Packing arc-disjoint triangles in regular and almost regular tournaments

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Abstract

For a tournament T, let $\nu_3(T)$ denote the maximum number of pairwise arc-disjoint triangles in T. Let $\nu_3(n)$ denote the minimum of $\nu_3(T)$ ranging over all regular tournaments with n vertices (n odd). It is conjectured that $\nu_3(n) = (1 + o_n(1))n^2/9$ and proved that

$$\frac{n^2}{11.43}(1 - o_n(1)) \le \nu_3(n) \le \frac{n^2}{9}(1 + o_n(1))$$

improving upon the best known upper bound and lower bound. The result is generalized to tournaments where the indegree and outdegree at each vertex may differ by at most βn.

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1 Introduction

All graphs and digraphs considered here are finite and contain no parallel edges or anti-parallel arcs. For standard graph-theoretic terminology the reader is referred to [2]. *Tournaments* are orientations of complete graphs, and are a major object of study in combinatorics and social choice theory. However, while complete graphs are unique for each order, there are exponentially many tournaments with the same order. As perhaps the most obvious property of a complete graph is its regularity, it seems interesting to study the properties of *regular* tournaments. Indeed, regular tournaments have been studied by several researchers, see. e.g. [8, 10, 11, 14, 16]. As any connected undirected graph has an Eulerian orientation if and only if every vertex is of even degree, we have that there exist regular tournaments for every odd order. Eulerian tournaments are, therefore, the same as regular tournaments. In fact, there are exponentially many non-isomorphic regular tournaments with n vertices [11].

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All regular tournaments have the same number of triangles, and the same number of transitive triples where a triangle is a set of three arcs $\{(x,y),(y,z),(z,x)\}$ while a transitive triple is a set of three arcs $\{(x,y),(y,z),(x,z)\}$. This follows from the well-known fact observed in [5], that the number of transitive triples (and hence triangles) in any tournament is determined by the *score* of the tournament, which is the sorted outdegree sequence. For regular tournaments this amounts to $n(n-1)(n-3)/8$ transitive triples and therefore to $\binom{n}{3} - n(n-1)(n-3)/8 = n(n^2-1)/24$ triangles. Asymptotically, this means that a fraction of $1/4$ of the triples are triangles while $3/4$ of the triples are transitive. Throughout this paper a triangle is denoted by C_3.

An (edge) *triangle packing* of an undirected graph is a set of pairwise edge-disjoint subgraphs that are isomorphic to a triangle. The study of triangle packings in graphs was initiated in the classical result of Kirkman [9] who proved that K_n has a triangle packing of size $n(n-1)/6$ whenever $n \equiv 1,3 \bmod 6$. In other words, when $n \equiv 1,3 \bmod 6$ there always exists a *Steiner triple system* (STS). This clearly implies that for other moduli of n there are packings with $(1-o_n(1))n^2/6$ triangles, and this is asymptotically tight as such packings cover $(1-o_n(1))\binom{n}{2}$ edges. In the directed case, a triangle packing of a tournament requires each subgraph to be isomorphic to C_3. Triangle packings and packings by transitive triples of digraphs have been studied by several researchers (see, e.g., [4, 7, 13]).

For a tournament T, we denote by $\nu_3(T)$ the size of a largest triangle packing. Observe that $\nu_3(T) \ge c_3(T)/(n-2)$ for every tournament with $n \equiv 1,3 \bmod 6$ where $c_3(T)$ is the total number of triangles. This can be seen by taking a random STS of n and observing that the expected number of directed triangles in the STS is $(n(n-1)/6)c_3(T)/\binom{n}{3}$. In particular, this means that $\nu_3(T) \ge (1-o_n(1))n^2/24$ for any regular tournament T with n vertices. On the other hand, we always have the trivial upper bound $\nu_3(T) \le (1-o_n(1))n^2/6$.

Let, therefore $\nu_3(n)$ denote the minimum of $\nu_3(T)$ ranging over all regular tournaments with n vertices (assuming, of course, that n is odd). Hence, trivially

$$\frac{n^2}{24}(1-o_n(1)) \le \nu_3(n) \le \frac{n^2}{6}(1-o_n(1)) \; .$$

While exact small values of $\nu_3(n)$ are known by brute force computation, determining the asymptotic value of $\nu_3(n)$ seems to be a difficult problem. The best known bounds were given in [16]:

$$\frac{n^2}{11.5}(1-o_n(1)) \le \nu_3(n) \le \frac{n^2-1}{8} \; .$$

The present paper improves both the lower bound and the upper bound. While the upper bound is improved significantly, the improvement in the lower bound is milder.

Theorem 1.1

$$\left(\frac{1}{3} - \frac{7}{3}\ln(\frac{10}{9})\right) n^2(1-o_n(1)) \le \nu_3(n) \le \frac{n^2}{9}(1+o_n(1)) \; .$$

Notice that $\frac{1}{3} - \frac{7}{3}\ln(\frac{10}{9}) > 1/11.43$. As explained in Section 2, it is natural to suspect that the construction yielding the upper bound is, in a sense, a "worst case" construction. Thus, we make the following conjecture.

Conjecture 1.2

$$\nu_3(n) = \frac{n^2}{9}(1 + o_n(1)) \ .$$

While the proof of the upper bound in Theorem 1.1 is different from the one in [16], the proof of the lower bound is similar in many aspects. The additional important ingredient that enables us to obtain the improved lower bound is a strengthening of Lemma 3.3 there, replaced by the significantly more involved Lemma 3.4 here, which bounds the number of triangles containing "dense" arcs (arcs that appear in many triangles).

We are able to extend our results to not necessarily regular tournaments. We say that a tournament is β-almost-regular (or, for brevity, and slightly abusing terminology, β-regular) if the indegree and outdegree at each vertex differ by at most βn. Thus, $\beta = 0$ coincides with regular tournaments and $\beta = 1$ coincides with the family of all tournaments. Here we no longer need to require that n has a certain parity. Generalizing the above notation, we denote by $\nu_3(\beta, n)$ the minimum of $\nu_3(T)$ ranging over all β-regular tournaments with n vertices. The following extends Theorem 1.1.

Theorem 1.3

$$\nu_3(\beta, n) \le \min\left\{\frac{1-\beta^2}{9} \ , \ \frac{(1-\beta)^2}{8}\right\} n^2(1 + o_n(1)) \ .$$

$$\nu_3(\beta, n) \ge \ln\left(\frac{12(1+\beta)}{11 + 12\beta + 3\beta^2}\right) n^2(1 - o_n(1)) \quad \text{if } \beta \le \tfrac{1}{2} \ ,$$

$$\nu_3(\beta, n) \ge \ln\left(\frac{6(1+\beta)}{5 + 9\beta - 3\beta^2 + \beta^3}\right) n^2(1 - o_n(1)) \quad \text{if } \beta > \tfrac{1}{2} \ .$$

The rest of this paper is organized as follows. In Section 2, we prove the upper bound in Theorem 1.1. To this end, we need to define the fractional relaxation of the problem and consider its dual covering problem. We also prove that the upper bound we obtain cannot be improved using our construction. We explain why it is natural to suspect that this construction is "the worst", and hence the justification for conjecture 1.2. We also show how to generalize the construction to β-regular tournaments and obtain the upper bound in Theorem 1.3. In Section 3 we prove the lower bound in Theorem 1.1. As in [16] our main tool is a result of Haxell and Rödl [6] tailored to the directed setting in [12] connecting the fractional value of a maximum packing with its integral one. Section 4 addresses the changes needed in the statements given in Section 3 in order to apply them to the more general setting of β-regular tournaments, resulting in the proof of the lower bound in Theorem 1.3.

