Definability in the group of infinitesimals of a compact Lie group

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July 23, 2019

Abstract

We show that for G a simple compact Lie group, the infinitesimal subgroup G^{00} is bi-intepretable with a real closed convexly valued field. We deduce that for G an infinite definably compact group definable in an o-minimal expansion of a field, G^{00} is bi-interpretable with the disjoint union of a (possibly trivial) $\mathbb{Q}$-vector space and finitely many (possibly zero) real closed valued fields. We also describe the isomorphisms between such infinitesimal subgroups, and along the way prove that every *definable* field in a real closed convexly valued field R is definably isomorphic to R.

1 Introduction

Let G be compact linear Lie group, by which we mean a compact closed Lie subgroup of $\mathrm{GL}_d(\mathbb{R})$ for some $d \in \mathbb{N}$, e.g. $G = SO_d(\mathbb{R})$. By Fact 2.1(i) below, any compact Lie group is isomorphic to a linear Lie group.

Let $\mathcal{R} \succ \mathbb{R}$ be a real closed field properly extending the real field. By Fact 2.1(ii), G is the group of $\mathbb{R}$-points of an algebraic subgroup of GL_d over $\mathbb{R}$, and we write $G(\mathcal{R}) \leq \mathrm{GL}_d(\mathcal{R})$ for the $\mathcal{R}$-points of this algebraic group.

Let $\mathrm{st} : \mathcal{O} \to \mathbb{R}$ be the standard part map, the domain $\mathcal{O} = \bigcup_{n\in\mathbb{N}}[-n, n] \subseteq \mathcal{R}$ of which is a valuation ring in $\mathcal{R}$. Let

$$\mathfrak{m} = \mathrm{st}^{-1}(0) = \bigcap_{n\in\mathbb{N}_{>0}} [-\frac{1}{n}, \frac{1}{n}] \subseteq \mathcal{R}$$

be the maximal ideal of $\mathcal{O}$. A field equipped with a valuation ring, such as $(\mathcal{R}; +, \cdot, \mathcal{O})$, is known as a *valued field*. The complete first-order theory of $(\mathcal{R}; +, \cdot, \mathcal{O})$ is the theory RCVF of non-trivially convexly valued real closed fields [CD83].

Since G is compact, $\mathrm{st} : \mathcal{O} \to \mathbb{R}$ induces a totally defined homomorphism $\mathrm{st}_G : G(\mathcal{R}) \twoheadrightarrow G$. The kernel $\ker(\mathrm{st}_G) \triangleleft G$ is the "infinitesimal subgroup" of G in $\mathcal{R}$. Our main results below describe definability in this group.

In fact, by Fact 5.3(iv) below, $\ker(\mathrm{st}_G)$ is precisely $G^{00}(\mathcal{R})$, the set of $\mathcal{R}$-points of the smallest bounded-index $\bigwedge$-definable subgroup G^{00} of the semialgebraic group G. In terms of matrices,

$$G^{00}(\mathcal{R}) = G(\mathcal{R}) \cap (I + \mathrm{Mat}_d(\mathfrak{m})) = \bigcap_{n\in\mathbb{N}_{>0}} (G(\mathcal{R}) \cap (I + \mathrm{Mat}_d([-\frac{1}{n}, \frac{1}{n}]))) \quad (1)$$

(where we write $\mathrm{Mat}_d(X)$ for the set of matrices with entries in a set $X \subseteq \mathcal{R}$). Note in particular that $G^{00}(\mathcal{R})$ is definable in the valued field $(\mathcal{R}; +, \cdot, \mathcal{O})$.

We adopt the convention that "definable" always means definable with parameters: a *definable set* in a structure $\mathcal{A} = (A; \ldots)$ is a subset of a Cartesian power A^n defined by a first-order formula in the language of $\mathcal{A}$ with parameters from A.

Recall that an *interpretation* of a (one-sorted) structure $\mathcal{A}$ in a structure $\mathcal{B}$ is a bijection of the universe of $\mathcal{A}$ with a definable set in $\mathcal{B}$, or more generally with the quotient of a definable set by a definable equivalence relation, such that the image of any definable set in $\mathcal{A}$ is definable in $\mathcal{B}$. A *definition* of $\mathcal{A}$ in $\mathcal{B}$ is an interpretation whose codomain is a definable set rather than a definable quotient; in fact, these are the only interpretations which will arise in the main results of this article. There is an obvious notion of composition of interpretations. A pair of interpretations of $\mathcal{A}$ in $\mathcal{B}$ and of $\mathcal{B}$ in $\mathcal{A}$ form a *bi-interpretation* if the composed interpretations of $\mathcal{A}$ in $\mathcal{A}$ and of $\mathcal{B}$ in $\mathcal{B}$ are definable maps in $\mathcal{A}$ and $\mathcal{B}$ respectively. If $\mathcal{A}$ and $\mathcal{B}$ are bi-interpretable, then the structure induced by $\mathcal{B}$ on the image of $\mathcal{A}$ is precisely the structure of $\mathcal{A}$, i.e. the definable sets are the same whether viewed in $\mathcal{A}$ or in $\mathcal{B}$. Indeed, if (f, g) is a bi-interpretation, then given a subset $X \subseteq \mathcal{A}^n$, if X is $\mathcal{A}$-definable then $f(X)$ is $\mathcal{B}$-definable, and conversely if $f(X)$ is $\mathcal{B}$-definable then $g(f(X))$ and hence X is $\mathcal{A}$-definable.

We adopt the following terminology throughout the paper. A *linear* Lie group is a closed Lie subgroup of $\mathrm{GL}_d(\mathbb{R})$ for some $d \in \mathbb{N}$. A Lie group is *simple* if it is connected and its Lie algebra is simple; a Lie algebra is simple if it is non-abelian and has no proper non-trivial ideal. (A simple Lie group may have non-trivial discrete centre; however, the underlying abstract group of a *centreless* simple Lie group is simple.) Our first main result describes definability in $G^{00}(\mathcal{R})$ for G simple:

Theorem 1.1. *Let G be a simple compact linear Lie group.*

Let $\mathcal{R} \succ \mathbb{R}$ be a proper real closed field extension of $\mathbb{R}$.

Then the inclusion $\iota : G^{00}(\mathcal{R}) \hookrightarrow G(\mathcal{R})$, viewed as a definition of the group $(G^{00}(\mathcal{R}); *)$ in the valued field $(\mathcal{R}; +, \cdot, \mathcal{O})$, can be extended to a bi-interpretation: there is a definition θ of the valued field $(\mathcal{R}; +, \cdot, \mathcal{O})$ in the group $(G^{00}(\mathcal{R}); *)$ such that the pair (ι, θ) form a bi-interpretation.

In particular, the $(G^{00}(\mathcal{R}); *)$-definable subsets of powers $(G^{00}(\mathcal{R}))^n$ are precisely the $(\mathcal{R}; +, \cdot, \mathcal{O})$-definable subsets.

Remark 1.2. The bi-interpretation of Theorem 1.1 requires parameters. Indeed, $G^{00}(\mathcal{R})$ has non-trivial inner automorphisms (this follows from Lemma 2.2 below), which under a hypothetical parameter-free bi-interpretation would induce definable non-trivial automorphisms of $(\mathcal{R}; +, \cdot, \mathcal{O})$.

However, no such automorphism exists. We sketch a proof of this, following a method suggested by Martin Hils. By Fact 4.1(1), if σ is an $(\mathcal{R}; +, \cdot, \mathcal{O})$-definable automorphism, then it agrees on some infinite interval I with an $(\mathcal{R}; +, \cdot)$-definable map f. But then for $a, b, c, d \in I$ with $c \neq d$, $\sigma(\frac{a-b}{c-d}) = \frac{\sigma(a)-\sigma(b)}{\sigma(c)-\sigma(d)} = \frac{f(a)-f(b)}{f(c)-f(d)}$, so σ is an $(\mathcal{R}; +, \cdot)$-definable field automorphism, thus $\sigma = \mathrm{id}$.

Remark 1.3. For G a simple centreless compact Lie group, Nesin and Pillay [NP91] showed that the group itself, $(G; *)$, is bi-interpretable with a real closed field. They interpret the field by finding a copy of $(\mathrm{SO}_3; *)$ and using the geometry of its involutions. A similar project is carried out in [PPS00b] for definably simple and semisimple groups in o-minimal structures. In the case of G^{00}, we also find a field by first finding a copy of

SO_3^{00}, but the "global" approach of considering involutions is not available; in fact G^{00} is torsion-free. Instead, we work "locally", and obtain the field by applying the o-minimal trichotomy theorem to a definable interval on a curve within SO_3^{00}. This kind of local approach was previously mentioned in an "added in proof" remark at the end of [PPS00b] as an alternative method for the case of $(G;*)$, but we have to take care to ensure that structure we apply trichotomy to is definable both o-minimally and in $(G^{00};*)$.

Remark 1.4. One might also consider "smaller" infinitesimal neighbourhoods corresponding to larger valuation rings $\mathcal{O}' \supsetneq \mathcal{O}$; Theorem 1.1 holds for these too. More generally, if $(\mathcal{R}';+,\cdot,\mathcal{O}') \vDash \mathrm{RCVF}$ extends $\mathbb{R}$ as an ordered field, then the group $G(\mathcal{R}) \cap \mathrm{Mat}_n(\mu')$ is bi-intepretable as in Theorem 1.1 with $(\mathcal{R}';+,\cdot,\mathcal{O}')$. Indeed, the existence of suitable parameters for the bi-intepretation is expressed by a sentence in RCVF with parameters in $\mathbb{R}$, and since RCVF is complete we can apply Theorem 1.1 to deduce the result.

We prove Theorem 1.1 in §3.

In §4, we deduce in the spirit of Borel-Tits a characterisation of the group isomorphisms of groups of the form G^{00}, decomposing them as compositions of valued field isomorphisms and isomorphisms induced by isomorphisms of Lie groups. In particular, this shows that simple compact G_1 and G_2 have isomorphic infinitesimal subgroups if and only if they have isomorphic Lie algebras. The key technical tool is Theorem 4.2, which shows that there are no unexpected fields definable in RCVF $= \mathrm{Th}(\mathcal{R};+,\cdot,\mathcal{O})$.

