $H_1 \subseteq H$ be finite subgroups with $F \subseteq A_1 \times H_1$.

By Claim I above, there is a map $\sigma_A : \widehat{A} \rightarrow A$, from which is elementary for $\langle \widehat{A}, \cdot, \{\hat{a} : a \in A_1\} \rangle$ and $\langle A, \cdot, \{a \in A_1\} \rangle$. By Claim II, there is an elementary map $\sigma_H : \widehat{H} \rightarrow H$, elementary for $\langle \widehat{H}, \cdot, \{\hat{a} : a \in H_1\} \rangle$ and $\langle H, \cdot, \{a \in H_1\} \rangle$.

It follows that the pair $(\sigma_A, \sigma_H) : \widehat{A} \times \widehat{H} \rightarrow A \times H$ is an elementary section for the groups $\widehat{A} \times \widehat{H}$ and $A \times H$, expanded by names for $A_1 \times H_1$ (so also for $F \subseteq A_1 \times H_1$), which in particular sends every (\hat{a}, \hat{h}) to (a, h) .

By Fact 7.6, we get an elementary section from $\widehat{A \times H}/F$ to $(A \times H)/F$. These last two groups are just \widehat{G} and G , respectively. \square

8. FINITE EXTENSIONS OF O-MINIMAL GROUPS

In this section we consider finite (but not necessarily central) extensions of arbitrary groups definable in o-minimal structure. Finite extensions of groups in o-minimal structures are studied by Edmundo, Jones and Peatfield in [9]. The following was shown there, using universal covers (see Proposition 2.10): If G is a definable, definably connected group in an o-minimal structure \mathcal{M} expanding a real closed field, and if $\pi : \tilde{G} \rightarrow G$ is any finite definable extension of G , defined possibly in an o-minimal expansion \mathcal{N} of \mathcal{M} , then \tilde{G} is definably isomorphic in \mathcal{N} to a group definable in \mathcal{M} . This implies for example that if G is semialgebraic then so is every finite extension of G which is definable in an o-minimal expansion.

In this section we give two different proofs for similar results about the interpretability of finite extensions of definable groups and arbitrary topological covers of definable groups over the field of real numbers. Although the two main results, Theorem 8.3 and Theorem 8.4 overlap in the case of finite covers, the assumptions and techniques are different so we include both.

We first need the following fact about the structure of arbitrary definably connected groups in o-minimal structures. For G a group and $n \in \mathbb{N}$ we let $\sigma_n : G \rightarrow G$ be the map $\sigma_n(g) = g^n$.

Lemma 8.1. *Let G be a definably connected group in an o-minimal structure. Then,*

- (i) *The group $G/[G, G]$ is divisible, namely, for every $n \geq 1$, $\sigma_n(G) \cdot [G, G] = G$. In fact, there exists $k \in \mathbb{N}$ such that $\sigma_n(G) \cdot [G, G]_k = G$.*
- (ii) *For every n , let $\langle \sigma_n(G) \rangle$ be the group generated by all elements g^n , $g \in G$. Then $G = \langle \sigma_n(G) \rangle$. In fact, there is $k \in \mathbb{N}$ such that $G = \sigma_n(G) \cdots \sigma_n(G)$ (k -times).*

Proof. (i) We use induction on $\dim G$. If $\dim G = 1$ then G is abelian (see [20, Proposition 3.1]) and therefore divisible. We consider the general case.

Assume first that G has an infinite definable normal abelian subgroup A . By induction, G/A^0 satisfies the lemma and therefore, for every $n \in \mathbb{N}$, $A^0 \sigma_n(G)[G, G] =$

G . Because A^0 is divisible, it is contained in $\sigma_n(G)$ and therefore $\sigma_n(G)[G, G] = G$, as needed.

If G has no infinite definable normal abelian subgroup then G is semisimple and therefore, by Claim 3.1, we have $[G, G] = G$.

For the last clause of (i), we may work in a sufficiently saturated structure, where the existence of such a k is clear. Once proved there, the same k works for G in any structure.

(ii) As before we may work in a sufficiently saturated structure. For G abelian the result is clear since it is divisible.

Assume that G has an infinite definable normal abelian subgroup A . By induction on dimension we have $G/A^0 = \langle \sigma_n(G/A^0) \rangle$, which implies that $G = A^0 \langle \sigma_n(G) \rangle$. However, since A^0 is divisible it is contained already in $\sigma_n(G)$ and hence $G = \langle \sigma_n(G) \rangle$.

If G has no infinite definable normal abelian subgroup then it is semisimple. Let us see why the theorem is indeed true in this case.

We first assume that G is definably simple. If G is not definably compact then it is abstractly simple (see 1.2(5)). The group $\langle \sigma_n(G) \rangle$ is clearly invariant under all automorphisms of G hence normal, so $G = \langle \sigma_n(G) \rangle$. If G is definably compact, then by 1.2 it is elementarily equivalent to a simple compact real Lie group H . By simplicity, $H = \langle \sigma_n(H) \rangle = \bigcup_{k=1}^{\infty} \sigma_n(H) \cdots \sigma_n(H)$ (k -times). It follows from compactness that for some k we have $H = \sigma_n(H) \cdots \sigma_n(H)$ (k -times). This implies that the same is true for G .

If G is semisimple then $Z(G)$ is finite and we have $G/Z(G) = H_1 \times \cdots \times H_r$, for H_i definably simple. By the above, each H_i satisfies $H_i = \langle \sigma_n(H_i) \rangle$, and hence we have $G = Z(G) \langle \sigma_n(G) \rangle$, so $\langle \sigma_n(G) \rangle$ has finite index in G . However, $\langle \sigma_n(G) \rangle$ is a countable union of definable sets and therefore it follows that G is a countable union of such sets. Because of saturation, this implies that $\langle \sigma_n(G) \rangle$ is actually definable (in finitely many steps) and by the definable connectedness of G we have $G = \langle \sigma_n(G) \rangle$. \square

We prove:

Theorem 8.2. *Let \mathcal{M} be an arbitrary structure, sufficiently saturated, and let \mathcal{R} be a definable o -minimal structure in \mathcal{M} . Let G be an \mathcal{R} -definable group, \tilde{G} an \mathcal{M} -definable group and let $\pi : \tilde{G} \rightarrow G$ be an \mathcal{M} -definable surjective homomorphism with finite kernel N .*

Then \tilde{G} is internal to G in the reduct containing $\langle \tilde{G}, \cdot \rangle$, $\langle G, \cdot \rangle$, π and a predicate for G^0 the definably connected component of G (denote this reduct by \mathcal{M}'). Namely, there is a definable map (possibly with parameters) in this reduct sending G^k , for some k , onto \tilde{G} .

said differently, \tilde{G} is in the \mathcal{M}' -definable closure of G and a finite subset of \tilde{G} .