2 Upper bounds

2.1 Fractional relaxation of packing and covering

We start this section by defining the fractional relaxation of the triangle packing problem together with its dual fractional covering problem, and define the parameters $\nu_3^*(n)$ and $\tau_3^*(n)$ that are the fractional analogue of $\nu_3(n)$ and its dual, respectively.

Let R_+ denote the set of nonnegative reals. A *fractional triangle packing* of a digraph G is a function ψ from the set $\mathcal{F}_3$ of copies of C_3 in G to R_+, satisfying $\sum_{e \in X \in \mathcal{F}_3} \psi(X) \le 1$ for each arc $e \in E(G)$. Letting $|\psi| = \sum_{X \in \mathcal{F}_3} \psi(X)$, the *fractional triangle packing number*, denoted $\nu_3^*(G)$, is defined to be the maximum of $|\psi|$ taken over all fractional triangle packings ψ. Since a triangle packing is also a fractional triangle packing (by letting $\psi = 1$ for elements of $\mathcal{F}_3$ in the packing and $\psi = 0$ for the other elements), we always have $\nu_3^*(G) \ge \nu_3(G)$. However, the two parameters may differ. In particular, they may differ for regular tournaments. Consider, for example, the 5-vertex regular tournament obtained by the following orientation of K_5 on the vertex set $\{1, 2, 3, 4, 5\}$. Orient a Hamilton cycle $(1, 2, 3, 4, 5)$ and another Hamilton cycle as $(1, 4, 2, 5, 3)$. Clearly, $\nu_3(T) = 2$. On the other hand, we may assign each of the five triangles $(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 1), (5, 1, 2)$ the value $1/2$ thereby obtaining a fractional triangle packing of value 2.5.

A *fractional triangle cover* of a digraph G is a function ϕ from the set of arcs $E(G)$ of G to R_+, satisfying $\sum_{e \in X \in \mathcal{F}_3} \phi(e) \ge 1$ for each triangle $X \in \mathcal{F}_3$. Letting $|\phi| = \sum_{e \in E(G)} \phi(e)$, the *fractional triangle cover number*, denoted $\tau_3^*(G)$, is defined to be the minimum of $|\phi|$ taken over all fractional triangle covers ϕ. By linear programming duality, $\tau_3^*(G) = \nu_3^*(G)$. For example, in the 5-vertex regular tournament of the previous paragraph, we may assign the value $1/2$ to each arc on the cycle $(1, 2, 3, 4, 5)$ and obtain a valid fractional triangle cover of value 2.5.

2.2 Upper bound for regular tournaments

In order to obtain a good upper bound, we must first construct a regular tournament which is "as transitive as possible" so that it will not be able to accommodate many pairwise arc disjoint triangles. Naturally, any regular tournament on n vertices cannot have a transitive subset on more than $(n + 1)/2$ vertices, since in such a subset the outdegree of the source would already be more than $(n - 1)/2$. The following regular tournament, denoted R_n, does have a transitive subset on $(n + 1)/2$ vertices, in fact it has many such subsets. It even has many pairs of arc-disjoint such subsets (each pair sharing exactly one vertex). It is reasonable to suspect that a maximum triangle packing of R_n yields the value of $\nu_3(n)$.

For n odd, we define R_n as follows. Its vertices are $\{0, \ldots, n - 1\}$ (one can view them as elements of the cyclic group Z_n). Vertex i has an outgoing arc towards vertex j if and only if $1 \le (j - i) \bmod n \le (n - 1)/2$. Thus, if we think of the vertices as lying on a directed cycle of length n, each vertex sends outgoing arcs to the $(n-1)/2$ vertices following it on the cycle. Observe that R_n is a regular tournament and that for any vertex i, the set of vertices $\{i, i+1, \ldots, i+(n-1)/2\}$ (indices modulo n) forms a transitive subset. We will prove that $\nu(R_n) \leq \frac{(n+o(n))^2}{9}$, which implies that $\nu_3(n) \leq \frac{(n+o(n))^2}{9}$. Since, by the previous subsection, $\tau_3^*(R_n) = \nu_3^*(R_n) \geq \nu_3(R_n)$, it suffices to prove the following.

Lemma 2.1

$$\tau_3^*(R_n) \leq \frac{(n+o(n))^2}{9} .$$

Proof. We consider first case where $n \equiv 1 \bmod 6$. We will construct a particular covering which attains the bound stated in the lemma. Define the *length* of an arc of R_n from i to j by $length(i,j) = (j-i) \bmod n$. We give all the arcs of length $1, \ldots, \lfloor \frac{n}{6} \rfloor$ the weight 0 (i.e. $\phi(e) = 0$ for $length(e) \in \{1, \ldots, \lfloor \frac{n}{6} \rfloor\}$.) To each arc e of length $\ell > \lfloor \frac{n}{6} \rfloor$ we give the weight $\phi(e) = \frac{2}{n+1}\left(\ell - \lfloor \frac{n}{6} \rfloor\right)$.

Proposition 2.2 *The assignment ϕ is a fractional triangle cover.*

Proof. Let (h,i,j) be a triangle, without loss of generality $h = 0$ so the triangle is $(0,i,j)$.
First case: $i \in \{1, \ldots, \lfloor \frac{n}{6} \rfloor\}$. Then the other arcs of the triangle must have length larger than $\lfloor \frac{n}{6} \rfloor$ and hence $\phi((i,j)) = \frac{2}{n+1}\left(j - i - \lfloor \frac{n}{6} \rfloor\right)$ and $\phi((j,0)) = \frac{2}{n+1}\left(n - j - \lfloor \frac{n}{6} \rfloor\right)$. The sum of weights of the arcs of the triangle $(0,i,j)$ is:

$$\begin{aligned}
& 0 + \frac{2}{n+1}\left(j - i - \left\lfloor \frac{n}{6} \right\rfloor\right) + \frac{2}{n+1}\left(n - j - \left\lfloor \frac{n}{6} \right\rfloor\right) \\
= \ & \frac{2}{n+1}\left(n - i - 2\left\lfloor \frac{n}{6} \right\rfloor\right) \geq \frac{2}{n+1}\left(n - 3\left\lfloor \frac{n}{6} \right\rfloor\right) \\
= \ & \frac{2}{n+1}\left(n - 3\frac{n-1}{6}\right) = \frac{2}{n+1}\left(n - \frac{n-1}{2}\right) = 1 .
\end{aligned}$$

Second case: $i \notin \{1, \ldots, \lfloor \frac{n}{6} \rfloor\}$. Then $\phi((0,i)) = \frac{2}{n+1}\left(i - \lfloor \frac{n}{6} \rfloor\right)$ and we have three subcases for the weight of arc (i,j) and arc $(j,0)$: the first subcase is $\phi((i,j)) = \frac{2}{n+1}\left(j - i - \lfloor \frac{n}{6} \rfloor\right)$ and $\phi((j,0)) = 0$, the second subcase is $\phi((i,j)) = 0$ and $\phi((j,0)) = \frac{2}{n+1}\left(n - j - \lfloor \frac{n}{6} \rfloor\right)$, and the last subcase is $\phi((i,j)) = \frac{2}{n+1}\left(j - i - \lfloor \frac{n}{6} \rfloor\right)$ and $\phi((j,0)) = \frac{2}{n+1}\left(n - j - \lfloor \frac{n}{6} \rfloor\right)$. Now we calculate the weight of the triangle $(0,i,j)$ in the three subcases:
First subcase:

$$\begin{aligned}
& \frac{2}{n+1}\left(i - \left\lfloor \frac{n}{6} \right\rfloor\right) + \frac{2}{n+1}\left(j - i - \left\lfloor \frac{n}{6} \right\rfloor\right) + 0 \\
= \ & \frac{2}{n+1}\left(j - 2\left\lfloor \frac{n}{6} \right\rfloor\right) \geq \frac{2}{n+1}\left(n - \frac{n-1}{6} - 2\frac{n-1}{6}\right) \\
= \ & \frac{2}{n+1}\left(\frac{n+1}{2}\right) = 1 .
\end{aligned}$$

We used the fact that in this subcase we must have $length(j,0) \leq \lfloor \frac{n}{6} \rfloor$ so $j \geq n - (n-1)/6$.