In §5, we generalise Theorem 1.1 to the setting of a definably compact group G definable in an o-minimal expansion of a field. Here, to say that G is *definably compact* means that any definable continuous map $[0,1) \to G$ can be completed to a continuous map $[0,1] \to G$; we refer to [PS99] for further details on this notion.

Define the **disjoint union** of 1-sorted structures M_i to be the structure $(M_i)_i$ consisting of a sort for each M_i equipped with its own structure, with no further structure between the sorts.

Theorem 1.5. *Let $(G;*)$ be an infinite definably compact group definable in a sufficiently saturated o-minimal expansion M of a field. Then $(G^{00}(M);*)$ is bi-interpretable with the disjoint union of a (possibly trivial) divisible torsion-free abelian group and finitely many (possibly zero) real closed convexly valued fields.*

To indicate why this is the correct statement, let us note that it can not be strengthened to bi-interpretability with a single real closed valued field as in Theorem 1.1: one reason is that G could be commutative, and then $(G^{00};*)$ is just a divisible torsion free abelian group and thus does not even define a field; another reason is that groups are orthogonal in their direct product, so e.g. if $G = H \times H$ for a semialgebraic compact group H then, viewing G^{00} as a definable set in a valued field $(\mathcal{R};+,\cdot,\mathcal{O})$ as above, the diagonal subgroup of $G^{00}(\mathcal{R}) = H^{00}(\mathcal{R}) \times H^{00}(\mathcal{R})$ is $(\mathcal{R};+,\cdot,\mathcal{O})$-definable but not $(G^{00}(\mathcal{R});*)$-definable.

Note that Theorem 1.5 applies in particular to an arbitrary compact linear Lie group, since by Fact 2.1(ii) any such group is definable in the real field, and is definably compact.

1.1 Acknowledgements

We would like to thank Mohammed Bardestani, Alessandro Berarducci, Emmanuel Breuillard, Itay Kaplan, and Martin Hils for useful discussion. We would also like to thank the Institut Henri Poincaré and the organisers of the trimester "Model theory, combinatorics and valued fields", where some of this work was done. We also thank the anonymous referee for suggestions which improved the presentation of this paper substantially. Finally, we would like to mention the following question of Itay Kaplan which initiated the discussions which led to our results (even though our results have in the end almost nothing to do with the question): is any $\bigwedge$-definable field in an NIP theory itself NIP as a pure field?

2 Preliminaries

2.1 Notation

We consider $\mathcal{R}$ and $\mathbb{R}$ as fields and $G^{00}(\mathcal{R})$ as a group, thus "$G^{00}(\mathcal{R})$-definable" means definable in the pure group $(G^{00}(\mathcal{R}); *)$, and "$\mathcal{R}$-definable" means definable in the field $(\mathcal{R}; +, \cdot)$, while we write "$(\mathcal{R}; \mathcal{O})$-definable" for definability in the valued field $(\mathcal{R}; +, \cdot, \mathcal{O})$.

We use exponential notation for group conjugation, $g^h := hgh^{-1}$. We write group commutators as $(a, b) := aba^{-1}b^{-1}$, reserving $[X, Y]$ for the Lie bracket. For G a group, we write $(G, G)_1$ for the set of commutators in G, $(G, G)_1 := \{(g, h) : g, h \in G\}$, and we write (G, G) for the commutator subgroup, the subgroup generated by $(G, G)_1$.

For G a group and A a subset, we write the centraliser of A in G as $C_G(A) = \{g \in G : \forall a \in A.\ (g, a) = e\}$, and we write $Z(G)$ for the centre $Z(G) = C_G(G)$. Similarly for $\mathfrak{g}$ a Lie algebra and A a subset, we write the centraliser of A in $\mathfrak{g}$ as $C_{\mathfrak{g}}(A) = \{X \in \mathfrak{g} : \forall Y \in A.\ [X, Y] = 0\}$. We write $C_G(g)$ for $C_G(\{g\})$ and $C_{\mathfrak{g}}(X)$ for $C_{\mathfrak{g}}(\{X\})$.

2.2 Compact Lie groups

Proofs of the statements in the following Fact can be found in [OV90] as Theorem 5.2.10 and Theorem 3.4.5 respectively.

Fact 2.1 (Chevalley). *(i) Any compact Lie group G is linear; that is, G is isomorphic to a Lie subgroup of $\mathrm{GL}_d(\mathbb{R})$ for some d.*

(ii) Compact linear groups are algebraic; that is, any compact subgroup of any $\mathrm{GL}_d(\mathbb{R})$ is of the form $G(\mathbb{R})$ for some algebraic subgroup $G \leq \mathrm{GL}_d$ over $\mathbb{R}$.

Lemma 2.2. *Suppose H is a connected closed Lie subgroup of a compact linear Lie group G.*

Then $C_{G(\mathcal{R})}(H(\mathcal{R})) = C_{G(\mathcal{R})}(H^{00}(\mathcal{R}))$.

Proof. Since G and H are algebraic by Fact 2.1(ii), the conclusion can be expressed as a first-order sentence in the complete theory $\mathrm{RCVF}_{\mathbb{R}}$ of a non-trivially convexly valued real closed field extension $(\mathcal{R}; +, \cdot, \mathcal{O})$ of the trivially valued field $\mathbb{R}$, so we may assume without loss that $\mathcal{R}$ is $\aleph_1$-saturated.

Suppose $x \in C_{G(\mathcal{R})}(H^{00}(\mathcal{R}))$. It follows from $\aleph_1$-saturation of $\mathcal{R}$ that $x \in C_{G(\mathcal{R})}(U(\mathcal{R}))$ for some $\mathbb{R}$-definable neighbourhood of the identity $U \subseteq H$, since $H^{00}(\mathcal{R})$ is the intersection of such (e.g. as in (1)). Now we could argue from general results on o-minimal groups (see [Pil88, Lemma 2.11]) and connectedness that $C_{G(\mathcal{R})}(U(\mathcal{R})) = C_{G(\mathcal{R})}(H(\mathcal{R}))$, but we can also argue directly as follows. H is compact and connected, thus is generated in finitely many steps from U. Since $\mathcal{R}$ is an elementary extension of $\mathbb{R}$, it follows that $H(\mathcal{R})$ is generated by $U(\mathcal{R})$. Hence $x \in C_{G(\mathcal{R})}(H(\mathcal{R}))$. ☐

2.3 SO(3)

We recall some elementary facts about the group of spatial rotations $\mathrm{SO}(3) = \mathrm{SO}_3(\mathbb{R})$, its universal cover Spin(3), and their common Lie algebra $\mathfrak{so}(3)$, as discussed in e.g. [Woi17, Chapter 6].

Any rotation $g \in \mathrm{SO}(3)$ can be completely described as a planar rotation $\alpha \in \mathrm{SO}(2)$ around an axis L, where L is a ray from the origin in $\mathbb{R}^3$. We can identify such a ray L with the unique element of the unit sphere S^2 which lies on the ray. Writing $\rho : \mathrm{SO}(2) \times S^2 \twoheadrightarrow \mathrm{SO}(3)$ for the corresponding map, the only ambiguities in this description are that $\rho(\alpha, L) = \rho(-\alpha, -L)$, and the trivial rotation is $e = \rho(0, L)$ for any L. The centraliser in SO(3) of a non-trivial non-involutary rotation $g = \rho(\alpha, L)$, $2\alpha \neq 0$, is the subgroup $C_{\mathrm{SO}(3)}(g) = \rho(\mathrm{SO}(2), L) \cong \mathrm{SO}(2)$ of rotations around L. The conjugation action of SO(3) on itself is by rotation of the axis: $\rho(\alpha, L)^g = \rho(\alpha, gL)$, where gL is the image of L under the canonical (matrix) left action of g on $\mathbb{R}^3$. The map ρ is continuous and $\mathbb{R}$-definable.

This description transfers to non-standard rotations: given $\mathcal{R} \succ \mathbb{R}$, ρ extends to a map $\rho : \mathrm{SO}_2(\mathcal{R}) \times S^2(\mathcal{R}) \twoheadrightarrow \mathrm{SO}_3(\mathcal{R})$. The infinitesimal rotations are then the rotations about any (non-standard) axis by an infinitesimal angle; i.e. ρ restricts to a map $\mathrm{SO}_2^{00}(\mathcal{R}) \times S^2(\mathcal{R}) \twoheadrightarrow \mathrm{SO}_3^{00}(\mathcal{R})$.

The universal group cover of SO(3) is denoted Spin(3); the corresponding continuous covering homomorphism $\pi : \mathrm{Spin}(3) \to \mathrm{SO}(3)$ is a local isomorphism with kernel $Z(\mathrm{Spin}(3)) \cong \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z}$. Since Spin(3) is compact, by Fact 2.1(i) we may represent it as $\mathrm{Spin}(3) = \mathrm{Spin}_3(\mathbb{R})$ where Spin_3 is a linear Lie group. Since π is an $\mathbb{R}$-definable local isomorphism, it induces an isomorphism of Lie algebras and an isomorphism of infinitesimal subgroups $\mathrm{Spin}_3^{00}(\mathcal{R}) \cong \mathrm{SO}_3^{00}(\mathcal{R})$, with respect to which the action by conjugation of $g \in \mathrm{Spin}_3(\mathcal{R})$ on $\mathrm{Spin}_3^{00}(\mathcal{R})$ agrees with the action by conjugation of $\pi(g)$ on $\mathrm{SO}_3^{00}(\mathcal{R})$.

The Lie algebra $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ has $\mathbb{R}$-basis $\{H, U, V\}$ and bracket relations

$$[U, V] = H, \ [H, U] = V, \ [V, H] = U. \quad (2)$$

The adjoint action Ad of $g \in \mathrm{SO}(3)$ on $\mathfrak{so}(3)$ is by the left matrix action with respect to this basis, $\mathrm{Ad}_g(X) = gX$, and similarly for $g \in \mathrm{Spin}(3)$ the adjoint representation is π, i.e. $\mathrm{Ad}_g(X) = \pi(g)X$.