Proof. We may assume that G is definably connected (in the sense of \mathcal{R}). Indeed, since N is finite, $\pi^{-1}(G^0)$ has finite index in \tilde{G} , so if $F \subseteq G$ is a finite set such that $G = FG^0$ then \tilde{G} is in the \mathcal{M}' -definable closure of $\pi^{-1}(G^0)$ and the finite set $\pi^{-1}(F)$. It follows that if $\pi^{-1}(G^0)$ is \mathcal{M}' -internal to G then so is \tilde{G} .

We may also assume that \tilde{G} has no \mathcal{M}' -definable subgroups of finite index. Indeed, if $\tilde{G}_1 \subseteq \tilde{G}$ is definable of finite index then $\pi(\tilde{G}_1)$ has finite index in G , and therefore (we assume G is definably connected) $\pi(\tilde{G}_1) = G$. This in turn implies that N is not contained in \tilde{G}_1 , and therefore $|N \cap \tilde{G}_1| < |N|$. Using induction on $|N|$ we can now finish using the same argument as in the previous paragraph.

The assumption that \tilde{G} has no \mathcal{M}' -definable subgroups of finite index implies that N is central in G . As we will show, under these assumptions, \tilde{G} is in the definable closure of G and N .

Let $n = |N|$.

For $k \in \mathbb{N}$, let f_k be the term $f_k(x_1, \dots, x_k) = x_1^n \cdots x_k^n$, and for a group H , let $f_{k,H} : H^k \rightarrow H$ be the evaluation of the term in H .

Let $\pi : \tilde{G}^k \rightarrow G^k$ be the projection map in each of the coordinates. Similarly to Claim 2.2, we claim that for $\bar{g}_1, \bar{g}_2 \in \tilde{G}^k$, if $\pi(\bar{g}_1) = \pi(\bar{g}_2)$ then $f_{k,\tilde{G}}(\bar{g}_1) = f_{k,\tilde{G}}(\bar{g}_2)$ (we use the fact that N is central and for every $h \in N$ we have $h^n = 1h$).

It now follows that there is an \mathcal{M}' -definable surjective map $h_k : G^k \rightarrow \sigma_n(\tilde{G}) \cdots \sigma_n(\tilde{G})$ (k -times) such that $f_{k,\tilde{G}}$ factors through π and h_k .

By Theorem 8.1, we may choose k such that $G = \sigma_n(G) \cdots \sigma_n(G)$ (k -times). Said differently, the map $f_{k,G} : G^k \rightarrow G$ is surjective. It easily follows that $\tilde{G} = Nh_k(G^k)$. \square

Theorem 8.3. *Let \mathcal{M} be an o -minimal structure and assume that $1 \rightarrow N \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ is an \mathcal{M} -definable extension with N finite and \tilde{G} definably connected.*

Let \mathbf{G} be some expansion of $\langle G, \cdot \rangle$ with property ρ .

Then $1 \rightarrow N \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ is definably isomorphic in \mathcal{M} to an extension $1 \rightarrow N_1 \rightarrow \tilde{G}' \rightarrow G \rightarrow 1$ which is definable in \mathbf{G} over parameters (with $h_G : G \rightarrow G$ the identity map).

The parameters name a bijection between a partition \mathcal{W} of a finite collection \tilde{W} of \mathbf{G} -definable sets and a finite subset of N . The collection \tilde{W} (but not necessarily \mathcal{W} and its members) is \emptyset -definable in the pure group G .

Proof. Note that because \tilde{G} is definably connected and N is normal and finite then it is necessarily central. The proof of the theorem is very similar to that of Theorem 2.1. Instead of products of k commutators (i.e. the function $F_{k,G}$) we use products of k -many n -powers of G (the function $f_{k,G}$ defined above), with $n = |N|$. Also, instead of k_n we use here the function $h_k : G^k \rightarrow \tilde{G}$ defined above and instead of the set $G(k)$ defined there we use the set

$$G_k = \{\bar{g} \in G^k : f_{k,G}(\bar{g}) = 1\}$$

and its definably connected components.

Finally, instead of using the fact there that every element of the perfect group G was a product of k commutators, we use Theorem 8.1 which implies that every element of G is a product of k n th-powers. The other details are identical to the proof of Theorem 2.1. The collection \tilde{W} is the set of definably connected components of $G(2r)$ and $G(3r)$ as described there. \square

8.1. The real case. Just like in case of Theorem 2.8, if one works over the field of real numbers then there is no need to assume that \tilde{G} is definable and we obtain the following version of Theorem 8.3:

Theorem 8.4. *Let \mathcal{M} be an o -minimal structure over the ordered set of real numbers, G an \mathcal{M} -definable group and assume that $E : 1 \rightarrow N \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ is a topological extension with N finite and \tilde{G} connected (again, assuming that the homomorphisms are quotient maps).*

Let \mathbf{G} be some expansion of $\langle G, \cdot \rangle$ with property ρ .

Then E is isomorphic as a topological extension, to an extension $E' : 1 \rightarrow N_1 \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ definable in the structure \mathbf{G} , (with the isomorphism being the identity on G).

Proof. The arguments as to why this version of 8.4 is true are identical to those explained in the proof of 2.8. In both cases the only o-minimal facts that are being used apply to G (rather than \tilde{G}). \square

We end this diversion into extensions of definable real Lie groups by considering topological covers and related central extensions. This is really an application of work by Edmundo ([7]) and Edmundo-Eleftheriou ([8]) on universal covers and local definability in an o-minimal setting, as well as work on definable fundamental groups by Berarducci and Otero ([4]). We include the material because we could not find it precisely stated in the literature. In any case thanks to Edmundo for his explanations to us of results implicit in his work, some of which we repeat in the proof below.