Second subcase:

$$
\begin{aligned}
& \frac{2}{n+1}\left(i-\left\lfloor\frac{n}{6}\right\rfloor\right)+0+\frac{2}{n+1}\left(n-j-\left\lfloor\frac{n}{6}\right\rfloor\right) \\
= & \frac{2}{n+1}\left(n-j+i-2\left\lfloor\frac{n}{6}\right\rfloor\right) \geq \frac{2}{n+1}\left(n-\frac{n-1}{6}-2\frac{n-1}{6}\right) \\
= & \frac{2}{n+1}\left(\frac{n+1}{2}\right)=1 .
\end{aligned}
$$

Recall that in this subcase $length(i,j) = j - i \leq \lfloor \frac{n}{6} \rfloor$.
Third subcase:

$$
\begin{aligned}
& \frac{2}{n+1}\left(i-\left\lfloor\frac{n}{6}\right\rfloor\right)+\frac{2}{n+1}\left(j-i-\left\lfloor\frac{n}{6}\right\rfloor\right)+\frac{2}{n+1}\left(n-j-\left\lfloor\frac{n}{6}\right\rfloor\right) \\
= & \frac{2}{n+1}\left(n-3\left\lfloor\frac{n}{6}\right\rfloor\right)=\frac{2}{n+1}\left(n-3\frac{n-1}{6}\right)=1 .
\end{aligned}
$$

End of Proposition 2.2.
We calculate the value of this fractional triangle cover. Observe that only lengths between $\lfloor n/6 \rfloor + 1$ until $\lfloor n/2 \rfloor$ (which is the maximum possible length of an arc by the definition of R_n) receive nonzero weight which is the length minus $\lfloor n/6 \rfloor$, normalized by multiplying it with $2/(n+1)$. Thus,

$$
\begin{aligned}
|\phi| = \sum_{e \in E} \phi(e) & = n\frac{2}{n+1}\left(1+2+3+\ldots+\frac{n-1}{3}\right) \\
& = \frac{2n}{n+1}\left(\frac{\frac{n-1}{3}\left(1+\frac{n-1}{3}\right)}{2}\right) \\
& = \frac{n}{n+1}\left(\frac{(n-1)(n+2)}{9}\right) \\
& < \frac{n^2}{9} .
\end{aligned} \tag{1}
$$

Hence $\tau_3^*(R_n) < \frac{n^2}{9}$ for $n \equiv 1 \bmod 6$.

Now, if $n \not\equiv 1 \bmod 6$, then either $n \equiv 3 \bmod 6$ or $n \equiv 5 \bmod 6$. Observe that since R_n is a subgraph of R_{n+2} (just delete vertices 0 and $(n+1)/2$ from R_{n+2} to obtain a subgraph isomorphic to R_n) we have $\tau_3^*(R_n) \leq \tau_3^*(R_{n+2}) \leq \tau_3^*(R_{n+4})$. Thus, for the case $n \equiv 5 \bmod 6$, we have that $n+2 \equiv 1 \bmod 6$ hence $\tau_3^*(R_n) \leq \tau_3^*(R_{n+2}) \leq \frac{(n+2)^2}{9} = \frac{(n+o(n))^2}{9}$. For the case $n \equiv 3 \bmod 6$, we have that $n+4 \equiv 1 \bmod 6$ hence $\tau_3^*(R_n) \leq \tau_3^*(R_{n+4}) \leq \frac{(n+4)^2}{9} = \frac{(n+o(n))^2}{9}$. This completes the proof of Lemma 2.1 and hence the upper bound in Theorem 1.1. ■

One may wonder whether the fractional cover constructed in Lemma 2.1 is optimal for R_n. Perhaps we can do better and improve the upper bound (regardless of whether one believes that R_n is a worst case example). In the following lemma we show that our constructed covering is asymptotically optimal for R_n.

Lemma 2.3
$$\nu(R_n) \geq \frac{(n-o(n))^2}{9} \ .$$

Proof. We prove this for $n = 9k$ (k odd). In this case we will show that we can pack exactly $\frac{n^2}{9} = 9k^2$ pairwise arc-disjoint triangles.

We define the packing as follows. It consists of $n = 9k$ sets of triangles, denoted $S_0, \ldots, S_{n-1}$. Each set will contain k pairwise arc-disjoint triangles. Overall, the construction consists of $nk = n^2/9$ triangles. Furthermore, for any two sets S_i, S_j, their triangles are pairwise arc-disjoint.

We describe S_j for $j = 0, \ldots, n-1$. It consists of the triangles $(j, (j+a_i) \bmod n, (j+a_i+b_i) \bmod n)$ for $i = 0, \ldots, k-1$ where:
$b_i = (n-1)/2 - 3k/2 + (i+2)/2$ for i odd.
$b_i = (n-1)/2 - 2k + i/2 + 1$ for i even.
$a_i = 2k + i/2$ for i even.
$a_i = 3k/2 + i/2$ for i odd.
For example, if $k = 9$ (hence $n = 81$) we have that S_0 is:

$$\{(0,18,41),(0,14,42),(0,19,43),(0,15,44),(0,20,45),(0,16,46),(0,21,47),(0,17,48),(0,22,49)\} \ .$$

We need to prove that each of the listed triples in each of the S_j is indeed a directed triangle of R_n, and that no arc repeats twice in any of the S_j.

Each triple is of the form $(j, (j+a_i) \bmod n, (j+a_i+b_i) \bmod n)$. The lengths of the arcs in this triangle are a_i, b_i and $c_i = n - a_i - b_i$. Observe that a_i is always between 1 and $(n-1)/2$ by its definition. Indeed, if i is even, then

$$\frac{2n}{9} = 2k \leq a_i \leq 2k + \frac{k-1}{2} = \frac{5k-1}{2} = \frac{5n}{18} - \frac{1}{2} \leq \frac{n-1}{2} \ .$$

If i is odd, then

$$\frac{n}{6} + \frac{1}{2} = \frac{3k}{2} + \frac{1}{2} \leq a_i \leq \frac{3k}{2} + \frac{k-2}{2} = \frac{4k-2}{2} = \frac{2n}{9} - 1 \leq \frac{n-1}{2} \ .$$

In any case, the first arc of each triangle whose length is a_i, is indeed an arc of R_n.