2.4 Lie theory in o-minimal structures

We recall briefly the Lie theory of a group G definable in an o-minimal expansion of a real closed field R; see [PPS00a] for further details, but in fact we apply it only to linear algebraic groups, for which it agrees with the usual theory for such groups. The Lie algebra of G is the tangent space at the identity $\mathfrak{g} = L(G) = T_e(G)$, a finite dimensional R-vector space. For $h \in G(R)$, define $\mathrm{Ad}_h : \mathfrak{g} \to \mathfrak{g}$ to be the differential of conjugation by h at the identity, $\mathrm{Ad}_h := d_e(\cdot^h)$, and define $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_R(\mathfrak{g})$ as the differential of $\mathrm{Ad} : G \to \mathrm{Aut}_R(\mathfrak{g})$ at the identity, $\mathrm{ad} := d_e(\mathrm{Ad})$. Then the Lie bracket of $\mathfrak{g}$ is defined as $[X, Y] := \mathrm{ad}_X(Y)$.

The statements above about the adjoint action of SO_3 and Spin_3 on $\mathfrak{so}_3$ transfer to $\mathcal{R}$: for $X \in \mathfrak{so}_3(\mathcal{R})$, we have that $\mathrm{Ad}_g(X) = gX$ for $g \in \mathrm{SO}_3(\mathcal{R})$, and $\mathrm{Ad}_g(X) = \pi(g)X$ for $g \in \mathrm{Spin}_3(\mathcal{R})$.

3 Proof of Theorem 1.1

Let G and $\mathcal{R}$ be as in Theorem 1.1, namely G is a simple compact linear Lie group and $\mathcal{R}$ is a proper real closed field extension[1] of $\mathbb{R}$.

In this section, we write G^{00} for the group $G^{00}(\mathcal{R})$.

We first give an overview of the proof. We begin in §3.1 by finding a copy of SO(3) or Spin(3) in G which is definable in such a way that its infinitesimal subgroup is G^{00}-definable. A reader who is interested already in the case $G = \mathrm{SO}(3)$ may prefer to skip that section on a first reading. In §3.2 we use the structure of SO_3^{00} to find in it an interval on a centraliser which is definable both in the group and the field. In §3.3 we see that the non-abelianity of G endows this interval with a rich enough structure to trigger the existence of a field by the o-minimal trichotomy theorem. Finally, in §3.4, we use an adjoint embedding to see the valuation on this field and obtain bi-interpretability.

3.1 Finding an SO_3^{00}

In this subsection, we find a copy of $\mathfrak{so}(3)$ in the Lie algebra of G which is the Lie algebra of a Lie subgroup $S \leq G$, and which moreover is defined in such a way that the infinitesimal subgroup $S^{00} \leq G^{00}$ is G^{00}-definable.

Let $\mathfrak{g}_0 := L(G)$ be the Lie algebra of G.

Lemma 3.1. *There exist Lie subalgebras $\mathfrak{s} \leq \mathfrak{s}' \leq \mathfrak{g}_0$ such that*

(i) $\mathfrak{s}' = C_{\mathfrak{g}_0}(C_{\mathfrak{g}_0}(\mathfrak{s}'))$

(ii) $\mathfrak{s} = [\mathfrak{s}', \mathfrak{s}']$

(iii) $\mathfrak{s} \cong \mathfrak{so}(3)$.

Proof. In this proof, and in this proof alone, we assume familiarity with the basic theory and terminology of the root space decomposition of a semisimple Lie algebra. This can be found in e.g. [Kna02, §§II.4, II.5].

We write V^* for the dual space of a vector space V.

Let $\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}_0$, and let $\bar{\cdot} : \mathfrak{g} \to \mathfrak{g}$ be the corresponding complex conjugation.

Let $\mathfrak{h}_0 \leq \mathfrak{g}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$, meaning that the complexification $\mathfrak{h} := \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$; we can take $\mathfrak{h}_0 := L(T)$ where T is a maximal torus of G (where a torus of G is a Lie subgroup isomorphic to a power of the circle group).

Now since G is simple, $\mathfrak{g}$ is semisimple and so admits a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ where each root space $\mathfrak{g}_\alpha$ is the 1-dimensional eigenspace of $\mathrm{ad}(\mathfrak{h})$ with eigenvalue $\alpha \in \mathfrak{h}^* \setminus \{0\}$, i.e. if $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$ then $[H, X] = \alpha(H)X$. The roots α span $\mathfrak{h}^*$. If $\alpha \in \Delta$ then $\mathbb{C}\alpha \cap \Delta = \{\alpha, -\alpha\}$. We have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \leq \mathfrak{h}$.

Since G is compact, each root $\alpha \in \Delta$ takes purely imaginary values on $\mathfrak{h}_0$, thus $i\alpha\restriction_{\mathfrak{h}_0} \in \mathfrak{h}_0^*$, and the subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}_0$ generated by $\mathfrak{l}_\alpha :=$

[1] Readers familiar with model theory might be made more comfortable by an assumption that $\mathcal{R}$ is (sufficiently) saturated. They may in fact freely assume this, since the conclusion of the theorem can be seen to not depend on the choice of $\mathcal{R}$.

$\mathfrak{g}_0 \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ is isomorphic to $\mathfrak{so}(3)$ (see [FH91, Proposition 26.4], or [Kna02, (4.61)]).

Explicitly, if $E_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, so $\overline{E_\alpha} \in \mathfrak{g}_{-\alpha}$, then $\mathfrak{l}_\alpha$ is spanned by $U_\alpha := iE_\alpha - i\overline{E_\alpha}$ and $V_\alpha := E_\alpha + \overline{E_\alpha}$. Then for $H \in \mathfrak{h}_0$ we have

$$[H, U_\alpha] = i\alpha(H)V_\alpha \text{ and } [H, V_\alpha] = -i\alpha(H)U_\alpha, \tag{3}$$

and $[U_\alpha, V_\alpha] = 2i[E_\alpha, \overline{E_\alpha}] \in \mathfrak{s}_\alpha \cap \mathfrak{h}_0$. Using that $\mathfrak{g}_0$ admits an $\mathrm{Ad}(G)$-invariant inner product, one can argue (see the proof of [Kna02, (4.56)] for details) that $\alpha([E_\alpha, \overline{E_\alpha}]) < 0$, and hence that after renormalising E_α, the $\mathbb{R}$-basis $\{[U_\alpha, V_\alpha], U_\alpha, V_\alpha\}$ for $\mathfrak{s}_\alpha$ satisfies the standard bracket relations (2) of $\mathfrak{so}(3)$.

Let $\alpha_0 \in \Delta$. Let $\mathfrak{s} := \mathfrak{s}_{\alpha_0}$ and $\mathfrak{l} := \mathfrak{l}_{\alpha_0}$, and let $\mathfrak{s}' := \mathfrak{h}_0 \oplus \mathfrak{l}$. It follows from the bracket relations above that $\mathfrak{s}'$ is a subalgebra of $\mathfrak{g}_0$, and the commutator subalgebra $[\mathfrak{s}', \mathfrak{s}']$ is precisely $\mathfrak{s}$.

It remains to show that $\mathfrak{s}' = C_{\mathfrak{g}_0}(C_{\mathfrak{g}_0}(\mathfrak{s}'))$.

It follows from (3) and $C_{\mathfrak{g}_0}(\mathfrak{h}_0) = \mathfrak{h}_0$ that $C_{\mathfrak{g}_0}(\mathfrak{s}') = \mathfrak{h}_0 \cap C_{\mathfrak{g}_0}(\mathfrak{l}) = \ker(\alpha_0 \restriction_{\mathfrak{h}_0})$.

Now for $\alpha' \in \Delta$ and $W \in \mathfrak{g}_{\alpha'} \setminus \{0\}$, we have $[W, \ker(\alpha_0 \restriction_{\mathfrak{h}_0})] = 0$ if and only if $\ker(\alpha' \restriction_{\mathfrak{h}_0}) \supseteq \ker(\alpha_0 \restriction_{\mathfrak{h}_0})$; since $i\alpha' \restriction_{\mathfrak{h}_0}, i\alpha_0 \restriction_{\mathfrak{h}_0} \in \mathfrak{h}_0^*$, this holds if and only if $\alpha' \in \mathbb{R}\alpha_0 \cap \Delta = \{\alpha_0, -\alpha_0\}$. Thus $C_{\mathfrak{g}_0}(C_{\mathfrak{g}_0}(\mathfrak{s}')) = \mathfrak{g}_0 \cap (\mathfrak{h} + \mathfrak{g}_{\alpha_0} + \mathfrak{g}_{-\alpha_0}) = \mathfrak{s}'$, as required. □

Before the next lemma we recall some facts and terminology from Lie theory. An *integral subgroup* of G is a connected Lie group which is an abstract subgroup of G via an inclusion map which is an immersion. An integral subgroup is a Lie subgroup iff it is closed in G (see [Bou89, Proposition III.6.2.2]). The map $H \mapsto L(H)$ of taking the Lie algebra is a bijection between integral subgroups of G and Lie subalgebras of $\mathfrak{g}_0 = L(G)$ (see [Bou89, Theorem III.6.2.2]).

Lemma 3.2. *There exists a closed Lie subgroup $S \leq G$ isomorphic to either* $\mathrm{Spin}(3)$ *or* $\mathrm{SO}(3)$*, such that the infinitesimal subgroup $S^{00} = S(\mathcal{R}) \cap G^{00} \cong \mathrm{SO}_3^{00}$ is G^{00}-definable as a subgroup of G^{00}.*

Proof. Let $\mathfrak{s}$ and $\mathfrak{s}'$ be as in Lemma 3.1. Let S and S' be the integral subgroups of G with Lie algebras $\mathfrak{s}$ and $\mathfrak{s}'$, respectively. Since $\mathfrak{s}' = C_{\mathfrak{g}_0}(C_{\mathfrak{g}_0}(\mathfrak{s}'))$, we have by [Bou89, Proposition III.9.3.3] that $S' = C_G(C_G(S'))$. In particular, S' is closed.

Since $[\mathfrak{s}', \mathfrak{s}'] = \mathfrak{s}$ and $\mathfrak{s}$ is an ideal in $\mathfrak{s}'$, we have by [Bou89, Proposition III.9.2.4] that the commutator subgroup (S', S') is equal to S. Now $\mathfrak{s}$ is isomorphic to $\mathfrak{so}(3)$ and $\mathrm{Spin}(3)$ is simply connected, thus (by [Bou89, Theorem III.6.3.3]) S is the image of a non-singular homomorphism $\mathrm{Spin}(3) \to G$ with central kernel, and so since $\mathrm{Spin}(3)$ is compact and its centre is of order 2, S is a closed Lie subgroup of G isomorphic either to $\mathrm{Spin}(3)$ or to $\mathrm{SO}(3) \cong \mathrm{Spin}(3)/Z(\mathrm{Spin}(3))$.