Let us now set up notation for Theorem 8.5 below. $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, \dots \rangle$ will be an o-minimal expansion of the real field, and G a definably connected group definable in \mathcal{M} (so G is what we have called a definable real Lie group). \tilde{G} will be the topological universal cover of G (also a connected real Lie group) and Γ denotes the kernel of $\tilde{G} \rightarrow G$, namely the fundamental group $\pi_1(G)$ of G . So Γ is a central discrete closed subgroup of \tilde{G} . If $f : \Gamma \rightarrow A$ is a homomorphism from Γ into an abelian group A , we form as usual the group $\tilde{G}_A = \tilde{G} \times_{\Gamma} A$, and we have a central extension $1 \rightarrow A \rightarrow \tilde{G}_A \rightarrow G \rightarrow 1$ of G (as abstract groups). We will refer to locally definable groups, for which the reader can consult [8], although we give an explanation inside the proof.

Theorem 8.5. (i) \tilde{G} and the covering homomorphism can be realized, even topologically, as a locally definable group and homomorphism in \mathcal{M} .

(ii) \tilde{G}_A with its group structure, the extension $1 \rightarrow A \rightarrow \tilde{G}_A \rightarrow G \rightarrow 1$, together with a section $G \rightarrow \tilde{G}_A$, can be interpreted with parameters in the two-sorted structure consisting of \mathcal{M} and $\langle A, + \rangle$.

Proof. We will be brief. But note first that taking $A = \Gamma$ and h the identity, (ii) says that \tilde{G} can be interpreted in the two-sorted structure consisting of \mathcal{M} and $\langle \Gamma, + \rangle$.

Recall first that for an arbitrary central group extension $E : 1 \rightarrow A \rightarrow H \xrightarrow{\pi} G \rightarrow 1$, if $s : G \rightarrow H$ is a section for π , and $h_s(x, y) = s(xy)^{-1}s(x)s(y)$, then h_s (which is called a 2 cocycle) is a map from $G \times G$ into A and the group H is isomorphic to the group H' whose underlying set is $G \times A$ and whose group operation is given by $(x, a) \cdot (y, b) = (xy, h_s(x, y) + a + b)$ (we can write the second coordinate additively because A is abelian). Moreover $\pi' : H' \rightarrow G$ is the usual projection, the embedding of A into H' is given by $a \mapsto (1, -a - h_s(1, 1))$, and the section s is just $x \mapsto (x, 0)$. Hence, in order to recover E we only need to find such a definable cocycle h_s .

We now prove part (i). The statement of (i) is that there exists in \mathcal{M} a locally definable group $\mathcal{U} = \bigcup_{i \in I} X_i$ (i.e. a bounded directed union of definable sets) with a locally definable group operation (i.e. it is definable when restricted to each $X_i \times X_j$), and a locally definable surjective homomorphism $w : \mathcal{U} \rightarrow G$, and moreover the group \mathcal{U} with its topology as a locally definable group in \mathcal{M} is precisely the universal covering of G . The reasoning is as follows: Choose $w : \mathcal{U} \rightarrow G$ to be the

universal locally definable cover of G as described in [8]. \mathcal{U} has of course a topology as a locally definable group, and as the underlying set of \mathcal{M} is \mathbb{R} , it will be locally Euclidean, connected, and a topological cover of G . So w induces an embedding w_* of the topological fundamental group $\pi_1(\mathcal{U})$ of \mathcal{U} into $\pi_1(G)$. We will point out that $\pi_1(\mathcal{U}) = 0$, whereby \mathcal{U} will be the universal cover \tilde{G} of G . Let $c \in \pi_1(\mathcal{U})$. So $w_*(c) \in \pi_1(G)$. By [4], $\pi_1(G) = \pi_1^{def}(G)$ (the definable fundamental group of G), whereby $w_*(c)$ is represented by a definable path γ (beginning and ending at the identity). By Lemma 2.7(1) of [8], γ lifts to a definable path γ' in \mathcal{U} starting at the identity. On general topological grounds, γ' is a loop, and represents c . But $\pi_1^{def}(\mathcal{U})$ is trivial, whereby γ' is definably homotopic to the identity. Thus $c = 0$.

For (ii) let us first prove the special case that \mathcal{U} is definable in $\langle \mathcal{M}; \Gamma \rangle$. Because I is bounded, there is an $i_0 \in I$ such that X_{i_0} projects onto G . Because of definable choice, there is an \mathcal{M} -definable $Y \subseteq X_{i_0}$ and an \mathcal{M} -definable $s : G \rightarrow Y$ which is a section for w . Moreover, because of local definability, there is $j \in I$ such that $Y^{-1} \cdot Y \cdot Y \subseteq X_j$, hence the associated 2-cocycle $h_s : G \times G \rightarrow X_j$ is also \mathcal{M} -definable and its image is contained in X_j . Note that since $w|_{X_j} : X_j \rightarrow G$ is definable it follows that the image of h_s in Γ must be finite (otherwise the kernel of π in X_j will be an infinite definable discrete set). Finally, as mentioned above, given the cocycle h_s we can recover a definable covering $1 \rightarrow \Gamma' \rightarrow \mathcal{U}' \rightarrow G \rightarrow 1$ in $\langle \mathcal{M}; \langle \Gamma, + \rangle \rangle$ which is isomorphic to the original one. Because Γ is a bounded set (independently of the model \mathcal{M}) the set $\mathcal{U}' = \Gamma \times G$ can be written as a directed union of definable sets. Since h_s is an \mathcal{M} -definable map the group structure on \mathcal{U}' is locally definable. Finally, the isomorphism $(x, a) \mapsto x \cdot a$ from \mathcal{U}' to \mathcal{U} is locally definable as well and therefore a homeomorphism.

