

LECTURES ON INFINITY CATEGORIES

VLADIMIR HINICH

To the memory of Michael Roytberg

ABSTRACT. These are lecture notes of the course in infinity categories given at Weizmann Institute in 2016–2017.

0. INTRODUCTION

0.1. These are lecture notes of the course in infinity categories given at Weizmann institute in the fall semester of 2016–2017 year.

The aim of the course was to give some relevant background from homotopy theory, to present different formal approaches to infinity categories, and to allow the audience to develop an understanding of the subject, which would not rely on a specific model of infinity categories.

Infinity categories have already proven an indispensable tool in derived algebraic geometry, factorization homology and geometric representation theory. There is no doubt that the importance of the language of infinity categories will continue growing in the near future. We hope these lecture notes may serve a less technical introduction to the subject, as compared to the remarkable (but quite voluminous) Lurie’s treatise [L.T].

The idea of infinity category owes, to a much extent, to a certain dissatisfaction with the classical language of derived categories developed by Grothendieck and Verdier in early 60-ies.¹ An infinity-category should have, apart of objects and morphisms, a sort of “higher morphisms” between the morphisms, as well as morphisms between these “higher morphisms”, and so on. Once it was realized that higher groupoids should correspond to homotopy types, and that for a wide range of applications it is sufficient to assume that all “higher morphisms” are invertible, a number of different definitions of these so-called $(\infty, 1)$ -categories was suggested.

¹Here is what S. Gelfand and Y. Manin wrote in the introduction to [GM] published in late 1980-ies. “We worked on this book with the disquieting feeling that the development of homological algebra is currently in a state of flux, and that the basic definitions and constructions of the theory of triangulated categories, despite their widespread use, are of only preliminary nature (this applies even more to homotopic algebra)”.

0.2. We will now present a few examples where infinity categories appear naturally as an “upgrade” of conventionally used categories.

0.2.1. Category theory appeared in algebraic topology which studies algebraic invariants of topological spaces. From the very beginning it was well understood that what is important is not just to assign, say, an abelian group to a topological space, but to make sure that this assignment is functorial, that is, that it carries a continuous map of topological spaces to a homomorphism of groups. Thus, singular homology appears as a functor

$$H : \mathbf{Top} \rightarrow \mathbf{Ab}$$

from the category of topological spaces to the category of abelian groups. The next thing to do is to realize that the map $H(f) : H(X) \rightarrow H(Y)$ does not really depend on $f : X \rightarrow Y$, but on the equivalence class of f up to homotopy. To make our language as close as possible to the problems we are trying to solve, we may replace the category \mathbf{Top} of topological spaces, factoring the sets $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ by the homotopy relation. We will get another meaningful category which should better describe the object of study of algebraic topology — but this category has some very unpleasant properties (lack of limits). Another approach, which is closer to the one advocated by infinity-category theory, is to think of the sets $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ as topological spaces, so that information on homotopies between the maps is encoded in the topology of Hom-spaces.

This will lead us, in this course, to defining an infinity category of spaces describing, roughly speaking, topological spaces up to homotopy. This infinity category is a fundamental object in the theory, playing the role similar to that of the category of sets \mathbf{Set} in the conventional category theory. Interestingly, this infinity category can be defined without mentioning a definition of topological space.

0.2.2. We are still thinking about topology, but we will now look at a single topological space X instead of the totality of all topological spaces.

Let X be a topological space. Following Poincaré, we assign to X a groupoid ² $\Pi_1(X)$ whose objects are the points of X and arrows are the homotopy classes of paths. The groupoid $\Pi_1(X)$ has a very nice homotopic property: it retains information on $\pi_0(X)$ and $\pi_1(X)$ of X but forgets all higher homotopy groups of X . And here is the reason: we took homotopy classes of paths as arrows. Could we have taken instead topological spaces of paths, we would have chance to retain all information about the homotopy type of X . This is, however, easier said than done. It is easy to compose homotopy classes of paths, but there is no canonical way of composing the actual paths.

²A groupoid is just a category whose arrows are invertible.

This sort of composition makes sense in infinity category theory. There spaces and infinity groupoids become just the same thing.

0.2.3. Here is a more algebraic example.

An important notion of homological algebra is the notion of derived functor. Study of derived functors led to the notion of derived category in which the standard ambiguity in the choice of resolutions used to calculate derived functors, disappears.