Now S is a closed Lie subgroup of G, and thus is a compact linear Lie group, and its infinitesimal subgroup S^{00} is correspondingly the subgroup $S(\mathcal{R}) \cap G^{00}$ of G^{00}. The same goes for the closed Lie subgroups S' and $S'' := C_G(S')$ of G.

Claim 3.3. $(S')^{00} = C_{G^{00}}(C_{G^{00}}(S'(\mathcal{R})))$.

Proof.

$$\begin{aligned}(S')^{00} &= S'(\mathcal{R}) \cap G^{00} \\ &= C_{G(\mathcal{R})}(C_{G(\mathcal{R})}(S'(\mathcal{R}))) \cap G^{00} \\ &= C_{G^{00}}(C_{G(\mathcal{R})}(S'(\mathcal{R}))) \\ &= C_{G^{00}}(S''(\mathcal{R})) \\ &= C_{G^{00}}((S'')^{00}) && \text{(by Lemma 2.2)} \\ &= C_{G^{00}}(C_{G(\mathcal{R})}(S'(\mathcal{R})) \cap G^{00}) \\ &= C_{G^{00}}(C_{G^{00}}(S'(\mathcal{R}))).\end{aligned}$$

□

By the Descending Chain Condition for definable groups in o-minimal structures, [PPS00a, Corollary 1.16], there exists a finite set $X_0 \subseteq C_{G^{00}}(S') \subseteq G^{00}$ such that $C_G(C_{G^{00}}(S')) = C_G(X_0)$. Thus,

$$S^{00} = C_G(X_0) \cap G^{00} = C_{G^{00}}(X_0)$$

is G^{00}-definable.

Meanwhile, for the commutator subgroup, we have $((S')^{00}, (S')^{00}) \subseteq (S', S')^{00} = S^{00}$. Now $(\mathrm{SO}_3^{00}, \mathrm{SO}_3^{00})_1 = \mathrm{SO}_3^{00}$; in fact [DM16] proves $(G^{00}, G^{00})_1 = G^{00}$ for any compact semisimple Lie group, but one can also see it directly in this case, since $(\mathrm{SO}_3^{00}, \mathrm{SO}_3^{00})_1$ is invariant under conjugation by $\mathrm{SO}_3(\mathcal{R})$ and can be seen to contain infinitesimal rotations of all infinitesimal angles. So since $(S')^{00} \supseteq S^{00} \cong \mathrm{SO}_3^{00}$, we have $((S')^{00}, (S')^{00})_1 = S^{00}$. Hence S^{00} is G^{00}-definable.

Thus S is as required. □

3.2 Finding a group interval in S^{00}

Let $S \leq G$ be as given by Lemma 3.2. Now S is isomorphic to SO_3 or Spin_3 via an isomorphism which has compact graph and hence is $\mathbb{R}$-definable, so let the $\mathbb{R}$-definable map $\pi : S \twoheadrightarrow \mathrm{SO}_3$ be this isomorphism or its composition with the universal covering homomorphism respectively. Then π induces an isomorphism $\pi : S^{00} \xrightarrow{\cong} \mathrm{SO}_3^{00}$.

From now on, we work in $\mathcal{R}$, and consider S, G, SO_2, SO_3, and Spin_3 as $\mathbb{R}$-definable (linear algebraic) groups rather than as Lie groups; moreover, **we write** S **for** $S(\mathcal{R})$, and similarly with G, SO_2, SO_3, and Spin_3.

Let $g \in S^{00} \setminus \{e\}$. Then $C_S(g) = \pi^{-1}(C_{\mathrm{SO}_3}(\pi(g)))$, and $C_{\mathrm{SO}_3}(\pi(g)) \cong \mathrm{SO}_2$ since g is not torsion. Thus the circular order on SO_2 induces[2] an $\mathcal{R}$-definable linear order on a neighbourhood of the identity in $C_S(g)$ containing $C_{S^{00}}(g)$, and the induced linear order on $C_{S^{00}}(g)$ makes it a linearly ordered abelian group. Moreover, the order topology on $C_{S^{00}}(g)$ coincides with the group topology. We may assume that g is positive with respect to this ordering.

Lemma 3.4. *There exists an open interval $J \subseteq C_{S^{00}}(g)$ containing e such that J and the restriction to J of the order on $C_{S^{00}}(g)$ are both S^{00}-definable and $\mathcal{R}$-definable.*

[2] Explicitly, say $\pi(g) = \rho(\alpha, L)$; then if $\pi_{(1,2)} : \mathrm{SO}_2 \to \mathcal{R}$ extracts the top-right matrix element, then $x_1 < x_2 \Leftrightarrow \pi_{(1,2)}(\beta_1) < \pi_{(1,2)}(\beta_2)$ where $\pi(x_i) = \rho(\beta_i, L)$ defines an order on such a neighbourhood of the identity in $C_S(g)$.

Proof. First, consider the $\mathcal{R}$-definable set $I := g^S g^S \cap C_S(g)$, where $g^S g^S := \{g^{a_1} g^{a_2} : a_1, a_2 \in S\}$. Since S^{00} is normal in S, in fact $I \subseteq C_{S^{00}}(g)$.

Say a subset X of a group is *symmetric* if it is closed under inversion, i.e. $X^{-1} = X$.

Claim 3.5. *I is a closed symmetric interval in $C_{S^{00}}(g)$.*

Proof. Recall that a definable set in an o-minimal structure is *definably connected* if it is not the union of disjoint open definable subsets.

$X := g^S g^S$ is the image under a definable continuous map of the definably connected closed bounded set $(g^S) \times (g^S)$, and hence ([vdD98, 1.3.6,6.1.10]) X is closed and definably connected.

Now X is invariant under conjugation by S. Thus $\pi(X) \subseteq \mathrm{SO}_3^{00}$ consists of the rotations around arbitrary axes by the elements of some $\mathcal{R}$-definable set $\Theta \subseteq \mathrm{SO}_2^{00}$, i.e. $\pi(X) = \rho(\Theta, S^2(\mathcal{R}))$. Since $\rho(\theta, L)^{-1} = \rho(\theta, -L)$, it follows that $\pi(X)$ is symmetric, and hence Θ is symmetric. Similarly, $\pi(g^S)$ is symmetric, and hence $e \in \pi(X)$ and so $e \in \Theta$.

Recall that if L is the axis of rotation of $\pi(g)$ (i.e. $\pi(g) = \rho(\alpha, L)$ for some α), then $\pi(C_S(g)) = \rho(\mathrm{SO}_2, L)$. So then $\pi(I) = \rho(\Theta, L)$. Since $\pi(X)$ is closed and definably connected, Θ is of the form $\Theta' \cup \Theta'^{-1}$ where Θ' is a closed interval. Thus since Θ contains the identity, Θ is itself a closed symmetric interval. So I is a closed symmetric interval in $C_{S^{00}}(g)$. ☐

Say $I = [h^{-1}, h]$. Write the group operation on $C_{S^{00}}(g)$ additively. For $g_1, g_2 \in S^{00}$, let $Y_{g_1,g_2} := g_1^{S^{00}} g_2^{S^{00}} \cap C_{S^{00}}(g)$ (an S^{00}-definable set). Clearly, $Y_{g_1,g_2} \subset I = [-h, h]$.

Claim 3.6. *There exist*[3] *$h'' < h$ and $g_1, g_2 \in S^{00}$ such that the interval $(h'', h]$ is contained in Y_{g_1,g_2}.*

Proof. Consider the map $f : S^2 \to S$ defined by $f(a_1, a_2) := g^{a_1} g^{a_2}$. By definable choice in $\mathcal{R}$, f admits a $\mathcal{R}$-definable section over the set I, and hence f admits a continuous $\mathcal{R}$-definable section on an open interval $\theta : (h', h) \to S^2$ for some $h' \in C_{S^{00}}(g)$. By definable compactness of S^2, θ extends to a continuous $\mathcal{R}$-definable section $\theta : (h', h] \to S^2$. Say $\theta(h) = (a_1, a_2) \in S^2$. Let $g_i := g^{a_i}$ for $i = 1, 2$. Note that $g_i \in S^{00}$.

So by continuity of θ, for some $h'' < h$ we have

$$(h'', h] \subseteq \theta^{-1}((S^{00} a_1) \times (S^{00} a_2)) \subseteq f((S^{00} a_1) \times (S^{00} a_2)) = g_1^{S^{00}} g_2^{S^{00}}.$$

☐

Thus

$$P := h - Y_{g_1,g_2} \subseteq h - I = [0, 2h]$$

is an S^{00}-definable subset of the non-negative part of $C_{S^{00}}(g)$.

So set $p := h - h'' > 0$. Note that $[0, p) \subseteq P \subseteq [0, 2h]$. It is now easy to see that the open interval $(0, p)$ is equal to $P \cap (p - P)$, thus is definable in S^{00}.

[3] Although this existence statement suffices for our purposes, in fact one may calculate that we may take $h = g^2$ and $g_1 = g = g_2$. This can be seen by combining the proof of the claim with the following observations on Spin_3 considered as the group of unit quaternions. Firstly, if $a, b \in \mathrm{Spin}_3(\mathbb{R})$ have the same scalar part, $\Re(a) = \Re(b)$, then $\Re(a * b) \geq \Re(a * a)$, with equality iff $a = b$. Secondly, conjugation in Spin_3 preserves scalar part. Finally, the order on $C_{S^{00}}(g)$ agrees (up to inversion) with the order on the scalar part.

So $J := (-p, p) = [0, p) \cup -[0, p)$ and its order are S^{00}-definable, since for $a, b \in J$ we have $a \geq b$ iff $a - b \in [0, p)$.

Finally, $J = (-p, p)$ and its order are also $\mathcal{R}$-definable as an interval in the $\mathcal{R}$-definable order on an $\mathbb{R}$-definable neighbourhood of the identity in $C_S(g)$. This ends the proof of Lemma 3.4. □

3.3 Defining the field

Let J be as given by Lemma 3.4. We now return to working with the full simple compact group G, of which J is a subset. Let $n := \dim(G)$. Recall that J consists of a neighborhood of the identity in the one-dimensional real algebraic group group $C_S(g)$, thus it is a one-dimensional smooth sub-manifold of G. For $h \in G$, the set J^h is an open neighborhood of e in the group $C_S(g)^h$. We let $T_e(J^h)$ denote its tangent space at e with respect to the real closed field R.