Observe similarly that b_i is always between 1 and $(n-1)/2$ by its definition. Indeed, if i is even, then

$$\frac{5n}{18} + \frac{1}{2} = \frac{n-1}{2} - 2k + 1 \leq b_i \leq \frac{n-1}{2} - 2k + 1 + \frac{k-1}{2} = \frac{n}{3} \leq \frac{n-1}{2} \ .$$

If i is odd, then

$$\frac{n}{3} + 1 = \frac{n-1}{2} - \frac{3k}{2} + \frac{3}{2} \leq b_i \leq \frac{n-1}{2} - \frac{3k}{2} + \frac{k}{2} = \frac{7n}{18} - \frac{1}{2} \leq \frac{n-1}{2} \ .$$

In any case, the second arc of each triangle whose length is b_i, is indeed an arc of R_n.

Finally, c_i is always between 1 and $(n-1)/2$ since by the definitions of a_i and b_i we have $c_i = (n-1)/2 - i$.

We have proved that each triple in each S_j is a directed triangle of R_n. Observe also that the interval of values of the a_i is always between $n/6 + 1/2$ and $5n/18 - 1/2$. The interval of values of the b_i is always between $5n/18 + 1/2$ and $7n/18 - 1/2$. The interval of values of the c_i is always between $7n/18 + 1/2$ and $(n-1)/2$. As these three intervals are disjoint, this proves that no arc is repeated twice in the construction. This proves the lemma when $n = 9k$ and k is odd. Now, for any other odd number n, let k be the largest odd number such that $9k \le n$. Recalling that R_{9k} is a subgraph of R_n we have that

$$\nu(R_n) \ge \nu(R_{9k}) \ge 9k^2 = \frac{(n - o(n))^2}{9} .$$

$\blacksquare$

2.3 Upper bound for β-regular tournaments

In this subsection we prove the upper bound for $\nu_3(\beta, n)$ given in Theorem 1.3. Consider the regular tournament graph $R_{(1+\beta)n}$ defined in the previous subsection. We can assume $(1+\beta)n$ is an odd integer as rounding issues do not affect the asymptotic claim. Delete from $R_{(1+\beta)n}$ the vertices $\{0, 1, \ldots, \beta n - 1\}$ and denote the resulting tournament by T. Notice that T has n vertices and since R_n is regular, and we have removed only βn vertices from it, we have that T is a β-regular tournament.

We first consider the case where $\beta \le 1/5$. Let ϕ be the fractional triangle cover defined on $R_{(1+\beta)n}$, proved in (1) to satisfy $|\phi| \le (1 + o_n(1))(1+\beta)^2 n^2/9$. Let ϕ' be the fractional triangle cover of T induced by ϕ. Namely, each arc of T retains its weight under ϕ. Now, $|\phi| - |\phi'|$ is just the sum of the weights of the arcs incident with the removed vertices $\{0, 1, \ldots, \beta n - 1\}$. By (1), the sum of the weights of the arcs emanating from each vertex of $R_{(1+\beta)n}$ is $(1 - o_n(1))(1+\beta)n/9$ and, by symmetry, the sum of the weights of the arcs entering each vertex of $R_{(1+\beta)n}$ is also $(1 - o_n(1))(1+\beta)n/9$. Now, for all $\beta \le 1/5$ we have that $\beta n \le (1+\beta)n/6$. Hence all the arcs (i, j) where $i, j \in \{0, 1, \ldots, \beta n - 1\}$ have $\phi((i, j)) = 0$. Thus,

$$|\phi| - |\phi'| \ge (\beta n) \cdot 2(1 - o_n(1)) \frac{1+\beta}{9} n .$$

It follows that

$$\begin{aligned}
|\phi'| &\le |\phi| - (1 - o_n(1)) \frac{2\beta(1+\beta)}{9} n^2 \\
&\le (1 + o_n(1)) \frac{(1+\beta)^2}{9} n^2 - (1 - o_n(1)) \frac{2\beta(1+\beta)}{9} n^2 \\
&\le (1 + o_n(1)) \frac{1 - \beta^2}{9} n^2 .
\end{aligned}$$

Since $\nu_3(\beta,n) \le \nu_3(T) \le \nu_3^*(T) = \tau_3^*(T) \le |\phi'|$ we have that $\nu_3(\beta,n) \le (1+o_n(1))(1-\beta^2)n^2/9$ for $\beta \le 1/5$.

The following triangle cover, denoted ϕ'' is valid for all $\beta < 1$. Assign the weight 1 to all the arcs of T of the form (i,j) where $i > j$. All other arcs receive the weight 0. Notice that each directed triangle must contain an arc having weight 1 and hence ϕ'' is a valid triangle cover (in fact, an integral cover). We count the number of arcs receiving weight 1. Vertex $(1+\beta)n-1$ (the vertex with largest index) has an outgoing arc in $R_{(1+\beta)n}$ to all vertices j with $j < (1+\beta)n/2$. Hence, it has at most $(1+\beta)n/2 - \beta n - 1 = n/2 - \beta n/2 - 1$ arcs emanating from it in T having weight 1. Similarly, for all $k = 1,\dots,n/2-\beta n/2$, vertex $(1+\beta)n-k$ has at most $n/2-\beta n/2-k$ arcs emanating from it in T having weight 1. Hence,

$$|\phi''| \le \sum_{k=1}^{n/2-\beta n/2} (n/2-\beta n/2-k) \le (1+o_n(1))\frac{(1-\beta)^2}{8}n^2 .$$

Since $\nu_3(\beta,n) \le \nu_3(T) \le \nu_3^*(T) = \tau_3^*(T) \le |\phi''|$ we have that $\nu_3(\beta,n) \le (1+o_n(1))(1-\beta)^2n^2/8$ for $\beta \le 1$. Observe that for all $\beta \le 1/17 \le 1/5$ the bound obtained via ϕ' is better than the bound obtained via ϕ'' hence we may summarize that

$$\nu_3(\beta,n) \le \min\left\{\frac{1-\beta^2}{9}\ ,\ \frac{(1-\beta)^2}{8}\right\} n^2(1+o_n(1)) .$$

3 A lower bound for regular tournaments

3.1 Integer versus fractional packings

A result of Nutov and Yuster [12] asserts that the integral and fractional parameters differ by $o(n^2)$. The following is a very spacial case of their result.

Theorem 3.1 *If T is an n-vertex tournament, then $\nu_3^*(T) - \nu_3(T) = o(n^2)$.*

An *undirected* version of Theorem 3.1 has been proved by Haxell and Rödl [6] who were the first to prove this interesting relationship between integral and fractional packings. The proof of Theorem 3.1 makes use of the *directed* version of Szemerédi's regularity lemma [15] that has been used implicitly in [3] and proved in [1].

Let $\nu_3^*(n)$ be the minimum of $\nu_3^*(T)$ ranging over all n-vertex regular tournaments T. Similarly, let $\nu_3^*(\beta,n)$ be the minimum of $\nu_3^*(T)$ ranging over all n-vertex β-regular tournaments T. By Theorem 3.1 and the fact that fractional packings are at least as large as integral packings we have:

Corollary 3.2 $\nu_3^*(n) \ge \nu_3(n) \ge \nu_3^*(n) - o(n^2)$. *Similarly,* $\nu_3^*(\beta,n) \ge \nu_3(\beta,n) \ge \nu_3^*(\beta,n) - o(n^2)$.

3.2 Proof of the lower bound in Theorem 1.1

In this section we prove the following theorem that, together with Corollary 3.2, yields the lower bound in Theorem 1.1.

Theorem 3.3 *A regular tournament T with n vertices has $\nu_3^*(T) \geq (1 - o_n(1))(\frac{1}{3} - \frac{7}{3}\ln(\frac{10}{9}))n^2$.*

As in [16], we call an arc *α-dense* if it is contained in at least αn triangles. Observe that no arc is 1/2-dense as any arc of a regular tournament appears in at most $(n-1)/2$ triangles. We require the following lemma that bounds the number of triangles that contain α-dense arcs where α is relatively large. It is an improvement over Lemma 3.3 in [16].