Here is, in two paragraphs, the construction of derived category of an abelian category \mathcal{A} . As a first step, one constructs the category of complexes $C(\mathcal{A})$ where all projective resolutions, their images after application of functors, live. The second step is similar to the notion of localization of a ring: given a ring R and a collection S of its elements, one defines a ring homomorphism $R \rightarrow R[S^{-1}]$ such that the image of each element in S becomes invertible in $R[S^{-1}]$ and universal for this property. Localizing $C(\mathcal{A})$ with respect to the collection of quasiisomorphisms, we get $D(\mathcal{A})$, the derived category of \mathcal{A} .

The construction of derived category $D(\mathcal{A})$ is a close relative to the construction of the homotopy category of topological spaces, when we factor the set of continuous maps by an equivalence relation. The notion of derived category is not very convenient, approximately for the same reasons we already mentioned. Localizing a category, we destroy an important information, similarly to destroying information about higher homotopy groups of X in the construction of the fundamental groupoid $\Pi_1(X)$. Here is another inconvenience.

0.2.4. Let \mathcal{A} be the category of abelian sheaves on a topological space X . For an open subset $U \subset X$ let \mathcal{A}_U be the category of sheaves on U . There is a very precise procedure how one can glue, given sheaves $F_U \in \mathcal{A}_U$ and some “gluing data”, a sheaf $F \in \mathcal{A}$ whose restrictions to U are F_U . The collection of \mathcal{A}_U is also a sort of a sheaf (of categories). It is still possible to glue \mathcal{A} from the collection of \mathcal{A}_U (even though one needs “2-gluing data” for this ³). We would be happy to be able to glue the derived categories $D(\mathcal{A}_U)$ into the global $D(\mathcal{A})$. It turns out this is in fact possible, but one needs to replace the derived category with a more refined infinity notion, and of course, use all “higher” gluing data.

0.2.5. There is a pleasant “side effect” in replacing the derived category $D(\mathcal{A})$ with an infinity category. As it is well-known, $D(\mathcal{A})$ is a triangulated category, that is an additive category endowed with a shift endofunctor, with a chosen collection of diagrams called distinguished triangles, satisfying a list of properties. The notion of triangulated category is a very important, but very unnatural one. Fortunately, the respective infinity categorical notion is very natural: this is just a property of infinity category (called *stability*, see Section 10) rather than a collection of extra structures (shift functor, distinguished triangles, etc.)

³The reason is that categories \mathcal{A}_U are the objects of a 2-category, that of categories.

0.3. The abundance of definitions of $(\infty, 1)$ -category is somewhat similar to the abundance of programming languages or of models of computation; all of them have in mind the same idea of computability, but realize this idea differently. Some programming languages are more convenient in specific applications than others. The same holds for different definitions of $(\infty, 1)$ -categories.

There is an easy way to compare expressive power of two programming languages: it is enough to write an interpreter for language I in language II and vice versa.

It is less obvious how to compare different formalizations of infinity categories. In all existing approaches, $(\infty, 1)$ -categories are realized as fibrant-cofibrant objects in a certain Quillen model category (simplicial sets with Joyal model structure, bisimplicial sets with Segal category or complete Segal model structure, simplicial categories, etc.). The graph of Quillen equivalences between different model categories in the above list is connected, which implies that at least the homotopy categories of different versions of $(\infty, 1)$ -categories are equivalent. This, however, seems too weak at first sight. On the other hand, if we choose our favorite definition of ∞ -category, we can see that, first of all, any model category gives rise to an ∞ -category (we call it *underlying ∞ -category*), and, furthermore, Quillen equivalent model categories give rise to equivalent ∞ -categories. In particular, ∞ -categories underlying different models of infinity categories, are equivalent.

This already sounds very similar to what happens when one compares different models of computation, and supports our belief that there exists a notion of $(\infty, 1)$ -category as Plato's idea, so that the different models are merely different (but equivalent) realizations of this idea.

0.4. Here is a detailed description of the course.

In the Section 1 we remind some standard notions and constructions of conventional category theory, define simplicial sets and compare them to topological spaces.

In Section 2 we discuss quasicategories, simplicial categories and homotopy coherent nerve.