Lemma 3.7. *There exists an open neighbourhood $U \subseteq G^{00}$ of the identity, an open interval $e \in J' \subseteq J$, and a bijection $\phi : J'^n \to U$ which is both G^{00}-definable and $\mathcal{R}$-definable, with $\phi(x, e, \ldots, e) = x$.*

Proof. Let $\mathfrak{g} = T_e(G)$ be the Lie algebra of G (as discussed in §2.4). Consider the $\mathcal{R}$-subspace $V \leq \mathfrak{g}(\mathcal{R})$ generated by

$$\bigcup_{h \in G^{00}} T_e(J^h) = \bigcup_{h \in G^{00}} \mathrm{Ad}_h(T_e(J)).$$

Then V is $\mathrm{Ad}_{G^{00}}$-invariant. Thus Ad restricts to $\mathrm{Ad} : G^{00} \to \mathrm{Aut}_{\mathcal{R}}(V)$, and hence the differential at the identity is a map $\mathrm{ad} = d_e \mathrm{Ad} : \mathfrak{g}(\mathcal{R}) \to \mathrm{End}_{\mathcal{R}}(V)$, i.e. it follows that V is $\mathrm{ad}_{\mathfrak{g}(\mathcal{R})}$-invariant. So V is a non-trivial ideal in $\mathfrak{g}(\mathcal{R})$. But $\mathfrak{g}$ is simple, thus we have $V = \mathfrak{g}(\mathcal{R})$.

So say $h_1, h_2, \ldots, h_n \in G^{00}$ are such that $T_e(J^{h_i})$ span T_eG. Conjugating by h_1^{-1}, we may assume $h_1 = e$.

Define $\phi : J^n \to G^{00}$ by

$$\phi(x_1, \ldots, x_n) := x_1^{h_1} x_2^{h_2} \ldots x_n^{h_n},$$

which is clearly both G^{00}-definable and $\mathcal{R}$-definable. Then the differential $d_{(e,\ldots,e)}\phi : T_eJ^n \to T_eG(\mathcal{R})$ is an isomorphism. Thus by the implicit function theorem (for the real closed field $\mathcal{R}$), for some open interval $e \in J' \subseteq J$, the restriction $\phi\restriction_{(J')^n}$ is a bijection with some open neighbourhood U of e, as required.

□

Redefine J to be J' as provided by the lemma.

We shall consider the o-minimal structure obtained by expanding the interval $(J; <)$ by the pullback of the group operation near e via the chart ϕ. As we will now verify, it follows from the non-abelianity of G that the resulting structure on J is "rich" in the sense of the o-minimal trichotomy theorem [PS98] (see below), and so by that theorem it defines a field. Let us first recall the relevant notions:

Definition 3.8. Let $\mathcal{M} = (M; <, \cdots)$ be an o-minimal structure. A *definable family of curves* is given by a definable set $F \subseteq M^n \times T$, for some definable $T \subseteq M^k$, such that for every $t \in T$, the set $F_t = \{a \in M^n : (a, t) \in F\}$ is of dimension 1.

The family is called *normal* if for $t_1 \neq t_2$, the set $F_{t_1} \cap F_{t_2}$ is finite. In this case, the dimension of the family is taken to be $\dim T$.

A linearly ordered set $(I; <)$, together with a partial binary function $+$ and a constant 0, is called *a group-interval* if $+$ is continuous, definable in a neighborhood of $(0, 0)$, associative and commutative when defined, order preserving in each coordinate, has 0 as a neutral element and each element near 0 has additive inverse in I.

We shall use Theorem 1.2 in [PS98]:

Fact 3.9. *Let $\mathcal{I} = (I; <, +, 0, \cdots)$ be an ω_1-saturated o-minimal expansion of a group-interval. Then one and only one of the following holds:*

1. *There exists an ordered vector space $\mathcal{V}$ over an ordered division ring, such that $(I; <, +, 0)$ is definably isomorphic to a group-interval in V and the isomorphism takes every $\mathcal{I}$-definable set to a definable set in $\mathcal{V}$.*
2. *A real closed field is definable in $\mathcal{I}$, with its underlying set a subinterval of I and its ordering compatible with $<$.*

We now return to our interval $J \subseteq C_S(g) \subseteq G^{00}$ and to the definable bijection $\phi : J^n \to U$. We let $\star : J^n \times J^n \dashrightarrow J^n$ be the partial function obtained as the pullback via ϕ of the group operation in G. Namely, for $a, b, c \in J^n$,

$$a \star b = c \Leftrightarrow \phi(a) \cdot \phi(b) = \phi(c).$$

Since ϕ and the group operation on U are definable in both G^{00} and $\mathcal{R}$, then so is $\star$. Because J is an open interval around the identity inside the one-dimensional group $C_S(g) \subseteq G$, the restriction to J of the group operation of G makes J into a group-interval, and we let $+$ denote this restriction to J. Note that the ordering on J is definable using $+$ and therefore definable in both G^{00} and in $\mathcal{R}$.

Lemma 3.10. *The structure $\mathcal{J} = (J; <, +, \star)$ is an o-minimal expansion of a group-interval, not of type (1) in the sense of Fact 3.9. It is definable in both G^{00} and in $\mathcal{R}$.*

Proof. The structure $\mathcal{J}$ is o-minimal since it is definable in the o-minimal structure $\mathcal{R}$ and the ordered interval $(J; <)$ is definably isomorphic via a projection map with an ordered interval in $(\mathcal{R}; <)$. We want to show that $\mathcal{J}$ is not of type (1).

To simplify notation we denote below the group $C_S(g)$ by H. Because G is a simple group, there are infinitely many distinct conjugates of the one-dimensional group H. More precisely, $\dim N_G(H) < \dim G$ and if $h_1, h_2 \in G$ are not in the same right-coset of $N_G(H)$ then $H^{h_1} \cap H^{h_2}$ and hence also $J^{h_1} \cap I^{h_2}$ is finite. Since $\dim(N_G(H)) < \dim G$ one can find infinitely many $h \in G$, arbitrarily close to e, which belong to different right-cosets of $N_G(H)$. In fact, by Definable Choice in o-minimal expansions of groups or group-intervals, we can find in $\mathcal{J}$ a definably connected one-dimensional set $C \subseteq J^n$ with $(e, \ldots, e) \in C$, such that no two elements of $\phi(C)$ belong to the same right-coset of $N_G(H)$.

The family $\{H^h \cap U : h \in \phi(C)\}$ is a one-dimensional normal family of definably connected curves in U, all containing e, and we want to "pull it back" to the structure $\mathcal{J}$. In order to do that we first note that by replacing J by a subinterval $J_0 \subseteq J$, and replacing C by a possibly smaller definably connected set, we may assume that for every $h \in \phi(C)$ and every $g \in J_0$, each of $h, h^{-1}g$ and $h^{-1}gh$ is inside U. Thus, the one-dimensional normal family of definably connected curves

$$\{\phi^{-1}(J_0^h) : h \in \phi(C)\}$$

is definable in $\mathcal{J}$, and all of these curves contain the point $(e, \ldots, e)$ $(=\phi^{-1}(e))$.

Now, if $\mathcal{J}$ was of type (1) in Fact 3.9, then up to a change of signature it would be a reduct of an ordered vector space. However, it easily follows from quantifier elimination in ordered vector spaces that in such structures there is no definable infinite normal family of one-dimensional definably connected sets, all going through the same point. Hence, $\mathcal{J}$ is not of type (1). □

By Fact 3.9, there is a $\mathcal{J}$-definable real closed field K on an open interval in J containing e. Thus K with its field structure is also G^{00}-definable and $\mathcal{R}$-definable.

3.4 Obtaining the valuation, and bi-interpetability

Let $K \subseteq J$ be the definable real closed field obtained in the previous subsection, considered as a pure field. By [OPP96], there is an $\mathcal{R}$-definable field isomorphism $\theta_R : \mathcal{R} \to K$. Let $G_K := G^{\theta_R}(K)$ be the group of K-points of the K-definable (linear algebraic) group obtained by applying θ_R to the parameters defining G, so θ_R induces an $\mathcal{R}$-definable group isomorphism $\theta_G : G \to G_K$. Let $G_K^{00} := (G^{\theta_R})^{00}(K)$ be the corresponding infinitesimal subgroup, thus θ_R restricts to an isomorphism $\theta_G\restriction_{G^{00}} : G^{00} \to G_K^{00}$.

Denote by $(\mathcal{R}; G^{00})$ the expansion of the field $\mathcal{R}$ by a predicate for $G^{00} \leq G$.

Lemma 3.11. *$(\mathcal{R}; G^{00})$ and $(\mathcal{R}; \mathcal{O})$ have the same definable sets.*

Proof. Since G is defined over $\mathbb{R}$, it admits a chart at the identity defined over $\mathbb{R}$, that is, an $\mathbb{R}$-definable homeomorphism $\psi : I^n \to U$, where $I \subseteq \mathcal{R}$ is an open interval around 0, and $U \subseteq G$ is an open neighbourhood of e, and $\psi((0, \ldots, 0)) = e$.

Then $\mathrm{st}_G(\psi(x)) = \psi(\mathrm{st}(x))$, and so $G^{00} = \psi(\mathfrak{m}^n)$. Thus G^{00} is definable in $(\mathcal{R}; \mathcal{O})$, and conversely $\mathfrak{m}^n$, and hence $\mathfrak{m}$, and so also $\mathcal{O} = \mathcal{R} \setminus \frac{1}{\mathfrak{m}}$, are definable in $(\mathcal{R}; G^{00})$. □

Lemma 3.12. *$G_K^{00} \leq G_K$ is G^{00}-definable, and moreover $\theta_G\restriction_{G^{00}} : G^{00} \to G_K^{00}$ is G^{00}-definable.*

Proof. Let $n := \dim(G)$.