Lemma 3.4 *For all $\alpha \geq 1/4$, the number of triangles that contain α-dense arcs is at most $(1 - 2\alpha)(\frac{5}{3}\alpha - \frac{1}{3})n^3$.*

Proof. As shown in [16], the total number of α-dense arcs entering each vertex is at most $n(1-2\alpha)$. We repeat the details of this observation for completeness. For a vertex v, we compute the number of α-dense arcs entering it. Let $B_v \subset N^-(v)$ be the set of vertices x such that (x, v) is α-dense. Consider a vertex x of maximum indegree in the sub-tournament $T[B_v]$ induced by B_v. Since in any tournament with $|B_v|$ vertices the maximum indegree is at least $(|B_v| - 1)/2$ we have that x has at least $(|B_v| - 1)/2$ arcs entering it in $T[B_v]$. On the other hand, as (x, v) is α-dense, we also have that x has at least αn vertices of $N^+(v)$ entering it. Since $N^+(v) \cap B_v = \emptyset$ we have that the indegree of x in T is at least $(|B_v| - 1)/2 + \alpha n$. But the indegree of x in T is $(n-1)/2$ and thus

$$(|B_v| - 1)/2 + \alpha n \leq (n-1)/2 \ .$$

It follows that $|B_v| \leq n(1 - 2\alpha)$. Similarly, if $C_v \subset N^+(v)$ is the set of vertices x such that (v, x) is α-dense, we have that $|C_v| \leq n(1 - 2\alpha)$.

But we are not interested in counting the number of α-dense arcs incident with a vertex, rather we wish to count the number of triangles containing α-dense arcs. To this end, we need to define certain parameters.

1. Let $r(v)$ denote the number of triangles of the form (v, x, y) such that (y, v) is α-dense and (v, x) is not α-dense.
2. Let $s(v)$ denote the number of triangles of the form (v, x, y) such that (y, v) is not α-dense and (v, x) is α-dense.
3. Let $t(v)$ denote the number of triangles of the form (v, x, y) such that (y, v) is α-dense and (v, x) is α-dense.
4. Let $b(v) = r(v) + t(v)$ denote the number of triangles of the form (v, x, y) such that (y, v) is α-dense.

5. Let $c(v) = s(v) + t(v)$ denote the number of triangles of the form (v, x, y) such that (v, x) is α-dense.

6. Let $q(v) = \frac{1}{2}r(v) + \frac{1}{2}s(v) + \frac{1}{3}t(v)$.

We claim that $\sum_{v \in V} q(v)$ is an upper bound for the total number of triangles containing an α-dense arc. Indeed, consider some triangle (x, y, z) containing an α-dense arc. If it contains a single α-dense arc, say (x, y), then this triangle is counted $1/2$ for $s(x)$ and $1/2$ for $r(y)$. If it contains three α-dense arcs, then it is counted $1/3$ for each of $t(x), t(y), t(z)$. If it contains precisely two α-dense arcs, say (x, y) and (y, z), then it is counted $1/2$ for $s(x)$, $1/2$ for $r(z)$ and $1/3$ for $t(y)$, so it contributes more than 1. In any case, each triangle containing an α-dense arc contributes at least 1 to the sum $\sum_{v \in V} q(v)$.

It remains to upper bound $\sum_{v \in V} q(v)$. We will upper bound each $q(v)$ separately, and multiply the bound by n. Notice that by the definitions of $b(v)$ and $c(v)$,

$$q(v) = \frac{1}{2}b(v) + \frac{1}{2}c(v) - \frac{2}{3}t(v) \ . \tag{2}$$

Let $\beta n = |B_v|$ and $\gamma n = |C_v|$ and recall that $\beta \le 1 - 2\alpha$ and $\gamma \le 1 - 2\alpha$. We start by giving upper bounds for $b(v)$ and $c(v)$ in terms of β and γ respectively. For any $x \in B_v$, let $f(x)$ denote the number of triangles containing the α-dense arc (x, v). By the definition of B_v we have that $f(x) \ge \alpha n$. Let $d(x)$ denote the indegree of x in $T[B_v]$. As in the argument at the beginning of the proof, we have that $d(x) + f(x) \le (n-1)/2$. Now, by the definition of $b(v)$ we have that $b(v) = \sum_{x \in B_v} f(x)$ and therefore

$$b(v) = \sum_{x \in B_v} f(x) \le \sum_{x \in B_v} (\frac{n-1}{2} - d(x)) \ .$$

On the other hand, $\sum_{x \in B_v} d(x) = |B_v|(|B_v| - 1)/2$. Hence,

$$b(v) \le |B_v|\frac{n-1}{2} - \frac{|B_v|(|B_v|-1)}{2} = \frac{\beta(1-\beta)}{2}n^2 \ . \tag{3}$$

Analogously, we have that

$$c(v) \le |C_v|\frac{n-1}{2} - \frac{|C_v|(|C_v|-1)}{2} = \frac{\gamma(1-\gamma)}{2}n^2 \ . \tag{4}$$

We next give a lower bound for $t(v)$. Consider any arc (x, y) that goes from C_v to B_v. This means that (v, x, y) is a triangle where both (y, v) and (v, x) are α-dense. Hence, this triangle contributes to $t(v)$. Thus, the number of arcs going from C_v to B_v is equal to $t(v)$. There are at least $|B_v| \times \alpha n$ arcs going from $N^+(v)$ to B_v. At most $(|N^+(v)| - |C_v|)|B_v|$ of them go from $N^+(v) \setminus C_v$ to B_v. Hence,

$$t(v) \ge |B_v|\alpha n - (|N^+(v)| - |C_v|)|B_v| = \alpha\beta n^2 - (\frac{n-1}{2} - \gamma n)\beta n \ge \beta(\alpha - \frac{1}{2} + \gamma)n^2 \ .$$

We can similarly estimate $t(v)$ by the fact that there are at least $|C_v| \times \alpha n$ arcs going from C_v to $N^-(v)$. At most $(|N^-(v)| - |B_v|)|C_v|$ of them go from C_v to $N^-(v) \setminus B_v$. Hence,

$$t(v) \geq |C_v|\alpha n - (|N^-(v)| - |B_v|)|C_v| = \alpha\gamma n^2 - (\frac{n-1}{2} - \beta n)\gamma n \geq \gamma(\alpha - \frac{1}{2} + \beta)n^2 .$$

Using the last two inequalities we obtain that

$$t(v) \geq \left(\beta\gamma - \frac{(\frac{1}{2} - \alpha)(\beta + \gamma)}{2}\right) n^2 . \tag{5}$$

By (2), (3), (4), (5) we get that

$$q(v) \leq \left((\frac{5}{12} - \frac{\alpha}{3})(\beta + \gamma) - \frac{(\beta + \gamma)^2}{4} - \frac{\beta\gamma}{6} \right) n^2 . \tag{6}$$