In Section 3 we define model categories and Quillen adjunctions; in Section 4 we present a Quillen equivalence between the simplicial sets and the topological spaces.

In Section 5 we discuss a Quillen equivalence between the simplicial sets with Joyal model structure (quasicategories) and simplicial categories with Bergner model structure; we also mention Dwyer-Kan localization which realizes the ∞ -categorical notion of localization.

In Section 6 we study Rezk's complete Segal spaces (CSS) and discuss equivalence of different models.

In Sections 7 and 8 we present the Grothendieck construction for left fibrations, using the CSS model and deduce the infinity version of Yoneda lemma (following Kazhdan-Varshavsky). This allows one to study limits, adjunction and other universal constructions.

Once we understood Yoneda lemma, we feel ready to gradually introduce the language of infinity categories without direct reference to a concrete model ⁴. Starting Section 9 (devoted to cocartesian fibrations), we use this language. We discuss stable infinity categories in Section 10 and monoidal structures in Section 11.

0.5. **Acknowledgements.** I am grateful to Rami Aizenbud from the Weizmann institute who persuaded me to give this course. I think I now understand the subject much better than back in 2016. I am grateful to everybody who attended the course, but especially to Shachar Carmeli, Gal Dor and Sasha Shamov who corrected some of the errors I made during the lectures. These notes are dedicated to Michael Roytberg who explained to me (some 40 years ago) how to prove equivalence of standard models of computation.

⁴instead of thinking, what is infinity category, we think what can be done with them.

1. CATEGORIES AND SIMPLICIAL SETS

1.1. **Categories. Simplicial sets.** This short introduction is not intended to teach those who never heard about categories; but to fix notation and remind the main points of the language.

1.1.1. *First definitions.* A category \mathcal{C} has a set (sometimes big) of objects denoted $\text{Ob } \mathcal{C}$, together with a set of morphisms $\text{Hom}_{\mathcal{C}}(x, y)$ for each pair of objects x, y of \mathcal{C} , an associative composition

$$\text{Hom}(y, z) \times \text{Hom}(x, y) \rightarrow \text{Hom}(x, z)$$

for each triple of objects, units $\text{id}_x \in \text{Hom}(x, x)$.

We will not write down here the full list of axioms (but you should know them). For two categories \mathcal{C}, \mathcal{D} a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is a map $f : \text{Ob } \mathcal{C} \rightarrow \text{Ob } \mathcal{D}$, together with a collection of maps $\text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(fx, fy)$ compatible with the compositions.

One can compose functors — so that the categories form a category \mathbf{Cat} . However, this notion is not very useful.

The reason for this is that most of categorical constructions are defined “up to” canonical isomorphism. For instance,

1.1.2. **Definition.** Let $x, y \in \mathcal{C}$. Their product is an object p together with a pair of arrows $p \rightarrow x$ and $p \rightarrow y$, satisfying a universal property: for any $q, q \rightarrow x, q \rightarrow y$ there exists a unique arrow $q \rightarrow p$ such that the diagrams are commutative.

1.1.3. **Lemma.** *A product, if exists, is unique up to a unique isomorphism.*

□

This allows one to construct a functor $\mathcal{C} \rightarrow \mathcal{C}$ carrying an object $y \in \mathcal{C}$ to $x \times y$ where x is a fixed object of \mathcal{C} (we assume \mathcal{C} has products, for instance, $\mathcal{C} = \mathbf{Set}$). The problem, however, is that such functor is not unique — it is unique only up to a unique isomorphism. This persuades us that the notion of morphism (or, at least, of isomorphism) of functors is very important. We will define them now.

A morphism $u : f \rightarrow g$ assigns to each $x \in \text{Ob } \mathcal{C}$ an arrow $u(x) \in \text{Hom}_{\mathcal{D}}(f(x), g(x))$ such that for any arrow $a \in \text{Hom}_{\mathcal{C}}(x, y)$ the diagram in \mathcal{D} presented below is commutative.

$$\begin{array}{ccc} f(x) & \xrightarrow{u(x)} & g(x) \\ f(a) \downarrow & & g(a) \downarrow \\ f(y) & \xrightarrow{u(y)} & g(y) \end{array} .$$

The functors from \mathcal{C} to \mathcal{D} form a new category denoted $\text{Fun}(\mathcal{C}, \mathcal{D})$: its objects are the functors, and $\text{Hom}(f, g)$ is defined as the collection of morphisms of functors.