Precisely as in [PPS00a, 3.2.2], translating by an element of G^{00} if necessary we may assume that the group operation is C_1 for K on a neighbourhood of the identity according to the chart $\phi\restriction_{K^n}$ (where ϕ is the map from Lemma 3.7), and then the adjoint representation yields an $\mathcal{R}$-definable homomorphism $\mathrm{Ad} : G \to \mathrm{GL}_n(K)$. Since G^{00} defines ϕ and the field K and the conjugation maps $x \mapsto x^g$ for $g, x \in G^{00}$, the restriction $\mathrm{Ad}\restriction_{G^{00}} : G^{00} \hookrightarrow \mathrm{GL}_n(K)$ is G^{00}-definable, and is an embedding since G has finite centre, and G^{00} is torsion-free.

Define $\eta := \mathrm{Ad} \circ \theta_G^{-1} : G_K \hookrightarrow \mathrm{GL}_n(K)$. So η is $\mathcal{R}$-definable. Since K is $\mathcal{R}$-definably isomorphic to $\mathcal{R}$, the $\mathcal{R}$-definable structure on K is just the field structure. Thus η is also K-definable, and hence G^{00}-definable.

So $\theta_G\restriction_{G^{00}} = \eta^{-1} \circ \mathrm{Ad}\restriction_{G^{00}}$ is G^{00}-definable.

□

Proof of Theorem 1.1. By Lemma 3.12, θ_R provides a definition of $(\mathcal{R}; G^{00})$ in G^{00} with universe K. This forms a bi-interpretation together with the tautological interpretation of G^{00} in $(\mathcal{R}; G^{00})$; indeed, the composed interpretations are θ_R and $\theta_G \restriction_{G^{00}}$, which are $\mathcal{R}$-definable and G^{00}-definable respectively.

Combining this with Lemma 3.11 concludes the proof of Theorem 1.1. □

4 Isomorphisms of infinitesimal subgroups

Cartan [Car30] and van der Waerden [vdW33] showed that any abstract group isomorphism between compact semisimple Lie groups is continuous. In a similar spirit, Theorem 4.10 below shows that every abstract group isomorphism of two infinitesimal subgroups of simple compact Lie groups is, up to field isomorphisms, given by an algebraic map.

We preface the proof with two self-contained preliminary subsections. In outline, the proof is as follows. We first prove in Theorem 4.2 that given $R \models \mathrm{RCVF}$, any model of RCVF definable R is definably isomorphic to R. As in other cases of the "model-theoretic Borel-Tits phenomenon", first described for ACF in [Poi88], it follows that every abstract group isomorphism of the infinitesimal subgroups is the composition of a valued field isomorphism with an RCVF-definable group isomorphism; we give a general form of this argument in Lemma 4.8. Finally, in Section 4.3 we deduce the final statement by seeing that any RCVF-definable group isomorphism of the infinitesimal subgroups is induced by an algebraic isomorphism of the Lie groups.

4.1 Definable fields in RCVF

Here, we show that there are no unexpected definable fields in R^n for $R \models \mathrm{RCVF}$.

In this subsection we reserve the term 'semialgebraic' for R-semialgebraic sets. Let $\mathcal{R}_v = \langle R; \mathcal{O} \rangle \models \mathrm{RCVF}$. We say a point a of an $\mathcal{R}_v$-definable set X over A is *generic* over A if $\mathrm{trd}(a/A)$ is maximal for points in $X(R')$ for R' an elementary extension of R. Such an a exists if R is $(\aleph_0 + |A|^+)$-saturated. This maximal transcendence degree is the dimension $\dim(X)$ of X, which coincides ([MMS00, Theorem 4.12]) with the largest d such that the image of X under a projection to d co-ordinates has non-empty interior.

Fact 4.1. 1. *Let $X \subseteq R^n$ be an $\mathcal{R}_v$-definable set over A, and $a \in X$ a generic element over A. Then there exists a semialgebraic neighborhood $U \subseteq R^n$ of a (possibly defined over additional parameters, which may be taken to be independent of Aa) such that $U \cap X$ is semialgebraic.*

2. *Let $U \subseteq R^d$ be an open $\mathcal{R}_v$-definable set over A, and $a \in U$ generic over A. Let $F : U \to R$ be an $\mathcal{R}_v$-definable function. Then there exists a semialgebraic neighborhood U of a, such that $F|U$ is semialgebraic and C^1 with respect to R, meaning that all partial derivatives of F with respect to R exist and are continuous on U.*

Proof. 1. Permuting co-ordinates, we may assume $a = (b, c)$ where $b \in R^d$ is generic over A and $c \in R^{n-d}$ is in $\mathrm{dcl}(bA)$. Let $\pi : R^n \to R^d$ be the projection to the first d co-ordinates.

By [MMS00, Theorem 4.11], X admits a decomposition into finitely many disjoint A-definable cells each of which is the graph of a definable function on an open subset of some R^t.

Definable closure in $\mathcal{R}_v$ coincides with definable closure in R [Mel06, Theorem 8.1(1)]. Thus if C is the cell containing a, then C is the graph of a semialgebraic function on a neighbourhood of b. We now claim that locally near a, the set X is equal to C.

Suppose for a contradiction that $C' \neq C$ is another cell in the decomposition, and $a \in \mathrm{cl}(C')$, the topological closure of C'. The cell decomposition of X induces a cell decomposition of $\pi(X) \subseteq R^d$, and since b is generic in R^d over A, it must belong to the interior of $\pi(C)$. It follows that $\pi(C') = \pi(C)$, and so there exists $c' \neq c$ such that $(b, c') \in C'$.

By the inductive definition of a cell and the genericity of a in X, also C' is the graph of a semialgebraic function on a neighbourhood of b, thus in particular is locally closed, contradicting $a \in \mathrm{cl}(C')$.

2. By (1), the graph of F is a semialgebraic set in a neighborhood of $(a, f(a))$, and since a is generic in its domain, the function F is C^1 in a neighborhood of a.

□

Theorem 4.2. *Let $\mathcal{R}_v = \langle R; \mathcal{O}\rangle \models \mathrm{RCVF}$.*

(a) If $F \subseteq R^n$ is a definable field in $\mathcal{R}_v$ then it is $\mathcal{R}_v$-definably isomorphic to either R or its algebraic closure $R(\sqrt{-1})$.

(b) If $F \subseteq R^n$ is a definable non-trivially valued field in $\mathcal{R}_v$ then it is $\mathcal{R}_v$-definably isomorphic to either $\mathcal{R}_v$ or its algebraic closure $\mathcal{R}_v(\sqrt{-1})$.

Proof. Passing to an elementary extension as necessary, we assume R is $(\aleph_0 + |A|^+)$-saturated for any parameter set A we consider.

We first prove (a). Let $d = \dim F$. We first show that the additive group of F can be endowed with the structure of a definable C^1 atlas (not necessarily finite). By that we mean: a definable family of subsets of F, $\{U_t : t \in T\}$, and a definable family of bijections $f_t : U_t \to V_t$, where each V_t is an open subset of R^d, such that for every $s, t \in T$, the set $f_t(U_t \cap U_s)$ is open in V_t and the transition maps $\sigma_{t,s} : f_t(U_t \cap U_s) \to f_s(U_t \cap U_s)$ are C^1 with respect to R. Moreover, the group operation and additive inverse are C^1-maps when read through the charts.

To see this we follow the strategy of the paper of Marikova, [Mař07]. Without loss of generality F is definable over $\emptyset$. We fix $g \in F$ generic and an open neighborhood $U \ni g$ as in Fact 4.1 (1). By the cell decomposition in real closed fields, we may assume that $U \cap F$ is a cell, so definably homeomorphic to some open subset V of R^d. By replacing $U \cap F$ with V we may assume that U is an open subset of R^d, and g is generic in F over the parameters defining U.

Claim 4.3. *The map $(x, y, z) \mapsto x - y + z$ is a C^1-map (as a map from U^3 into U) in some neighborhood of (g, g, g).*

Proof. The proof is identical to [Mař07, Lemma 2.10], with Fact 4.1 (2) above replacing Lemma 2.8 there. □

Thus, there exists $U_0 \ni g$, such that the map $(x, y, z) \mapsto x - y + z$ is a C^1 map from U_0^3 into U. We now consider the definable cover of F:

$$\mathcal{U} = \{h + U_0 : h \in F\},$$

(with $+$ the F-addition) and the associated family of chart maps $f_h : h+U_0 \to U_0$, $f_h(x) = x-h$. Using Claim 4.3, it is not hard to see that $\mathcal{U}$ endows $\langle F, +\rangle$ with a definable C^1-atlas; indeed, if $h+U_0 \cap h'+U_0 \neq \emptyset$, say $h+u_0 = h'+u_0'$, then $h-h' = u_0'-u_0$, so $\sigma_{h,h'}(u_0'') = h+u_0''-h' = u_0''-u_0+u_0'$. Similarly, the function $+$ is a C^1-map from F^2 into F (where F^2 is endowed with the product atlas), and $x \mapsto -x$ is a C^1-map as well. Indeed, in [Mař07] Marikova proves in exactly the same way that the same $\mathcal{U}$ endows the group with a topological group structure (using [Mař07, Lemma 2.10] in place of Claim 4.3).

By Fact 4.1 (2), every definable function from F to F is C^1 in a neighborhood of generic point of F. Thus, just as in [Mař07, Lemma 2.13], we have:

Fact 4.4. *If $\alpha : F \to F$ is a definable endomorphism of $\langle F, +\rangle$ then α is a C^1-map.*

For every $c \in F$, we consider the map $\lambda_c : F \to F$, defined by $\lambda_c(x) = cx$ (multiplication in F). By fact 4.4 each λ_c is a C^1-map and we consider its Jacobian matrix at 0, with respect to R, denoted by $J_0(\lambda_c)$. This is a matrix in $M_d(R)$, and the map $c \mapsto J_0(\lambda_c)$ is $\mathcal{R}_v$-definable.

As was discussed in [OPP96, Lemma 4.3], it follows from the chain rule that the map $c \mapsto J_0(\lambda_c)$ is a ring homomorphism into $M_d(R)$ (note that we do not use here the uniqueness of solutions of ODE as in [OPP96], thus we a-priori only obtain a ring homomorphism). However, since F is a field the map is injective.