Hence, our remaining task is to maximize the expression $(\frac{5}{12} - \frac{\alpha}{3})(\beta + \gamma) - \frac{(\beta+\gamma)^2}{4} - \frac{\beta\gamma}{6}$ subject to the constraints $0 \leq \beta \leq 1 - 2\alpha$ and $0 \leq \gamma \leq 1 - 2\alpha$ (and recall that $\alpha \leq 1/2$). Simple analysis of the partial derivatives show that for all $\alpha \geq 3/8$, the maximum is obtained when $\beta = \gamma = 1 - 2\alpha$. When $1/4 \leq \alpha \leq 3/8$ the bound in the statement of the lemma trivially holds as $(1 - 2\alpha)(\frac{5}{3}\alpha - \frac{1}{3}) \geq 1/24$ in this range (and recall that a regular tournament has less than $n^3/24$ triangles). Thus, in any case, plugging in $\beta = \gamma = 1 - 2\alpha$ in (6) and rearranging the terms we obtain that

$$q(v) \leq (1 - 2\alpha)(\frac{5}{3}\alpha - \frac{1}{3})n^2 .$$

Consequently, for all $\alpha \geq 1/4$, the number of triangles that contain α-dense arcs is at most

$$\sum_{v \in V} q(v) \leq (1 - 2\alpha)(\frac{5}{3}\alpha - \frac{1}{3})n^3 .$$

■

For an arc e let $f(e)$ denote the number of triangles that contain e. We define a fractional triangle packing ψ as in [16] by assigning to a triangle X the value

$$\psi(X) = \frac{1}{\max_{e \in X} f(e)} . \tag{7}$$

In other words, we consider the three arcs of X and take the arc e with $f(e)$ maximal, setting $\psi(X)$ to $1/f(e)$. Notice that ψ is a valid fractional triangle packing. Indeed, the sum of the weights of triangles containing any arc e is at most $f(e) \cdot f(e)^{-1} = 1$.

Proof of Theorem 3.3: Let k be a positive integer, and let $1 > x > 3/4$ be a parameter to be chosen later. Define $c = x^{1/(k+1)}$ and let $\alpha_i = \frac{1}{2}c^{i+1}$ for $i = 0, \ldots, k$. Observe that $\alpha_k = x/2$ so $1/2 > \alpha_i \geq \alpha_k > 3/8$. Define as in [16]

$$E_i = \{e \in E(T) \ : \ f(e) \geq \alpha_i n\} .$$

So, E_i is the set of all α_i-dense arcs and notice that $E_0 \subset E_1 \subset \cdots \subset E_k$. For $i = 0, \ldots, k$, let S_i denote the set of all triangles that contain an arc from E_i and *do not* contain an arc from E_j where $j < i$. In particular, S_0 is just the set of triangles that contain an arc from E_0. Finally, let S_{k+1} be the triangles that are not in $\cup_{i=0}^{k} S_i$ and observe that $S_0, \ldots, S_{k+1}$ is a partition of the set of all $n(n^2-1)/24$ triangles of T.

For $i = 0, \ldots, k$, all the elements of $S_0 \cup \cdots \cup S_i$ contain arcs that are α_i-dense and therefore by Lemma 3.4 we have that for $i = 0, \ldots, k$:

$$t_i = |\cup_{j=0}^{i} S_j| \le (1-2\alpha_i)(\frac{5}{3}\alpha_i - \frac{1}{3})n^3 \ . \tag{8}$$

By the definition of t_i we have that for $i = 1, \ldots, k$, $|S_i| = t_i - t_{i-1}$ and that $|S_0| = t_0$. Thus, we also have that

$$|S_{k+1}| = \frac{n(n^2-1)}{24} - t_k \ . \tag{9}$$

For $i = 1, \ldots, k+1$, all the elements of S_i receive weight that is greater than $1/(\alpha_{i-1}n)$. Indeed, consider $X \in S_i$. We know that it does *not* contain an arc from E_j for $j < i$. So the maximum value of $f(e)$ for an arc e of X is smaller than $\alpha_{i-1}n$. By the definition of ψ we therefore have that $\psi(X) > 1/(\alpha_{i-1}n)$. For elements $X \in S_0$ we use the trivial bound $\psi(X) > 2/n$. Summing up the weights of all the triangles of T we find that:

$$|\psi| \ge t_0 \cdot \frac{2}{n} + \sum_{i=1}^{k} (t_i - t_{i-1}) \frac{1}{\alpha_{i-1}n} + \left(\frac{n(n^2-1)}{24} - t_k \right) \frac{1}{\alpha_k n} \ .$$

Rearranging the terms we have:

$$|\psi| \ge \frac{n^2-1}{24\alpha_k} - \frac{t_0}{n}\left(\frac{1}{\alpha_0} - 2\right) - \sum_{i=1}^{k} \frac{t_i}{n}\left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}}\right) \ . \tag{10}$$

Using (8) we have that:

$$|\psi| \ge \frac{n^2-1}{24\alpha_k} - n^2(1-2\alpha_0)(\frac{5}{3}\alpha_0 - \frac{1}{3})\left(\frac{1}{\alpha_0} - 2\right) - \sum_{i=1}^{k} n^2(1-2\alpha_i)(\frac{5}{3}\alpha_i - \frac{1}{3})\left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}}\right) \ .$$

Thus, we must choose k and x so as to maximize

$$\frac{1}{24\alpha_k} - (1-2\alpha_0)(\frac{5}{3}\alpha_0 - \frac{1}{3})\left(\frac{1}{\alpha_0} - 2\right) - \sum_{i=1}^{k}(1-2\alpha_i)(\frac{5}{3}\alpha_i - \frac{1}{3})\left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}}\right) \ .$$

Recalling that $a_i/a_{i-1} = c$ the last expression is identical to

$$\frac{1}{24\alpha_k} + \frac{1}{3\alpha_0} - 3 + 8\alpha_0 - \frac{20}{3}\alpha_0^2 + \frac{1}{3\alpha_k} - \frac{1}{3\alpha_0} - \frac{7}{3}k + \frac{7}{3}ck + (\frac{10}{3}\sum_{i=1}^{k}\alpha_i) - (\frac{10}{3}c\sum_{i=1}^{k}\alpha_i) \ .$$

Since $\sum_{i=1} k\alpha_i = 0.5c^2(c^k-1)/(c-1)$ the last expression is identical to

$$\frac{3}{4c^{k+1}} - 3 + 4c - \frac{5}{3}c^2 + \frac{7}{3}k(c-1) - \frac{5}{3}c^2(c^k-1) \; .$$

Finally, recalling that $c = x^{1/(k+1)}$, the last expression is identical to

$$\frac{3}{4x} - 3 + 4x^{1/(k+1)} - \frac{5}{3}x^{2/(k+1)} + \frac{7}{3}k(x^{1/(k+1)}-1) - \frac{5}{3}x^{2/(k+1)}(x^{k/(k+1)}-1) \; .$$

Taking the limit of the last expression as $k \to \infty$ we obtain

$$\frac{3}{4x} + 1 + \frac{7}{3}\ln x - \frac{5}{3}x \; .$$

The maximum of the last expression for $1 > x > 3/4$ is obtained at $x = 9/10$ in which case the expression amounts to

$$\frac{1}{3} - \frac{7}{3}\ln(\frac{10}{9}) \; .$$

This proves that

$$|\psi| \geq \left(\frac{1}{3} - \frac{7}{3}\ln(\frac{10}{9})\right) n^2(1 - o_n(1)) \; .$$

■

4 Lower bound for β-regular tournaments

In order to generalize the lower bound for β-regular tournaments we need to address three issues. The first is that the number of triangles in β-regular tournaments may not be the same for all such tournaments, (unlike regular tournaments which all have precisely $n(n^2-1)/24$ triangles), and we must therefore determine a tight lower bound in terms of β. The second issue requires an analogue of Lemma 3.4 suitable for β-regular tournaments. The third issue concerns the analysis of the fractional packing, generalizing the one given in the proof of Theorem 3.3. We start with a lower bound for the number of triangles in β-regular tournaments.