We now consider the general case of (ii). By what has been done so far we may identify \tilde{G} with \mathcal{U} . Let $h_s : G \times G \rightarrow \Gamma$ be the 2-cocycle from the previous paragraph. Define $h' : G \times G \rightarrow A$ to be $f \circ h_s$ (where recall f is the given homomorphism of Γ into A). Then h' is precisely the 2-cocycle determining \tilde{G}_A . As h_s was definable in \mathcal{M} with finite image in Γ , it follows that the group operation on $G \times A$ given in the first paragraph of the proof is definable in the two-sorted structure consisting of \mathcal{M} and $\langle A, + \rangle$. \square

Let us remark in closing this section that Theorem 8.5 gives an interesting twist on certain covering structures considered by Zilber, such as the two-sorted structure M_0 , say, consisting of $\langle \mathbb{C}, + \rangle$ in one sort, $\langle \mathbb{C}, +, \cdot \rangle$ in the other sort and the complex exponential map exp going from the first sort to the second. The kernel of exp can be identified with the (definable in M_0) subgroup \mathbb{Z} of the first sort. It is easy to see that the full structure M_0 cannot be interpreted in the reduct consisting of the sorts $\langle \mathbb{Z}, + \rangle$ and $\langle \mathbb{C}, +, \cdot \rangle$. But Theorem 8.5 says that M_0 can be so interpreted if we add a predicate for \mathbb{R} to the second sort.

We should also mention the Ph.D. thesis,[12], of Misha Gavrilovich on the model theory of the universal covering spaces of complex algebraic varieties, which contains ideas and constructions related to ours above.

9. GROUPS WHICH ARE NOT DEFINABLY CONNECTED

In this section we prove an analogue of Theorem 7.1 for definably compact groups which are not assumed to be definably connected.

We assume that \mathcal{M} is a sufficiently saturated o-minimal structure expanding a real closed field. We still use \widehat{G} to denote G/G^{00} . Here are some preliminaries:

Claim 9.1. (i) *Let G be any group, and H a normal subgroup of finite index. Then there are $g_1, \dots, g_n \in G$, and $h_{ij} \in H$, $1 \leq i, j \leq n$, such that the structures $\langle G, \cdot, H, g_1, \dots, g_n \rangle$ and $\langle H, \cdot, \alpha_1, \dots, \alpha_n, h_{ij} \rangle_{i,j}$ are bi-interpretable, where $\alpha_i : H \rightarrow H$ are defined by $\alpha_i(x) = g_i x g_i^{-1}$.*

(ii) *In the special case when G is definable in an o-minimal structure and $H = G^0$ (as always, G^0 is the definably connected component of G in the sense of \mathcal{M}), then G^0 is definable in $\langle G, \cdot \rangle$ without parameters so in fact $\langle G, \cdot, g_i \rangle_i$ and $\langle G^0, \cdot, \alpha_k, h_{ij} \rangle_{i,j,k}$ are bi-interpretable.*

Proof. (i) Let g_1, \dots, g_n be representatives of the cosets of H in G , and for each $1 \leq i, j \leq n$, let $g_i g_j = h_{ij} g_r$ for $r = r(i, j)$ and $h_{ij} \in H$.

To interpret G in $\mathcal{H} = \langle H, \cdot, \alpha_k, h_{ij} \rangle_{i,j,k}$, we fix some elements $a_1, \dots, a_n \in \text{dcl}_{\mathcal{H}}^{eq}(\emptyset)$ and interpret G on the set $H \times \{a_1, \dots, a_n\}$, with the group operation given by

$$(h, a_i)(h', a_j) = (h \cdot \alpha_i(h') \cdot h_{ij}, g_{r(i,j)}).$$

The elements g_1, \dots, g_n are then identified with $(1, a_1), \dots, (1, a_n)$.

The interpretation of H , the α_i 's and h_{ij} 's in $\langle G, \cdot, H, g_1, \dots, g_n \rangle$ is obvious, and it is easy to see that the two interpretations yield bi-interpretability

(ii) In the notation of Lemma 8.1, there is an n such that $\sigma_n(G) \subseteq G^0$ and therefore, by the same lemma, there is a k such that $G^0 = \sigma_n(G) \cdots \sigma_n(G)$ (k -times). This implies that G^0 is definable (without parameters) in G . \square

By the above, in order to understand an arbitrary definable group G we need to understand G^0 together with finitely many definable automorphisms.

By 6.4, every definably compact, definably connected group G is the almost direct product of the semisimple group $[G, G]$ and $Z(G)^0$.

Clearly, every definable automorphism of G leaves invariant both $[G, G]$ and $Z(G)^0$, so we need to understand each of the two groups, together with finitely many definable automorphisms.

Theorem 7.5 allows us to treat definable automorphisms of a definable abelian group A (by viewing their graph as a subgroup of $A \times A$). Hence, we now need to examine definable automorphisms of definable semisimple groups.

Claim 9.2. *Let \mathcal{M} be an o-minimal expansion of a real closed field R . If G is an R -semialgebraic, definably connected, definably compact semisimple group, then every definable automorphism of G in the structure \mathcal{M} is R -semialgebraic.*

Proof. Assume first that G is definably simple and $f : G \rightarrow G$ is a definable automorphism. Because G is definably compact it is bi-interpretable with a real closed field R_1 . Therefore, by [25, Proposition 4.8], (a Borel-Tits-style result), $f = g \circ h$, where g is an R -semialgebraic automorphism of G and h is induced by an automorphism σ of the semialgebraic field R_1 . The proof of Proposition 4.8 cited above shows that σ is definable, if f is definable. So σ is definable, fixes the rationals pointwise, and

thus equals the identity (its set of fixed points is an infinite definable subgroup of R_1). So f is semialgebraic, proving the claim in the special case.

Assume now that G is semisimple and centerless. Hence, by 1.2, G is definably isomorphic in R to $H_1 \times \cdots \times H_n$, where each H_i is a linear semialgebraic group, defined over the real algebraic numbers $R_{alg} \subseteq R$. Without loss of generality, $G = H_1 \times \cdots \times H_n$, and we consider each H_i as a subgroup of G . We want to show first that f must permute the H_i 's so we claim that H_1, \dots, H_n are the only definable, nontrivial, minimal normal subgroups of G . The definability of the H_i 's was shown in the proof of 4.4 and clearly each H_i is minimal among the normal definable subgroups. We want to show that these are the only ones. Take $H \subseteq G$ nontrivial, normal and definable and consider $H \cap H_i$ for the various H_i 's. If the intersection is nonempty for some i then clearly $H_i \subseteq H$ so we may assume that $H \cap H_i = \{1\}$ for all i . It follows that the projection of H on, say $H_1 \times H_2$, is the graph of an isomorphism $h : H_1 \rightarrow H_2$. Let's see that this is impossible. Take $x \neq 1$ in H_1 and $y = h(x) \in H_2$ and choose $z \in H_2$ which does not commute with y (H_2 is centerless). If we conjugate (x, y) by the elements $(1, z) \in H_1 \times H_2$ then we get a new element $(x, z^{-1}yz)$ in the graph of h , with $z^{-1}yz \neq y$, contradiction. We therefore proved our claim about the H_i 's and as a result showed that f must permute the H_i 's.