To summarize, we mapped F isomorphically and definably onto an $\mathcal{R}_v$-definable field, call it F_1, of matrices inside $M_d(R)$. Notice that now the field operations are just the usual matrix operations, 1_{F_1} is the identity matrix, so in particular, all non-zero elements of F_1 are invertible matrices in $M_d(R)$. Our next goal is to show that F_1 is semialgebraic.

By Fact 4.1 (2), there exists some non-empty relatively open subset of F_1 which is semialgebraic. By translating it to 0 (using now the semialgebraic F_1-addition), we find such a neighborhood, call it $W \subseteq F_1$, of the 0-matrix. But now, given any $a \in F$, by multiplying a by an invertible matrix $b \in W$ sufficiently close to 0, we obtain $ba \in W$. Thus, $F_1 = \{a^{-1}b : a, b \in W\}$. Because W is semialgebraic so is F_1.

Thus, we showed that F is definably isomorphic in $\mathcal{R}_v$ to a semialgebraic field F_1. We now apply Theorem [OPP96, Theorem 1.1] and conclude that F_1 is semialgebraically isomorphic to R or to $R(\sqrt{-1})$.

Finally, we address (b). This follows immediately from (a) once we observe that $\mathcal{O}$ is the only definable valuation ring in $\mathcal{R}_v$. So suppose $\mathcal{O}'$ is another. By weak o-minimality, $\mathcal{O}'$ is a finite union of convex sets, and then since it is a subring with unity it is easy to see that $\mathcal{O}'$ is convex. Thus either $\mathcal{O} \subseteq \mathcal{O}'$ or $\mathcal{O}' \subseteq \mathcal{O}$. Without loss of generality, $\mathcal{O}$ is the standard valuation ring $\cup_{n\in\mathbb{N}}[-n, n]$, so $\mathcal{O}$ properly contains no non-trivial convex valuation ring. Thus $\mathcal{O} \subseteq \mathcal{O}'$. But then if $v : R \to \Gamma$ is the valuation induced by $\mathcal{O}$, then the image of the units of $\mathcal{O}'$ is a definable subgroup $v((\mathcal{O}')^*) \leq \Gamma$. But Γ is a pure divisible ordered abelian group, and so has no non-trivial definable subgroup. Hence $v((\mathcal{O}')^*) = \{0\}$, and hence $\mathcal{O}' = \mathcal{O}$. ☐

Remark 4.5. The techniques we applied here will not readily adapt to handle imaginaries. In the case of algebraically closed valued fields, a result of [HR18] is that the only *interpretable* fields, up to definable isomorphism, are the valued field and its residue field. It would be natural to expect that, correspondingly, the only interpretable fields in RCVF up to definable isomorphism are the valued field, its residue field, and their algebraic closures.

4.2 Interpretations and general nonsense

We address the "Borel-Tits phenomenon" associated with bi-interpretations which require parameters. We spell out an abstract form of the argument given by Poizat [Poi88] in the case of algebraically closed fields. The ideas in this subsection are well-known. For convenience of exposition, we first give a name to the following key property.

Definition 4.6. Say a theory T is **self-recollecting** if any $\mathcal{B}' \models T$ interpreted in any $\mathcal{B} \models T$ is $\mathcal{B}$-definably isomorphic to $\mathcal{B}$.

Say T is **self-recollecting for definitions** if this holds for interpretations which are definitions (where recall a *definition* is an interpretation which doesn't involve non-trivial quotients).

Examples 4.7. ACF is self-recollecting by [Poi88], RCF is self-recollecting by [NP91], and $\mathrm{Th}(\mathbb{Q}_p)$ is self-recollecting for definitions by [Pil89]. It follows directly from the characterisation of interpretable fields in [HR18] that the theory of non-trivially valued algebraically closed fields ACVF is self-recollecting.

Theorem 4.2(b) proves that RCVF is self-recollecting for definitions, but we do not settle the question of whether it is self-recollecting.

We use the notation $\alpha : \mathcal{A} \rightsquigarrow \mathcal{B}$ to denote an interpretation of $\mathcal{A}$ in $\mathcal{B}$, which recall we consider to be a map from $\mathcal{A}$ to some definable quotient in $\mathcal{B}$. Note that any isomorphism is in particular an interpretation (and even a definition). We denote composition of interpretations by concatenation.

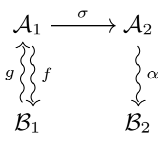
Lemma 4.8. *Suppose $\mathcal{A}_i$ is a structure interpreted in a structure $\mathcal{B}_i$ for $i = 1, 2$, and the interpretation of $\mathcal{A}_1$ in $\mathcal{B}_1$ can be completed to a bi-interpretation. Suppose further that* $\mathrm{Th}(\mathcal{B}_1) = \mathrm{Th}(\mathcal{B}_2)$*, and* $T_B := \mathrm{Th}(\mathcal{B}_1)$ *is self-recollecting.*

Suppose $\sigma : \mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism of structures.

Then there exist an isomorphism $\sigma' : \mathcal{B}_1 \xrightarrow{\cong} \mathcal{B}_2$ and a $\mathcal{B}_2$-definable isomorphism $\theta : \sigma'(\mathcal{A}_1) \to \mathcal{A}_2$ such that $\sigma = \theta(\sigma'{\restriction}_{\mathcal{A}_1})$.

If T_B is only self-recollecting for definitions but the given interpretations are definitions, then the same result holds.

Proof. Let (f, g) be the bi-interpretation of $\mathcal{A}_1$ with $\mathcal{B}_1$, and $\alpha : \mathcal{A}_2 \rightsquigarrow \mathcal{B}_2$ the interpretation.



For interpretations $\beta, \gamma : \mathcal{A} \rightsquigarrow \mathcal{B}$, we write $\beta \sim \gamma$ if $\gamma\beta^{-1} : \beta(\mathcal{A}) \to \gamma(\mathcal{A})$ is a $\mathcal{B}$-definable isomorphism between the two copies of $\mathcal{A}$ in $\mathcal{B}$.[4]

Now $\alpha\sigma g$ is an interpretation of $\mathcal{B}_1$ in $\mathcal{B}_2$, and thus by self-recollecting there is a $\mathcal{B}_2$-definable isomorphism $\tau : (\alpha\sigma g)(\mathcal{B}_1) \to \mathcal{B}_2$. Let $\sigma' := \tau\alpha\sigma g : \mathcal{B}_1 \to \mathcal{B}_2$. Then $\alpha\sigma g \sim \sigma'$, and so $\alpha\sigma g f \sim \sigma' f$.

[4] Interpretations satisfying this condition are sometimes called *homotopic*.

Now gf is definable, and it follows that $\alpha\sigma \sim \alpha\sigma gf$. Thus $\alpha\sigma \sim \sigma' f$. Then $\alpha' := \sigma' f \sigma^{-1} \sim \alpha$.

So $\theta := \alpha\alpha'^{-1}$ is a $\mathcal{B}_2$-definable isomorphism, and $\theta\sigma' f = \alpha\sigma$. Since we view $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{B}_2$ via f and α, this is as desired.

The proof in the case of definitions is identical. □

4.3 Characterisation of isomorphisms of infinitesimal subgroups

Lemma 4.9. *If G, H are compact connected centreless linear Lie groups, and $\mathcal{R} \succ \mathbb{R}$ is a proper real closed field extension of $\mathbb{R}$, and $\theta : G^{00}(\mathcal{R}) \xrightarrow{\cong} H^{00}(\mathcal{R})$ is an $(\mathcal{R}; \mathcal{O})$-definable group isomorphism, then θ extends to an $\mathcal{R}$-definable algebraic isomorphism $G(\mathcal{R}) \xrightarrow{\cong} H(\mathcal{R})$.*

Proof. We may assume $\mathcal{R}$ is $\aleph_0$-saturated. Let $\Gamma \subseteq (G \times H)(\mathcal{R})$ be the $\mathcal{R}$-Zariski closure of the graph Γ_θ of θ. Since Γ_θ is an abstract subgroup, Γ is (the set of $\mathcal{R}$-points of) an algebraic subgroup over $\mathcal{R}$. The image of the projection $\pi_G : \Gamma \to G(\mathcal{R})$ contains $G^{00}(\mathcal{R})$, which is Zariski-dense in G since G is connected, and thus $\pi_G(\Gamma) = G(\mathcal{R})$.

Let $k \leq \mathcal{R}$ be a finitely generated field over which θ is defined. Since Γ_θ is Zariski dense in Γ, there exists $(g, \theta(g)) \in \Gamma_\theta$ which is algebraically generic in Γ over k (i.e. of maximal transcendence degree); indeed, the Zariski density implies that Γ_θ is contained in no subvariety of Γ over k of lesser dimension, and so such a generic exists by $\aleph_0$-saturation of $\mathcal{R}$. But definable closure in RCVF agrees with field-theoretic algebraic closure [Mel06, Theorem 8.1(1)], so $\operatorname{trd}(\theta(g)/k(g)) = 0$ and thus $\pi_G^{-1}(g)$ is finite. Then also $\ker(\pi_G)$ is finite, and hence central. So since G is centreless, π_G is an isomorphism.

Similarly, π_H is an isomorphism. So θ extends to the algebraic isomorphism $\pi_H \circ \pi_G^{-1}$. □

Theorem 4.10. *Suppose G_1 and G_2 are compact simple centreless linear Lie groups, and $\mathcal{R}_i \succ \mathbb{R}$ is a proper real closed field extension of $\mathbb{R}$ for $i = 1, 2$, and $\sigma : G_1^{00}(\mathcal{R}_1) \xrightarrow{\cong} G_2^{00}(\mathcal{R}_2)$ is an abstract group isomorphism.*

Then there exist a valued field isomorphism $\sigma' : (\mathcal{R}_1, \mathcal{O}_1) \xrightarrow{\cong} (\mathcal{R}_2, \mathcal{O}_2)$ and an $\mathcal{R}_2$-definable isomorphism $\theta : G_3(\mathcal{R}_2) \xrightarrow{\cong} G_2(\mathcal{R}_2)$, where $G_3 = \sigma'(G_1)$, such that $\sigma = \theta\restriction_{G_3^{00}} \circ \sigma'\restriction_{G_1^{00}}$.

In particular, σ extends to an abstract group isomorphism $G_1(\mathcal{R}_1) \xrightarrow{\cong} G_2(\mathcal{R}_2)$.