Lemma 4.1 *The number of C_3 in a β-regular tournament with n vertices is at least $\frac{1-3\beta^2}{24}n^3(1-o_n(1))$ for $\beta \leq 1/2$ and at least $\frac{(1-\beta)^3}{12}n^3(1-o_n(1))$ for $\beta > 1/2$. This is asymptotically tight for all $0 \leq \beta \leq 1$.*

Proof. The number of transitive triples (and hence the number of triangles) in any tournaments is determined by the outdegrees of the vertices. Let d_i denote the outdegree of vertex i in a tournament with vertices $1, \ldots, n$. The number of transitive triples is clearly

$$\sum_{i=1}^{n} \binom{d_i}{2} \; .$$

and we wish to maximize this amount. In β-regular tournaments we have the additional restriction that $n(1-\beta)/2 \le d_i \le n(1+\beta)/2$. Now, suppose the degrees are sorted so that $d_i \le d_{i+1}$ for $i = 1, \ldots, n-1$. In order for the tournament to be realized we have the further restriction that $d_1 + \ldots + d_i \ge \binom{i}{2}$ since already the first i vertices induce a tournament whose outdegree sum is $\binom{i}{2}$. Similarly, $(n-1-d_{n-i+1}) + \ldots + (n-1-d_n) \ge \binom{i}{2}$ since already the last i vertices induce a tournament whose indegree sum is $\binom{i}{2}$.

As the statement of the lemma is asymptotic, it is more convenient to formulate the analogous continuous convex optimization problem.

$$\begin{array}{rl} \textit{maximize} & \displaystyle\int_0^1 \frac{f(x)^2}{2} dx \\ \textit{s.t.} & f(x) \textit{ is monotone nondecreasing} \\ & \displaystyle\frac{1-\beta}{2} \le f(x) \le \frac{1+\beta}{2} \\ & \displaystyle\int_0^\alpha f(x)dx \ge \frac{\alpha^2}{2} \\ & \displaystyle\int_\alpha^1 (1-f(x))dx \ge \frac{(1-\alpha)^2}{2} \ . \end{array}$$

When $\beta \le 1/2$ the obvious solution, by convexity, is obtained by setting $f(x) = (1-\beta)/2$ for $0 \le x \le 1/2$ and $f(x) = (1+\beta)/2$ for $1/2 \le x \le 1$. Observe that since $\beta \le 1/2$, the last two restrictions of the convex minimization problem trivially hold. In this case we obtain that

$$\int_0^1 \frac{f(x)^2}{2} dx = \frac{1+\beta^2}{8}$$

and correspondingly,

$$\sum_{i=1}^{n} \binom{d_i}{2} \le \frac{1+\beta^2}{8} n^3 (1 + o_n(1)) \ .$$

The number of triangles is therefore always at least

$$\left(\frac{1}{6} - \frac{1+\beta^2}{8}\right) n^3 (1 - o_n(1)) = \left(\frac{1-3\beta^2}{24}\right) n^3 (1 - o_n(1)) \ .$$

When $\beta > 1/2$, the last two restrictions of the convex minimization problem force $f(x)$ to linearly increase in the range $1-\beta \le x \le \beta$ and we obtain the optimal solution

$$f(x) = \begin{cases} \frac{1-\beta}{2} & 0 \le x \le 1-\beta \\ x & 1-\beta < x < \beta \\ \frac{1+\beta}{2} & \beta \le x \le 1 \ . \end{cases}$$

In this case we obtain that

$$\int_0^1 \frac{f(x)^2}{2} dx = \frac{(1-\beta)^2}{8}(1-\beta) + \frac{(1+\beta)^2}{8}(1-\beta) + \frac{\beta^3}{6} - \frac{(1-\beta)^3}{6} = \frac{1}{12} + \frac{1}{4}\beta - \frac{1}{4}\beta^2 + \frac{1}{12}\beta^3$$

and correspondingly,

$$\sum_{i=1}^{n} \binom{d_i}{2} \le \left(\frac{1}{12}+\frac{1}{4}\beta-\frac{1}{4}\beta^2+\frac{1}{12}\beta^3\right) n^3(1+o_n(1)) \ .$$

The number of triangles is therefore always at least

$$\left(\frac{1}{12}-\frac{1}{4}\beta+\frac{1}{4}\beta^2-\frac{1}{12}\beta^3\right) n^3(1-o_n(1)) = \frac{(1-\beta)^3}{12} n^3(1-o_n(1)) \ .$$

The result is asymptotically tight for every β as the extremal degree sequences are realizable as β-regular tournaments. For $\beta \le 1/2$ we can take two disjoint regular tournaments A and B on $n/2$ vertices each. We can then take $(1/4-\beta/2)n$ disjoint perfect matchings between A and B and direct all edges of these matchings from A to B. The remaining edges between A and B are directed from B to A. In the resulting tournament, each vertex of A has outdegree $n(1-\beta)/2-1/2$ and each vertex of B has outdegree $n(1+\beta)/2-1/2$, hence a β-regular tournament realizing the extremal degree sequence. For $\beta > 1/2$ we can take two disjoint regular tournaments A and B on βn vertices each, and an additional set of vertices denoted as $x_1, \ldots, x_{n(1-2\beta)}$. Now, for $i = 1, \ldots, n(1-2\beta)$, direct arcs from x_i to all vertices of A and to all vertices x_j with $j < i$. Direct arcs to x_i from all vertices of B and from all vertices x_j with $j > i$. Also direct all arcs from B to A. The resulting tournament has n vertices, is β-regular, and its degree sequence realizes the extremal case. ■

We next need to obtain an analogue of Lemma 3.4 that applies to β-regular tournaments. Although it is possible to generalize Lemma 3.4 directly, the (already involved) analysis become less tractable. We settle for a somewhat simpler version with only a small loss in the upper bound.

Lemma 4.2 *Let T be a β-regular tournament with n vertices. For all $0 < \alpha < (1+\beta)/2$, the number of triangles of T that contain α-dense arcs is at most $\frac{n^3(1+\beta-2\alpha)}{2}$.*

Proof. For a vertex v, we compute the number of α-dense arcs entering it. Let $B_v \subset N^-(v)$ be the set of vertices x such that (x,v) is α-dense. Consider a vertex x of maximum indegree in the sub-tournament $T[B_v]$ induced by B_v. Since in any tournament with $|B_v|$ vertices the maximum in-degree is at least $(|B_v|-1)/2$ we have that x has at least $(|B_v|-1)/2$ arcs entering it in $T[B_v]$. On the other hand, as (x,v) is α-dense, we also have that x has at least αn vertices of $N^+(v)$ entering it. Since $N^+(v) \cap B_v = \emptyset$ we have that the indegree of x in T is at least $(|B_v|-1)/2 + \alpha n$. But the in-degree of x in T is at most $(n(1+\beta)-1)/2$ and thus

$$(|B_v|-1)/2 + \alpha n \le (n(1+\beta)-1)/2 \ .$$

It follows that $|B_v| \le n(1+\beta-2\alpha)$. Similarly, if $C_v \subset N^+(v)$ is the set of vertices y such that (v,y) is α-dense, we have that $|C_v| \le n(1+\beta-2\alpha)$. Now, each $x \in B_v$ lies in at most $|N^+(v)|$ triangles and each $y \in C_v$ lies in at most $|N^-(v)|$ triangles. We therefore have that the number of triangles containing v and an α-dense arc incident with v (either entering v or emanating from v)

is at most $n(1+\beta-2\alpha)(|N^+(v)|+|N^-(v)|) < n^2(1+\beta-2\alpha)$. Summing this value for each $v \in V$ and observing that each triangle that contains an α-dense arc is counted at least twice, we obtain that the number of triangles containing α-dense arcs is at most $n^3(1+\beta-2\alpha)/2$. ■