It is therefore sufficient to prove that any definable isomorphism between two definably simple semialgebraic groups is itself semialgebraic. As above, this follows from [25, Proposition 4.8(2)].

Let G be a definably connected semisimple group. In case H is a group interpretable in G on which f induces a well defined map we denote this map by $f|_H$. Now, since $f : G \rightarrow G$ is an automorphism it leaves $Z(G)$ invariant and we denote the map induced on $H = G/Z(G)$ by $f|_H$. Because $f|(G/Z(G))$ is definable then, by the centerless case, $f|(G/Z(G))$ is semialgebraic (and clearly the finite map $f|Z(G)$ is also semialgebraic).

By Theorem 2.1, there is a surjection $\sigma_c : Z(G) \times (G/Z(G))^{2r} \rightarrow G$ which is definable in $\langle G, \cdot \rangle$, over a tuple of parameters c from G . We let $d = f(c)$. Then for every $x \in G$, if $x = \sigma_c(a, b)$, for $a \in Z(G)$ and $b \in (G/Z(G))^{2r}$ then

$$f(x) = f(\sigma_c(a, b)) = \sigma_d(f|Z(G)(a), f|(G/Z(G))(b)).$$

The map σ_d is semialgebraic and because $f|(G/Z(G))$ and $f|Z(G)$ are semialgebraic so is f . \square

The following lemma is general.

Lemma 9.3. *Let G be an arbitrary group, $A \subseteq G$ a central subgroup of G , such that for some number k , $G/A = [G/A, G/A]_k$.*

Let $f : G \rightarrow G$ be a group automorphism of G such that $f(A) = A$. Then f is definable in the structure $\mathbf{G} = \langle G, \cdot, A, f|_A, f|(G/A) \rangle$.

Proof. We use Beth definability theorem: Namely, we take $\langle \tilde{G}, \tilde{A}, \tilde{f}|_A, \tilde{f}|(G/A) \rangle$ elementarily equivalent to \mathbf{G} and show that there is a unique automorphism $g : \tilde{G} \rightarrow \tilde{G}$ leaving A invariant such that $g|_A = \tilde{f}|_A$ and $g|(G/A) = \tilde{f}|(G/A)$. Note that the assumption on G implies that \tilde{G} is still the product of \tilde{A} and $[\tilde{G}, \tilde{G}]$ (this is true in G and because $[G/A, G/A]$ is generated in finitely many steps, it becomes a first order statement true in \tilde{G} as well).

Assume that we have $g, h : \tilde{G} \rightarrow \tilde{G}$ automorphisms as above and consider gh^{-1} . Then $gh^{-1}|_A = id$ and $gh^{-1}|\tilde{G}/\tilde{A} = id$. We may therefore assume that $g|_A$ and $g|(\tilde{G}/\tilde{A})$ are the identity maps and aim to show that $g = id$.

Because $g|(\tilde{G}/\tilde{A}) = id$, for every $x \in \tilde{G}$, we have $x^{-1}g(x) \in \tilde{A}$ and hence there exists a function $a : \tilde{G} \rightarrow \tilde{A}$ such that $g(x) = xa(x)$. We claim that a is a group homomorphism: For $x, y \in \tilde{G}$ we have

$$xya(xy) = g(xy) = g(x)g(y) = xa(x)ya(y) = xya(x)a(y)$$

(because $a(x), a(y)$ are central elements). It follows that $a(xy) = a(x)a(y)$.

For every $x \in A$ we have $g(x) = x$, hence $a(x) = 1$. Also, if $b = xyx^{-1}y^{-1}$ is a commutator in \tilde{G} then $g(b) = b \cdot a([x, y]) = b[a(x), a(y)] = b$, hence $a(b) = 1$. But then $\ker(a)$ contains both A and the commutator subgroup of \tilde{G} . Because \tilde{G} is generated by these two groups, $a(x) = 1$ for all $x \in \tilde{G}$ and therefore $g = id$. \square

Theorem 9.4. *If G is a definably compact group in an o-minimal expansion \mathcal{M} of an ordered group (not necessarily definably connected) then it is elementarily equivalent to a definably compact, semialgebraic (over parameters) group H over a real closed field, with $\dim H = \dim G$.*

Proof. As we saw in 9.1, G is bi-interpretable, over parameters from G^0 , with G^0 together with the action of finitely many automorphisms f_1, \dots, f_k . For simplicity, we denote G^0 by H and $Z(H)^0$ by A .

We also saw, in 9.3, that the structure

$$\langle H, \cdot, \{f_1, \dots, f_k\}, \{c_1, \dots, c_r\} \rangle$$

(for constants $c_1, \dots, c_r \in H$) is definable in

$$\langle H, \cdot, A, \{f_i|_A : i = 1, \dots, k\}, \{f_i|(H/A) : i = 1, \dots, k\}, \{c_1, \dots, c_r\} \rangle.$$

We denote each $f_i|(H/A)$ by g_i and each $f_i|_A$ by h_i .

By Theorem 6.1, the group H is interpretable, over parameters, in the two-sorted structure $\langle H/A, A \rangle$. Putting it all together, we see that G is interpretable, over parameters, in $\langle H/A, \{g_i : i = 1, \dots, k\}, A, \{h_i : i = 1, \dots, k\} \rangle$, (where H/A and A are endowed with their group structure).

By 4.4 the semisimple group H/A is definably isomorphic to a semialgebraic group G_0 over $R_{alg} \subseteq R$, for a real closed field R and by 9.2, each g_i is sent by this isomorphism to an R -semialgebraic automorphism of G_0 , possibly defined over parameters.