The notion of isomorphism of functors allows us to formalize our feeling that it is not really important which model for the direct product of two objects to choose.

It allows as well to formulate that sometimes different, non-isomorphic categories should be seen as basically the same. Here is the notion which more appropriate than the notion of isomorphism in the world of categories.

1.1.4. Definition. A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence if there exists a functor $g : \mathcal{D} \rightarrow \mathcal{C}$ and a pair of isomorphisms of functors

$$g \circ f \xrightarrow{\sim} \text{id}_{\mathcal{C}}, \quad f \circ g \xrightarrow{\sim} \text{id}_{\mathcal{D}}.$$

If you believe in Axiom of choice (I do), here is an equivalent definition.

1.1.5. Definition. A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence if

- It is essentially surjective, that is for any $y \in \mathcal{D}$ there exists $x \in \mathcal{C}$ and an isomorphism $f(x) \xrightarrow{\sim} y$.
- It is fully faithful, that is for all $x, x' \in \mathcal{C}$ the map

$$\text{Hom}_{\mathcal{C}}(x, x') \rightarrow \text{Hom}_{\mathcal{D}}(fx, fx')$$

is an isomorphism.

Remark. For an equivalence f the functor g , “quasi-inverse” to f , is not defined uniquely — but uniquely up to unique isomorphism. We leave this as an exercise.

1.1.6. Yoneda lemma. Representable functors. Probably the most important example of category is the category of sets, denoted **Set**.

Sometimes functors do not preserve arrows, but invert them. This justifies the following definition.

Definition. Let \mathcal{C} be a category. The opposite category \mathcal{C}^{op} is defined as follows. It has the same objects and inverted morphisms:

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(x, y) = \text{Hom}_{\mathcal{C}}(y, x).$$

Functors $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ are sometimes called the contravariant functors.

Definition. $P(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ — the category of presheaves.

Here is the origin of the name. Let X be a topological space. A sheaf on X (of, say, abelian groups) F assigns to each open set $U \subset X$ an abelian group $F(U)$, and for $V \subset U$ a homomorphism $F(U) \rightarrow F(V)$ (restriction of a section to an open subset) such that certain additional gluing properties are satisfied. Presheaf is a collection of abelian groups $F(U)$ with restriction maps without extra gluing properties. In our terms, this is just a contravariant functor from the category of open subsets of X to the abelian groups.

We define Yoneda embedding as the functor

$$Y : \mathcal{C} \rightarrow P(\mathcal{C})$$

carrying $x \in \mathcal{C}$ to the functor $Y(x)$, $Y(x)(y) = \text{Hom}_{\mathcal{C}}(y, x)$.

Lemma. *Yoneda embedding is fully faithful.*

Meaning: in order to describe an object $x \in \mathcal{C}$ up to unique isomorphism, it suffices to describe the functor $Y(x)$ (called: the functor represented by x).

A presheaf isomorphic to $Y(x)$ for some x is called representable presheaf.

Yoneda lemma is a direct consequence of yet stronger result which is also called Yoneda lemma.

Lemma. *Let \mathcal{C} be a category and $F \in P(\mathcal{C})$. Then for any $x \in \mathcal{C}$ the map*

$$\text{Hom}_{P(\mathcal{C})}(Y(x), F) \rightarrow F(x)$$

carrying any morphism of functors $a : Y(x) \rightarrow F$ to $a(\text{id}_x) \in F(x)$, is a bijection.

We suggest to prove the lemma as an exercise. An important step in our course will be an infinity-categorical version of Yoneda lemma.

1.1.7. *Adjoint functors.* Here is a standard definition.