Finally, we need to generalize the analysis given in the proof of Theorem 3.3. We use the exact same fractional packing ψ defined in (7). As in the proof of Theorem 3.3 we let k be a positive integer, let $x < 1$ be a parameter to be chosen later, define $c = x^{1/(k+1)}$ and define $\alpha_i = (1+\beta)c^{i+1}/2$ for $i = 0, \ldots, k$. By Lemma 4.2, the upper bound for t_i given in (8) is replaced with:

$$t_i = |\cup_{j=0}^{i} S_j| \le \frac{(1+\beta-2\alpha)}{2} n^3 \ . \tag{11}$$

Similarly, using Lemma 4.1, the lower bound for S_{k+1} given in (9) is replaced with:

$$|S_{k+1}| \ge \frac{1-3\beta^2}{24} n^3(1-o_n(1)) - t_k \text{ if } \beta \le \frac{1}{2} \ \ , \ \ |S_{k+1}| \ge \frac{(1-\beta)^3}{12} n^3(1-o_n(1)) - t_k \text{ if } \beta > \frac{1}{2} \ .$$

As in (10) we have, after rearranging the terms:

$$|\psi| \ge \frac{1-3\beta^2}{24\alpha_k} n^2(1-o_n(1)) - \frac{t_0}{n}\left(\frac{1}{\alpha_0} - \frac{2}{1+\beta}\right) - \sum_{i=1}^{k} \frac{t_i}{n}\left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}}\right) \quad \text{if } \beta \le \tfrac{1}{2} \ ,$$
$$|\psi| \ge \frac{(1-\beta)^3}{12\alpha_k} n^2(1-o_n(1)) - \frac{t_0}{n}\left(\frac{1}{\alpha_0} - \frac{2}{1+\beta}\right) - \sum_{i=1}^{k} \frac{t_i}{n}\left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}}\right) \quad \text{if } \beta > \tfrac{1}{2} \ .$$

Using (11) we have that:

$$\begin{aligned} |\psi| \ge \ & \tfrac{1-3\beta^2}{24\alpha_k} n^2(1-o_n(1)) - n^2 \tfrac{(1+\beta-2\alpha_0)}{2}\left(\tfrac{1}{\alpha_0} - \tfrac{2}{1+\beta}\right) \\ & - \textstyle\sum_{i=1}^{k} n^2 \tfrac{(1+\beta-2\alpha_i)}{2}\left(\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_{i-1}}\right) && \text{if } \beta \le \frac{1}{2} \ , \\ |\psi| \ge \ & \tfrac{(1-\beta)^3}{12\alpha_k} n^2(1-o_n(1)) - n^2 \tfrac{(1+\beta-2\alpha_0)}{2}\left(\tfrac{1}{\alpha_0} - \tfrac{2}{1+\beta}\right) \\ & - \textstyle\sum_{i=1}^{k} n^2 \tfrac{(1+\beta-2\alpha_i)}{2}\left(\tfrac{1}{\alpha_i} - \tfrac{1}{\alpha_{i-1}}\right) && \text{if } \beta > \frac{1}{2} \ . \end{aligned}$$

Thus, we must choose k and x so as to maximize

$$|\psi| \ge \frac{1-3\beta^2}{24\alpha_k} - \frac{(1+\beta-2\alpha_0)}{2}\left(\frac{1}{\alpha_0} - \frac{2}{1+\beta}\right) - \sum_{i=1}^{k} \frac{(1+\beta-2\alpha_i)}{2}\left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}}\right) \quad \text{if } \beta \le \tfrac{1}{2} \ ,$$
$$|\psi| \ge \frac{(1-\beta)^3}{12\alpha_k} - \frac{(1+\beta-2\alpha_0)}{2}\left(\frac{1}{\alpha_0} - \frac{2}{1+\beta}\right) - \sum_{i=1}^{k} n^2 \frac{(1+\beta-2\alpha_i)}{2}\left(\frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}}\right) \quad \text{if } \beta > \tfrac{1}{2} \ .$$

Recalling that $a_i/a_{i-1} = c$ the last expression is identical to

$$\frac{-11-12\beta-3\beta^2}{24\alpha_k} + 2 - \frac{2\alpha_0}{1+\beta} + k(1-c) \quad \text{if } \beta \le 1/2 \ ,$$
$$\frac{-5-9\beta+3\beta^2-\beta^3}{12\alpha_k} + 2 - \frac{2\alpha_0}{1+\beta} + k(1-c) \quad \text{if } \beta > 1/2 \ .$$

Recalling that $c = x^{1/(k+1)}$, $\alpha_0 = (1+\beta)c/2$, $\alpha_k = (1+\beta)c^{k+1}/2$ we obtain that

$$\begin{aligned} &\frac{-11-12\beta-3\beta^2}{12x(1+\beta)} + 2 - x^{1/(k+1)} + k(1-x^{1/(k+1)}) \quad \text{if } \beta \le \tfrac{1}{2}\ , \\ &\frac{-5-9\beta+3\beta^2-\beta^3}{6x(1+\beta)} + 2 - x^{1/(k+1)} + k(1-x^{1/(k+1)}) \quad \text{if } \beta > \tfrac{1}{2}\ . \end{aligned}$$

Taking the limit of the last expression as $k \to \infty$ we obtain

$$\begin{aligned} &\frac{-11-12\beta-3\beta^2}{12x(1+\beta)} + 1 + \ln(1/x) \quad \text{if } \beta \le \tfrac{1}{2}\ , \\ &\frac{-5-9\beta+3\beta^2-\beta^3}{6x(1+\beta)} + 1 + \ln(1/x) \quad \text{if } \beta > \tfrac{1}{2}\ . \end{aligned}$$

The maximum of the last expression is obtained at $x = \frac{11+12\beta+3\beta^2}{12(1+\beta)}$ when $\beta \le 1/2$ and at $x = \frac{5+9\beta-3\beta^2+\beta^3}{6(1+\beta)}$ when $\beta > 1/2$ in which case the expression amounts to

$$\begin{aligned} &\ln\left(\frac{12(1+\beta)}{11+12\beta+3\beta^2}\right) \quad \text{if } \beta \le \tfrac{1}{2}\ , \\ &\ln\left(\frac{6(1+\beta)}{5+9\beta-3\beta^2+\beta^3}\right) \quad \text{if } \beta > \tfrac{1}{2}\ . \end{aligned}$$

This proves that

$$\begin{aligned} |\psi| &\ge \ln\left(\frac{12(1+\beta)}{11+12\beta+3\beta^2}\right) n^2(1-o_n(1)) \quad \text{if } \beta \le \tfrac{1}{2}\ , \\ |\psi| &\ge \ln\left(\frac{6(1+\beta)}{5+9\beta-3\beta^2+\beta^3}\right) n^2(1-o_n(1)) \quad \text{if } \beta > \tfrac{1}{2}\ . \end{aligned}$$

This completes the proof of the lower bound in Theorem 1.3 which, together with the upper bound proved in Section 2, yields the entire proof. ■

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