The structure $\langle A, +, \{h_1, \dots, h_k\} \rangle$ is clearly a reduct of the structure \mathbf{A}_{ab} considered in Theorem 7.5 (since every automorphism of G gives rise to a subgroup of $G \times G$). Therefore, it is elementarily equivalent to an expansion of a connected, compact, abelian real Lie group \hat{A} (with $\dim \hat{A} = \dim A$), by Lie group automorphisms $\hat{h}_1, \dots, \hat{h}_k$. Finally, \hat{A} , as a compact Lie group, is isomorphic to a real algebraic linear group L . This isomorphism takes each graph of \hat{h}_i to a Lie subgroup of L^2 , which itself must be semialgebraic (indeed, this last fact follows for example from [24, 3.3], applied to the o-minimal structure \mathbb{R}_{an} , in which every definable compact linear group is definable).

Hence, by going to a sufficiently saturated real closed field \tilde{R} , we can find constants $d_1, \dots, d_r \in \tilde{R}$ such that

$$\mathcal{M}_1 = \langle A, \{h_i : i = 1, \dots, k\}, H/A, \{g_i : i = 1, \dots, k\}, \{c_i : i = 1, \dots, r\} \rangle$$

is elementarily equivalent to

$$\mathcal{M}_2 = \langle L(\tilde{R}), \{\hat{h}_i : i = 1, \dots, k\}, G_0(\tilde{R}), \{\hat{g}_i : i = 1, \dots, k\}, \{d_i : i = 1, \dots, r\} \rangle,$$

with G_0 , L , and the automorphisms \hat{g}_i, \hat{h}_i all semialgebraic.

Because G is definable over parameters in \mathcal{M}_1 , it is elementarily equivalent to a group definable (over parameters) in \mathcal{M}_2 , and this last group must be semialgebraic. \square

Remark. By Lemma 11.2, the parameters in A can be realized as tuples of real algebraic elements in an elementarily equivalent real algebraic group over \mathbb{R} . However, we do not know how to do the same for the parameters in H/A .

10. COMPACT DOMINATION FOR DEFINABLY COMPACT GROUPS

Here we give another application of Corollary 6.4. The “compact domination conjecture” for *definably compact*, definably connected groups in (saturated) o-minimal expansions of real closed fields, was introduced in [16]. The conjecture says that G is dominated by G/G^{00} equipped with its Haar measure. Namely, writing $\pi : G \rightarrow G/G^{00}$ for the canonical surjective homomorphism, for any definable subset X of G in the structure \mathcal{M} , the set of $c \in G/G^{00}$ such that $\pi^{-1}(c)$ intersects both X and its complement, has Haar measure 0. We sometimes just say “ G is compactly dominated”. The conjecture was proved in [16] for G with “very good reduction”, and by part (ii) of Theorem 4.4 of the current paper, this is the case for semisimple definably connected groups. In [17] compact domination was proved for G commutative. With 6.4 we know that an arbitrary G (definably compact, definably connected) almost splits into its semisimple and abelian parts, and one would expect that this makes it easy to deduce compact domination of G from the two special cases.

Theorem 10.1. *Every definably connected, definably compact group in \mathcal{M} , a sufficiently saturated expansion of a real closed field, is compactly dominated.*

Proof. We first prove the result for a group $G \times H$, with G commutative and H semisimple and definably connected. It is clear that $(G \times H)^{00} = G^{00} \times H^{00}$. It is sufficient to prove the result for each definable set separately so we may assume that the language is countable. We fix $\mathcal{M}_0 \subseteq \mathcal{M}$ a countable model. We let

$$\pi : G \times H \rightarrow G/G^{00} \times H/H^{00},$$

with $\pi = (\pi_1, \pi_2)$ and $\pi_1 : G \rightarrow G/G^{00}$, $\pi_2 : H \rightarrow H/H^{00}$.

The following observation is true in greater generality (for any type-definable equivalence relation), but we only observe it in the o-minimal setting: If K is a definably compact group in \mathcal{M} , definable over M_0 and $a_1, a_2 \in K$ realize the same type over M_0 then they lie in the same K^{00} -coset.

Indeed, let σ be an automorphism of \mathcal{M} which fixes M_0 and takes a_1 to a_2 . The map σ induces a continuous (with respect to the logic topology) automorphism of K/K^{00} which fixes all the torsion points of K (since they belong to M_0). But $\pi(\text{Tor}(K))$ is dense in K/K^{00} , therefore σ induces the identity map on K/K^{00} . It follows that a_1, a_2 lie in the same K^{00} -coset.

Let $X \subseteq G \times H$ be a definable set over M_0 and assume, towards contradiction, that the set

$$B = \{(g', h') \in G/G^{00} \times H/H^{00} : \pi^{-1}(g', h') \cap X \neq \emptyset \& \pi^{-1}(g', h') \cap X^c \neq \emptyset\}$$

has positive Haar measure.

At this point we have to refer to and draw some conclusions from the published paper [16] and the preprint [17]. Let \mathcal{M}^* denote the expansion of \mathcal{M} obtained by adjoining relations for all externally definable sets, namely intersections of definable (with parameters) sets in an elementary extension of \mathcal{M} with the various M^n . As pointed out (with references) at the beginning of section 8 of [17], results of both Shelah and Baisalov-Poizat imply that \mathcal{M}^* is weakly o-minimal (but not o-minimal). As H has “very good reduction” (i.e. by 4.4 of the current paper) we may assume that H is a semialgebraic group defined over a copy \mathbb{R} of the real field in \mathcal{M} (and as such is already a topological group). Then H^{00} is definable in \mathcal{M}^* as the usual infinitesimals. By Lemma 8.2 of [17], G^{00} is also definable in \mathcal{M}^* . Hence $G/G^{00} \times H/H^{00}$ is a definable group in $(\mathcal{M}^*)^{eq}$. For the sake of this part of the paper we will be working in $(\mathcal{M}^*)^{eq}$ so will talk about “definable in” rather than “interpretable in”.

Claim. There is a set I and ordering $<$ (piecewise dense), definable in, and o-minimal in, \mathcal{M}^* such that

- (i) $G/G^{00} \times H/H^{00}$ is definably isomorphic (in \mathcal{M}^*) to a subset of I^m , and so is equipped with an appropriate o-minimal topology with respect to which the group operation is continuous.
- (ii) This o-minimal topology on $G/G^{00} \times H/H^{00}$ coincides with the logic topology.
- (iii) A definable subset of $G/G^{00} \times H/H^{00}$ has interior (in either topology) if and only if it has positive Haar measure.