Definition. A pair of functors $L : \mathcal{C} \rightarrow \mathcal{D}$ and $R : \mathcal{D} \rightarrow \mathcal{C}$, together with morphisms $\alpha : L \circ R \rightarrow \text{id}_{\mathcal{D}}$, $\beta : \text{id}_{\mathcal{C}} \rightarrow R \circ L$, is called an *adjoint pair* if the compositions below give identity of L and of R respectively.

$$(1) \quad L \xrightarrow{1 \circ \beta} LRL \xrightarrow{\alpha \circ 1} L$$

$$(2) \quad R \xrightarrow{\beta \circ 1} RLR \xrightarrow{1 \circ \alpha} R$$

Here is a more digestible definition: this is a pair of functors L, R , together with a natural isomorphism (=isomorphism of bi-functors)

$$\text{Hom}_{\mathcal{D}}(Lx, y) = \text{Hom}_{\mathcal{C}}(x, Ry).$$

Thus, a primary datum is a bifunctor ⁵

$$F : \mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}.$$

This bifunctor can be equivalently rewritten as a functor $\mathcal{C} \rightarrow P(\mathcal{D})$ or as a functor $\mathcal{D} \rightarrow P(\mathcal{C})$. By Yoneda lemma, if the first functor has its essential image in $\mathcal{D} \subset P(\mathcal{D})$ ⁶ there is a unique functor L up to unique isomorphism presenting F as

$$F(x, y) = \text{Hom}_{\mathcal{D}}(L(x), y).$$

Similarly, if the second functor has essential image in $\mathcal{C} \subset P(\mathcal{C})$, there exists $R : \mathcal{D} \rightarrow \mathcal{C}$ so that $F(x, y) = \text{Hom}_{\mathcal{C}}(x, R(y))$.

⁵You have to define a product of categories!

⁶That is, every object of the image is isomorphic to an object of \mathcal{D} .

- Corollary.** 1. If $L : \mathcal{C} \rightarrow \mathcal{D}$ admits a right adjoint functor, it is unique up to unique isomorphism.
2. L admits a right adjoint iff for any $y \in \mathcal{D}$ the functor $x \mapsto \text{Hom}_{\mathcal{D}}(L(x), y)$ is representable.

Proof. Exercise. □

1.2. **Exercises.** Prove everything formulated above without proof.

1.3. **Category Δ . Category \mathbf{sSet} .**

1.3.1. *Category Δ .* Δ is a very important category, “the category of combinatorial simplices”.

Its objects are $[n] = \{0, \dots, n\}$, considered as ordered sets. Morphisms are maps of ordered sets (preserving the order). In particular, $[0]$ consists of one element and so is the terminal object in Δ .

By the way,

Definition. An object $x \in \mathcal{C}$ is terminal if $\text{Hom}_{\mathcal{C}}(y, x)$ is a singleton for all y . An object x is initial if $\text{Hom}_{\mathcal{C}}(x, y)$ is a singleton for all y .

The category Δ has no nontrivial isomorphisms. This is sometimes convenient; otherwise I would prefer to define Δ as the category of totally ordered finite nonempty sets. It would be equivalent to the one we defined, but would look more natural.

Here are some special arrows in Δ .

Faces $\delta^i : [n-1] \rightarrow [n]$, the injective map missing the value $i \in [0, n]$.

Degeneracies $\sigma^i : [n] \rightarrow [n-1]$ the surjective map for which the value $i \in [0, n-1]$ is repeated twice.

Any map $[m] \rightarrow [n]$ can be uniquely presented as a surjective map followed by an injective map. Any injective map is a composition of faces, and any surjective map is a composition of degeneracies.

The latter presentations are not unique. For instance, $\delta^j \circ \delta^i = \delta^i \circ \delta^{j-1}$ for $i < j$.

Exercise. Prove this. Try to find and prove all the identities.

Definition. A simplicial object in a category \mathcal{C} is a functor $\Delta^{\text{op}} \rightarrow \mathcal{C}$. A simplicial object in sets is called a simplicial set. The category of simplicial sets will be denoted \mathbf{sSet} . In other words, $\mathbf{sSet} = P(\Delta)$.

The category of simplicial sets is the one where most of the homotopy theory lives. Let us describe in more detail what is a simplicial set.

To each $[n]$ it assigns a set X_n called “the set of n -simplices of X ”. Any map $\alpha : [m] \rightarrow [n]$ defines $\alpha^* : X_n \rightarrow X_m$. In particular, we will usually denote $d_i = (\delta^i)^*$ and $s_i = (\sigma^i)^*$. Here how a simplicial set looks like (this is only a small part of it):

$$(3) \quad X_0 \begin{array}{c} \xrightarrow{\text{dotted}} \\ \xleftarrow{\text{solid}} \\ \xrightarrow{\text{solid}} \\ \xleftarrow{\text{solid}} \end{array} X_1 \begin{array}{c} \xrightarrow{\text{dotted}} \\ \xleftarrow{\text{solid}} \\ \xrightarrow{\text{solid}} \\ \xleftarrow{\text{solid}} \end{array} X_2,$$

where the solid arrows denote the faces whereas the dotted arrows denote the degeneracies.