Proof. By Theorem 1.1, Theorem 4.2(b), and Lemma 4.8, there exist an isomorphism $\sigma' : (\mathcal{R}_1, \mathcal{O}_1) \xrightarrow{\cong} (\mathcal{R}_2, \mathcal{O}_2)$ and an $(\mathcal{R}_2, \mathcal{O}_2)$-definable isomorphism $\theta' : \sigma'(G_1^{00}(\mathcal{R}_1)) \xrightarrow{\cong} G_2^{00}(\mathcal{R}_2)$ such that $\sigma = \theta' \circ \sigma'\restriction_{G_1^{00}}$.

Then $\sigma'(G_1^{00}(\mathcal{R}_1)) = G_3^{00}(\mathcal{R}_2)$. Thus by Lemma 4.9, θ' extends to an $\mathcal{R}_2$-definable algebraic isomorphism $\theta : G_3(\mathcal{R}_2) \xrightarrow{\cong} G_2(\mathcal{R}_2)$, as required. □

Remark 4.11. We have stated the results of this section in terms of G^{00}, but it is easy to see that they apply equally to other infinitesimal subgroups as in Remark 1.4.

5 Infinitesimal subgroups of definably compact groups

In this section, we prove Theorem 1.5 by combining Theorem 1.1 with results in the literature on definably compact groups and G^{00}.

We work in a sufficiently saturated o-minimal expansion M of a real closed field, say κ-saturated where κ is sufficiently large. (In fact $\kappa = 2^{\aleph_0}$ suffices for the arguments below; moreover, it follows after the fact that Theorem 1.5 holds with only $\kappa = \aleph_0$, but we do not spell this out). For G a definable group, let G^{00} be the smallest $\bigwedge$-definable (in the sense of M) subgroup of bounded index. Here, a $\bigwedge$-*definable set* is a set defined by an infinite conjunction of formulas over a common parameter set $A \subseteq M$ with $|A| < \kappa$.

Lemma 5.1. *Let G and H be definably compact definable groups.*

(i) If $\theta : G \twoheadrightarrow H$ is a definable surjective homomorphism, then $\theta(G^{00}) = H^{00}$.

(ii) If H is a definable subgroup of G, then $H^{00} = G^{00} \cap H$.

(iii) G^{00} is the unique $\bigwedge$-definable subgroup of bounded index which is divisible and torsion-free.

(iv) $(G \times H)^{00} = G^{00} \times H^{00}$

Proof. (i) $\theta^{-1}(H^{00})$ resp. $\theta(G^{00})$ is a $\bigwedge$-definable bounded index subgroup of G resp. H.

(ii) [Ber07, Theorem 4.4].

(iii) [Ber07, Corollary 4.7].

(iv) This follows directly from (iii).

□

Say a group $(G; *)$ is the **definable internal direct product** of its subgroups $H_1, \ldots, H_n$ if each H_i is $(G; *)$-definable and $(h_1, h_2, \ldots, h_n) \mapsto h_1 h_2 \ldots h_n$ is an isomorphism $\prod_i H_i \to G$. The following lemma is an immediate consequence of the definition.

Lemma 5.2. *If a group $(G; *)$ is the definable internal direct product of subgroups $H_1, \ldots, H_n$, then $(G; *)$ is bi-intepretable with the disjoint union of the $(H_i; *)$.*

The following Fact extracts from the literature the key results we will need on the structure of definably connected definably compact groups. A definable group is *definably simple* if it contains no proper non-trivial normal definable subgroup. First recall that if $G \subseteq M^n$ is a definable group in an o-minimal structure M then, by [Pil88], it admits a topology with a definable basis which makes it into a topological group. Moreover, this is the unique topology which agrees with the ambient M^n-topology on a definable subset of G whose complement has smaller dimension. All topological notions below (e.g. definable compactness) are with respect to this topology.

Fact 5.3. *Let G be a definably connected definably compact definable group. Let $G' := (G, G)$ be the derived subgroup of G. Then:*

(i) G' and $Z(G)$ are definable and definably compact.

(ii) *G is the product of its subgroups G' and $Z(G)^0$, and $Z(G)^0 \cap G'$ is finite.*

(iii) *$Z(G')$ is finite, and $G'/Z(G')$ is the direct product of finitely many definably simple definably compact definable subgroups H_i.*

(iv) *If G is definably simple, then there exists a compact real linear Lie group H and a real closed field $\mathcal{R}$ extending $\mathbb{R}$ such that $(G^{00}; *)$ is isomorphic to $(H^{00}; *)$, where $H^{00} := H^{00}(\mathcal{R})$ is the infinitesimal neighbourhood of the identity as defined in §1.*

Proof. (i) By [HPP11, Corollary 6.4(i)], G' is definable. Since definable subgroups are closed, both groups are definably compact.

(ii) This is [HPP11, Corollary 6.4(ii)].

(iii) This is immediate from [HPP11, Corollary 6.4(i)] and [HPP11, Fact 1.2(3)] (based on [PPS00a, Theorem 4.1]).

(iv) This follows from the proof of [Pil04, Proposition 3.6]. Indeed, as discussed there (see also [HPP11, Fact 1.2(1)]), G is definably isomorphic to $H(R)$ for R a definable real closed field and H a semi-algebraic linear group over a copy of the real field within R. Since G is definably compact, H is compact (see [PS99, Theorem 2.1]). Now *Case II* in the proof of [Pil04, Proposition 3.6] shows that the smallest M-$\bigwedge$-definable subgroup $H^{00}(R)$ of H is precisely the infinitesimal neighbourhood $\mathrm{st}^{-1}(e)$, as required.

□

We now repeat the statement of Theorem 1.5, and prove it.

Theorem 1.5. *Let $(G; *)$ be an infinite definably compact group definable in a sufficiently saturated o-minimal expansion M of a field. Then $(G^{00}(M); *)$ is bi-interpretable with the disjoint union of a (possibly trivial) divisible torsion-free abelian group and finitely many (possibly zero) real closed convexly valued fields.*

Proof. It follows from Lemma 5.1(ii) that $G^{00} = (G^0)^{00}$ where G^0 is the smallest definable subgroup of finite index, so we may assume $G = G^0$, and hence (by [Pil88, Lemma 2.12]) that G is definably connected.

Let $G' := (G, G)$ be the derived subgroup of G, and let $(G^{00})' := (G^{00}, G^{00})$ be the derived subgroup of G^{00}.

By Fact 5.3(i), G' and $Z(G)$ are definable and definably compact, and so Lemma 5.1 applies to them.

Let $H := G'/Z(G')$. Let H_i be as in Fact 5.3(iii), so $H = \prod_i H_i$.

Claim 5.4. *(i) $(G')^{00} \cong H^{00}$ as groups.*

(ii) *$H^{00} = (\prod_i H_i)^{00}$ is the definable internal direct product of the H_i^{00}.*

Proof. (i) $(G')^{00}$ is torsion-free by Lemma 5.1(iii), and $Z(G')$ is finite by Fact 5.3(iii), thus $Z(G') \cap (G')^{00} = \{e\}$. So by Lemma 5.1(i), the quotient map induces such an isomorphism.

(ii) Given Lemma 5.1(iv), we need only show that H_i^{00} is $(H^{00}; *)$-definable.

By Lemma 2.2, $C_H(H_i^{00}) = C_H(H_i)$, and since each H_i is centreless we have $H_i = \bigcap_{j \neq i} C_H(H_j)$, thus $H_i^{00} = \bigcap_{j \neq i} C_{H^{00}}(H_j^{00})$ (using Lemma 5.1(iv) again).

□

Claim 5.5. *(i)* $G^{00} = Z(G)^{00}(G')^{00}$.

(ii) $(G')^{00} = (G^{00})'$.

(iii) $Z(G)^{00} = Z(G^{00})$.

(iv) G^{00} *is the definable internal direct product of* $Z(G)^{00}$ *and* $(G')^{00}$.

Proof. (i) Since $Z(G)^{00}$ is central and each of $Z(G)^{00}$ and $(G')^{00}$ is divisible and torsion-free, also $L := Z(G)^{00}(G')^{00}$ is divisible and torsion-free. Now $G = Z(G)G'$ by Fact 5.3(ii), so any coset of L can be written as $zaL = (zZ(G)^{00})(a(G')^{00})$ with $z \in Z(G)$ and $a \in G'$, thus the index of L in G is bounded by the product of the indices of $Z(G)^{00}$ in $Z(G)$ and of $(G')^{00}$ in G'. So L is a bounded index subgroup. Thus we conclude by Lemma 5.1(iii).

(ii) By Fact 5.3(iv) and [DM16], $H_i^{00} = (H_i^{00}, H_i^{00})_1$ for each i. So by Claim 5.4, $(G')^{00} = ((G')^{00}, (G')^{00})_1$. Also $G^{00} \cap G' = (G')^{00}$ by Lemma 5.1(ii), and so $(G^{00}, G^{00})_1 = (G')^{00}$, and then since this is a subgroup we also have $(G^{00})' = (G^{00}, G^{00})_1$.

(iii) By (i) it suffices to see that $Z((G')^{00}) = \{e\}$. But indeed, as in Lemma 2.2, $Z((G')^{00}) \leq C_{G'}((G')^{00} = C_{G'}(G') = Z(G') = \{e\}$. Alternatively, one can see this way that each $Z(H_i^{00}) = \{e\}$, and apply Claim 5.4(ii).

(iv) By Fact 5.3(ii), $Z(G)^0 \cap G'$ is finite. Hence also $Z(G)^{00} \cap (G')^{00}$ is finite, and thus by Lemma 5.1(iii) it is trivial. Combining this with the previous items of this Claim, we conclude.

□

Now each H_i^{00} is bi-interpretable with a model of RCVF by Fact 5.3(iv) and Theorem 1.1, and $Z(G)^{00}$ is (by Lemma 5.1(iii)) a divisible torsion free abelian group, so we conclude by Claim 5.5(iv), Claim 5.4, and Lemma 5.2.

□

Remark 5.6. Since M is an o-minimal expansion of a field, any M-definable real closed field is M-definably isomorphic to M as a field. Thus the valued fields R_i interpreted in the groups H_i^{00} in the above proof are M-definably isomorphic as fields. However, the disjoint union structure clearly does not define any such isomorphisms between the R_i, and hence nor does the group $(G^{00}; *)$.

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