Proof. (of claim). Firstly to say that $(I, <)$ is o-minimal in \mathcal{M}^* means that any subset of I definable (with parameters) in \mathcal{M}^* is a finite union of intervals and points, and moreover uniformly (namely holds also in a saturated model).

Secondly we note that (i) and (ii) hold for each of G/G^{00} and H/H^{00} separately. In the case of G/G^{00} this is 8.7 and 8.15 respectively of [17]. The case of H/H^{00} is more direct (although also follows from these results in [17]). Namely we know generally that identifying H/H^{00} with $H(\mathbb{R})$ via the standard part map shows that the logic topology on H/H^{00} coincides with the Lie group topology on $H(\mathbb{R})$. Also we take here I to be *Fin/Inf* (finite elements of M quotiented by infinitesimals) which identifies I with \mathbb{R} . (See section 10 of [17].)

Now the concatenation of two definable, definably ordered sets I, J , both o-minimal in \mathcal{M}^* is also o-minimal in \mathcal{M}^* . Hence we obtain (i). As both the o-minimal and logic topologies on the product $G/G^{00} \times H/H^{00}$ are the products of the respective topologies on G/G^{00} , H/H^{00} , we also obtain (ii).

For (iii): clearly any definable set X , being a Boolean combination of open sets, is Haar measurable. By compactness of the group, if X has interior then finitely many translates cover the group, so X has positive measure. The converse is contained in the proof of Lemma 8.9 of [17] for example. \square

We now return to the main thread of the proof. As the set B is definable in \mathcal{M}^* and has positive Haar measure, by (iii) of the Claim above, there are open (definable) sets $U \subseteq G/G^{00}$ and $V \subseteq H/H^{00}$ with $U \times V \subseteq B$. We claim that there exists

$g' \in U$ such that all elements of $\pi_1^{-1}(g')$ realize the same type in \mathcal{M} , over M_0 . Indeed, because G is compactly dominated, for every M_0 -definable subset X of G , the set of all $g' \in G/G^{00}$ such that $\pi_1^{-1}(g') \cap X \neq \emptyset$ and $\pi_1^{-1}(g') \cap X^c \neq \emptyset$ has Haar measure zero. So, after removing countably many such sets (each of measure zero), the pre-image of every $g' \in U$ under π_1 is contained in a complete \mathcal{M} -type over M_0 .

We fix one such $g' \in U$ as above, $g \in \pi_1^{-1}(g')$, and consider the set $X_g = \{h \in H : (g, h) \in X\}$. We claim that for every $h' \in V$, the sets $\pi_2^{-1}(h') \cap X_g$ and $\pi_2^{-1}(h') \cap X_g^c$ are both nonempty, and this will contradict the fact that H is compactly dominated, as V being open has positive measure. Indeed, if $h' \in V$ then, by assumption on B , there are $g_1, g_2 \in \pi_1^{-1}(g')$ and $h_1, h_2 \in \pi_2^{-1}(h')$ with $(g_1, h_1) \in X$ and $(g_2, h_2) \in X^c$. Because g_1, g_2 and g all realize the same type over M_0 , there are $h_3, h_4 \in H$ conjugates over M_0 of h_1, h_2 , respectively, with $(g, h_3) \in X$ and $(g, h_4) \in X^c$. By our earlier observation, h_3 and h_4 belong to the pre-image of h' , so $\pi_2^{-1}(h') \cap X_g$ and $\pi_2^{-1}(h') \cap X_g^c$ are non-empty. Contradiction. We thus showed that $G \times H$ is compactly dominated.

The result for an arbitrary definably compact group follows from the special case using Corollary 6.4, noting that compact domination is preserved under quotients (using the fact that a definable surjective homomorphism $\sigma : G_1 \rightarrow G_2$ of definably compact groups sends G_1^{00} onto G_2^{00} , see [2, Theorem 5.2]). \square

11. APPENDIX: ON ABELIAN GROUPS

Since all groups here are abelian we write them additively.

Lemma 11.1. *Let A, B be two divisible abelian groups such that B has unbounded exponent. Assume that $\phi : B \rightarrow A$ is a group embedding, with $\text{Tor}(A) \subseteq \phi(B)$. Then ϕ is an elementary map (in the language of groups).*

Proof. This is a well-known fact in the model theory of abelian groups. First by considering the Szmielew invariants, B and A are elementarily equivalent. Secondly as B is divisible, $\phi(B)$ is pure in A , hence an elementary substructure. See Appendix A.2 of [14] for an account of the background. \square

We also need the following claim on definable abelian groups in o-minimal structures.

Lemma 11.2. *Let \mathcal{M} be an o-minimal expansion of an ordered group. If G is a definable abelian group (possibly not definably connected) and C is a finite subset of G then $\langle G, +, \{b \in C\} \rangle$ (namely, we add a constant to every element of C) is elementarily equivalent to a real algebraic group of the same dimension, with finitely many real algebraic elements named. If G is definably compact then the real algebraic group can be chosen to be compact.*

Proof. Assume first that G is definably connected. It follows that it is divisible.

By [28], there exists $G_0 \subseteq G$ a torsion-free definable subgroup of G with G/G_0 definably compact. Because G_0 is divisible and torsion-free it is elementarily equivalent to $\mathbb{R}^{\dim G_0}$. The group $G_1 = G/G_0$ is a definably compact, definably connected group and therefore by [10] (and, in the case that \mathcal{M} expands an ordered group also by [11] and [22]), $\text{Tor}(G_1)$ is isomorphic to the torsion group of the real torus $\mathbb{T}^{\dim G_1}$. It follows (say, by 11.1) that G_1 is elementarily equivalent to the semialgebraic $\mathbb{T}^{\dim G_1}$ and G is elementarily equivalent to the group $\mathbb{R}^{\dim G_0} \times \mathbb{T}^{\dim G_1}$.

Finally, consider the divisible hull H in G of the group generated by the set C and $Tor(G)$. As any divisible subgroup of an abelian group is a direct summand, it follows that H can be written as the direct sum of $Tor(G)$ and $\mathbb{Q}c_1 \oplus \cdots \oplus \mathbb{Q}c_k$, for some $c_1, \dots, c_k \in C$. By 11.1, H is an elementary subgroup of G . It can be realized also as an elementary subgroup of $\mathbb{R}^{\dim G_0} \times \mathbb{T}^{\dim G_1}$. Moreover, since all torsion elements are given as tuples of real algebraic numbers and we can also choose real algebraic elements which are torsion-free and \mathbb{Q} -independent (using the well-known fact that the field of real algebraic numbers is infinite-dimensional as a \mathbb{Q} -vector space), it follows that $\langle G, +, \{g : g \in C \cup Tor(G)\} \rangle$ is elementarily equivalent to $\mathbb{R}^{\dim G_0} \times \mathbb{T}^{\dim G_1}$, with names for all torsion elements and finitely many other elements, all having real algebraic coordinates.