The first examples of simplicial sets we can easily produce are representable by the objects of Δ : any object $[n]$ defines a simplicial set Δ^n whose m -simplices are maps $[m] \rightarrow [n]$. This simplicial set is called *the standard n -simplex*.

1.4. Singular simplices. Nerve of a category. Simplicial sets and, more generally, simplicial objects, are everywhere. Let us look around and find them.

1.4.1. Singular simplices. Let X be a topological space. We assign to it a simplicial set $\text{Sing}(X)$ as follows. The set of n -simplices $\text{Sing}_n(X)$ is the set of continuous maps from the standard (topological) n -simplex

$$\Delta[n] = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum x_i = 1\}$$

to X .

The faces and the degeneracies are defined via maps

$$\delta^i : \Delta[n-1] \rightarrow \Delta[n]$$

and

$$\sigma^i : \Delta[n] \rightarrow \Delta[n-1]$$

where δ^i inserts 0 at the place i and σ^i puts $x_i + x_{i+1}$ at the place i .

1.4.2. Nerve of a category. There is a very similar construction in the world of categories — this is something that allows one to guess that categories and topological spaces are somehow connected.

Given a category \mathcal{C} , we define a simplicial set $N(\mathcal{C})$, the nerve of \mathcal{C} , as follows. Its n -simplices are functors $[n] \rightarrow \mathcal{C}$ where $[n]$ is now considered as the category defined by the corresponding ordered set (the objects are numbers $0, \dots, n$, and there is a unique arrow $i \rightarrow j$ for $i \leq j$.)

1.4.3. Example: BG . Let G be a discrete group. Denote BG the category having one object and G as its group of automorphisms. A functor $BG \rightarrow \mathbf{Vect}$ is the same as a representation of G .

Nerve of BG is the simplicial set whose n -simplices are sequences of n elements of G ; degeneracies insert $1 \in G$ and faces d_i , $i = 1, \dots, n-1$ multiply two neighboring elements of G .

What do d_0 and d_n do?

1.4.4. *Geometric realization.* The functor $\text{Sing} : \mathbf{Top} \rightarrow \mathbf{sSet}$ admits a left adjoint (called geometric realization and denoted $|X|$ for $X \in \mathbf{sSet}$.)

We already know that it is sufficient to check that for each $X \in \mathbf{sSet}$ the functor $\mathbf{Top} \rightarrow \mathbf{Set}$ carrying T to $\text{Hom}(X, \text{Sing}(T))$, is (co)representable. We definitely know this for $X = \Delta^n$ — then by definition the functor is corepresented by $\Delta[n]$.

We will prove existence of left adjoint after a discussion of colimits (=inductive limits). Meanwhile, we calculated $|\Delta^n| = \Delta[n]$ which is very nice.

1.5. Topological spaces versus simplicial sets.

1.5.1. *Generalities: limits and colimits.* Numerous important operations in categories, for instance

- Intersection of a decreasing family of sets.
- Union of an increasing family of sets.
- Coproduct, direct product, fiber product,

are special cases of the notion of limit (colimit).

Here is a general setup for a colimit: given a functor $F : I \rightarrow \mathcal{C}$, we are looking for an object $x \in \mathcal{C}$ endowed with compatible collection of maps $F(i) \rightarrow x$ (explained below) and universal with respect to this property (also explained below). This object x with all extra information (the maps $F(i) \rightarrow x$) is called the colimit of F .

Compatibility in the above definition means that for any arrow $a : i \rightarrow j$ in I the diagram

$$F(i) \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} F(j) \xrightarrow{\quad} x$$

is commutative.

Universality in the above definition means that for any $x' \in \mathcal{C}$ with the same compatible collection of maps $F(i) \rightarrow x'$, there exists a unique arrow $x \rightarrow x'$ such that the diagrams

$$F(i) \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} x \xrightarrow{\quad} x'$$

commute.

The notion of limit is obtained by dualization: Limit of a functor $F : I \rightarrow \mathcal{C}$ is the same as a colimit of $F^{\text{op}} : I^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$.

1.5.2. *(Co)limits via adjoint functors.* One can describe the notion of (co)limit using the language of adjoint functors.

Functors from I to \mathcal{C} live in $\text{Fun}(I, \mathcal{C})$; one has an obvious functor

$$(4) \quad \text{Fun}(I, \mathcal{C}) \leftarrow \mathcal{C} : \text{const}$$

assigning to any object $x \in \mathcal{C}$ the constant functor with value x . Then colimit and limit appear as left and right adjoint functors to const .

Of course, limits and colimits do not always exist.

Note that for $F \in \text{Fun}(I, \mathcal{C})$, a canonical adjunction yields a map $F \rightarrow \text{const}(\text{colim}(F))$. This is to stress that the compatible collection of maps $F(i) \rightarrow \text{colim } F$ is a part of the data for $\text{colim } F$.

1.5.3. *Colimits via adjoint functors: cont.* Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair of functors. Let $a : I \rightarrow \mathcal{C}$ be a functor and let $A = \text{colim } a$. Then $F(a)$ is a colimit of $F \circ a$. We explained that a left adjoint functor preserves all colimits (that exist in \mathcal{C}). Dually, a right adjoint functor preserves limits.

1.5.4. *Adjoint pair of functors between topological spaces and simplicial sets.* The functor $\text{Sing} : \text{Top} \rightarrow \mathbf{sSet}$ admits a left adjoint called geometric realization. By definition, geometric realization $|X|$ of a simplicial set X satisfies the following universal property:

$$(5) \quad \text{Hom}_{\text{Top}}(|X|, Y) = \text{Hom}_{\mathbf{sSet}}(X, \text{Sing}(Y)).$$

The right-hand side is a compatible collection of continuous maps $\tilde{x} : \Delta[n] \rightarrow Y$ given for each $x \in X_n$, the compatibility meaning that for any $a : [m] \rightarrow [n]$ and $x \in X_n$, $y = a^*(x) \in X_m$ one has $\tilde{y} = \tilde{x} \circ a$. This means that $|X|$ is a colimit of the functor we will now define.

The functor is defined on the category whose objects are pairs $(n, x \in X_n)$ and whose arrows are pairs (a, x) where $a : [m] \rightarrow [n]$ is an arrow in Δ and $x \in X_n$. We denote this category $N_*(X)$ — the category of simplices in X . The functor $F_X : N_*(X) \rightarrow \text{Top}$ assigns to (a, x) the standard (topological) n -simplex $\Delta[n]$. It is easy to see that one has

$$|X| = \text{colim } F_X.$$

1.6. First notions in homotopy theory.

Weak equivalences, Kan fibrations, Kan simplicial sets. Singular simplices of a topological space are Kan.

The adjoint pair of functors $(|-|, \text{Sing})$ does not produce an equivalence between topological spaces and simplicial sets; but both categories are good to study homotopical properties of topological spaces. Thus, we should not be surprised to find out that these categories are equivalent in another, “homotopic” sense. There are different ways to express this. We will present one of them in some detail later — they are Quillen equivalent.

Before formally presenting the necessary machinery, we will try to describe some of the features of this equivalence.

In a few words, there is a notion of weak equivalence in both categories so that the adjoint functors induce an equivalence of respective localizations. We will discuss localization later. We will now mention some standard notions of algebraic topology.

1.6.1. *Homotopy equivalence.* Two maps $f, g : X \rightarrow Y$ in \mathbf{Top} are homotopic if there is a (continuous) map $F : X \times [0, 1] \rightarrow Y$ which restricts to f and g at 0, 1. A map $f : X \rightarrow Y$ is a homotopy equivalence if there exists $g : Y \rightarrow X$ such that the two compositions are homotopic to the respective identity.

1.6.2. *Homotopy groups.* Fix $n > 0$. The n -th homotopy group $\pi_n(X, x)$ (of a pointed space $(X, x \in X)$) is the set of homotopy classes of maps $(D^n, \partial D^n) \rightarrow (X, x)$ ⁷.