Assume now that G is not definably connected. Then it equals a direct sum of its connected component G^0 and a finite group and therefore, by the above, it is elementarily equivalent to a semialgebraic group H of the same dimension, which can be defined over the real algebraic numbers. We can handle similarly finitely many named elements in G . The last clause follows from the proof. \square

Given an expansion \mathbf{A} of an abelian group A , consider the sub-language L_{ab} which has a predicate R_S for every \emptyset -definable (in \mathbf{A}) subgroup $S \subseteq A^n$, $n \in \mathbb{N}$, as well as symbols for $+$ and 0 . Let \mathbf{A}_{ab} denote the reduct of \mathbf{A} to L_{ab} . Such a structure has been sometimes called an *abelian structure*. We call the subgroups S above the *basic* ones in the structure \mathbf{A}_{ab} . If B is a subgroup of A then we denote by \mathbf{B}_{ab} the L_{ab} -induced structure on B , namely the interpretation of R_S is just its intersection with B^n . The next fact is a restatement of 7.2, but we add a bit more information in item 1. The results belong to the folklore around abelian structures but we give a few details.

Fact 11.3. *In the above setting (no o -minimality is assumed)*

- (1) (i) *The theory of the structure \mathbf{A}_{ab} eliminates quantifiers (in the language L_{ab}).*
(ii) *Moreover $Th(\mathbf{A}_{ab})$ is axiomatized as follows: (a) Axioms for abelian groups, (b) Each symbol R_S denotes a subgroup, (c) Axioms for the defining properties of R_S : If S_1 is a projection of S_2 then R_{S_1} denotes the corresponding projection of R_{S_2} , and if $S_1 = \{x \in A^n : S_2(x, 0)\}$ then R_{S_1} denotes the corresponding fiber of R_{S_2} , (d) Axioms about the index (a given finite number or ∞) of R_{S_1} in R_{S_2} whenever $S_1 \leq S_2 \leq A^n$ are basic.*
- (2) *Assume that $B \leq A$ is a subgroup of A .*

Then $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$ if and only if the following hold:

- (i) *For every \emptyset -definable (in \mathbf{A}_{ab}) subgroup $S \leq A^{n+k}$ and $b \in B^k$,*

$$S(B^n, b) \neq \emptyset \Leftrightarrow S(A^n, b) \neq \emptyset.$$

- (ii) *For all \emptyset -definable (in \mathbf{A}_{ab}) subgroups $S_1 \leq S_2 \leq A^n$,*

$$[S_2 : S_1] = [S_2 \cap B^n : S_1 \cap B^n],$$

with the meaning that if this index is infinite on one side then it is infinite on the other.

- (3) *Assume that \mathbf{A}_{ab} has DCC on \emptyset -definable subgroups. Then, for every $\mathbf{B}_{ab} \prec \mathbf{A}_{ab}$ there exists a surjective group homomorphism $\phi : A \rightarrow B$ which is the*

identity map on B and in addition sends every \emptyset -definable $S \subseteq A^n$ onto $S \cap B^n$. (We call such a ϕ a homomorphic retract).

Proof. (1)(i) is proved in [13].

1(ii) can be extracted from the proof of the quantifier elimination result in [13], in exactly the same way as the analogous statement for theories of modules is deduced from the proof of *pp* elimination for modules. See Theorem 1.1 and Corollary 1.5 of [32]. In fact in the statements on indices only basic subgroups of A itself (rather than A^n) need be considered. A direct proof of (ii) would go as follows: let \mathbf{B}, \mathbf{C} be ω -saturated models of the given axioms. Then try to show that the collection of partial isomorphisms between finitely generated substructures has the back-and-forth property. Namely suppose that \bar{b} and \bar{c} are finite tuples from \mathbf{B}, \mathbf{C} respectively, which have the same quantifier-free type, and suppose $d \in \mathbf{B}$. Then find $e \in \mathbf{C}$ such that (\bar{b}, d) and (\bar{c}, e) have the same quantifier-free type. By compactness, given some basic S and T_1, \dots, T_k such that $\mathbf{B} \models R_S(\bar{b}, d) \wedge \neg R_{T_1}(\bar{b}, d) \wedge \dots \wedge \neg R_{T_k}(\bar{b}, d)$, we must show that $\mathbf{C} \models \exists y (R_S(\bar{c}, y) \wedge \neg R_{T_1}(\bar{c}, y) \wedge \dots \wedge \neg R_{T_k}(\bar{c}, y))$. We can accomplish this by using the Neumann Lemma together with the information about the indices of $R_S(\bar{0}, y) \wedge \bigwedge_{i \in I} R_{T_i}(\bar{0}, y)$ in $R_s(\bar{0}, y)$ for all subsets $I = \{1, \dots, k\}$, as well as formulas in the quantifier-free type of \bar{b} . Going through the details would just be repeating the proof of *pp*-elimination in modules.

(2) follows from (1).

(3) Using the quantifier elimination result above, the proof of (3) is basically identical to that of Theorem 2.8, p.28, in [30]. The mathematical content here is that a totally transcendental abelian structure is algebraically compact, namely in the present set-up, that for any \mathbf{B} as in the statement of (iii), any positive quantifier-free type over B in possibly infinitely many variables (and which is consistent with the complete diagram of \mathbf{B}) is realized in \mathbf{B} . A direct proof would go as follows: Let $\Sigma(x_1, x_2, \dots)$ be a set of atomic formulas with parameters from \mathbf{B} which is consistent. We can assume Σ to be closed under finite conjunctions (in the obvious sense). The DCC assumption implies that the set of formulas $\exists x_2, x_3, \dots (\psi(x_1, x_2, x_3, \dots))$ (with free variable x_1) for ψ ranging over Σ , is equivalent to a single formula, which we can realize in \mathbf{B} . Continue. □

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