One defines $\pi_0(X)$ as the set of (path) connected components of X . One has

- π_1 has a group structure.
- π_n has a commutative group structure for $n > 1$ ⁸.

1.6.3. *Weak homotopy equivalence.* A map $f : X \rightarrow Y$ of topological spaces is called a weak homotopy equivalence if it induces a bijection of connected components, and for each $x \in X$ it also induces an isomorphism of all homotopy groups $\pi_n(X, x) \rightarrow \pi_n(Y, f(x))$. It is easy to verify that homotopy equivalences satisfy the above properties.

The notion of weak homotopy equivalence turns out to be more convenient to work with.

Fortunately, the two notions of homotopy equivalence coincide for good topological spaces, see below.

Theorem. (*Whitehead*) *Let $f : X \rightarrow Y$ be a weak homotopy equivalence. If X and Y are CW complexes, then f is a homotopy equivalence.*

1.6.4. *Homotopy groups of simplicial sets.* Let $X \in \mathbf{sSet}$. We define $\pi_n(X, x)$ as $\pi_n(|X|, x)$. In general, there is no easy combinatorial way to define $\pi_n(X, x)$ (homotopy groups of spheres are not easily calculated).

The homotopy groups are easily calculated for Kan simplicial sets (see definition below). One has

Theorem. *Let X be a Kan simplicial set. Then $\pi_n(X, x)$ is the set of equivalence classes of maps $(\Delta^n, \partial\Delta^n) \rightarrow (X, x)$, with equivalence given by homotopies.*

1.6.5. *Degenerate and nondegenerate simplices.* Even the smallest (nonempty) simplicial set has infinite number of simplices. This is because degeneracy maps $s_i : X_n \rightarrow X_{n+1}$ are injective. Therefore, it is interesting to look at the simplices $x \in X_n$ which are not degenerations of any simplex. Such simplices are called nondegenerate. The collection of all simplices of dimension $\leq n$ and of all their degenerations is a simplicial subset of X called n -th skeleton of X , $\text{sk}_n(X)$.

⁷We work here with pairs (space, subspace). Maps $(X, A) \rightarrow (Y, B)$ are continuous maps $X \rightarrow Y$ carrying A to B . There is a similar notion of homotopy of such maps.

⁸This is a very important point. Multiplication in π_1 is defined by a concatenation of paths; it defines a group law on the homotopy classes of paths. Commutativity of higher π_n is a manifestation of *Eckmann-Hilton argument*.

One has $\text{sk}_{n-1}(\Delta^n)$ contains all non-degenerate simplices of Δ^n except for the one of dimension n (corresponding to $\text{id}_{[n]}$). We will denote this simplicial set $\partial\Delta^n$; this is, by definition, the boundary of Δ^n . The following simplicial subsets will be also very important. These are Λ_i^n —simplicial subsets of Δ^n spanned by all nondegenerate simplices of $\partial\Delta^n$ but one — $d_i : [n-1] \rightarrow [n]$. (Λ_i^n is called *i-th horn of Δ^n*).

1.6.6. *Kan simplicial sets.* A simplicial set X is Kan if any map $\Lambda_i^n \rightarrow X$ extends to a map $\Delta^n \rightarrow X$.

Exercise. Prove that $\text{Sing}(X)$ is Kan for any topological space X . Prove that the nerve $N(\mathcal{C})$ of a category \mathcal{C} is Kan if and only if the category \mathcal{C} is a groupoid.

1.6.7. Here is an expression of the fact that simplicial sets and topological spaces are “practically equivalent”.

Theorem. *Let S be a simplicial set and let X be a topological space. A map $f : S \rightarrow \text{Sing}(X)$ is a weak equivalence of simplicial sets iff the corresponding map $|S| \rightarrow X$ is a weak homotopy equivalence.*

1.6.8. There is an important property of geometric realizations which does not follow from the adjunction.

Theorem. *The functor of geometric realization preserves products.*

Direct product of simplicial sets is given pointwise: $(X \times Y)_n = X_n \times Y_n$.

First of all, one has a canonical map $|X \times Y| \rightarrow |X| \times |Y|$. So it remains to verify the given map is a homeomorphism.

Step 1. Prove the claim when $X = \Delta^n$ and $Y = \Delta^m$.

It is worthwhile to explicitly describe $\Delta^n \times \Delta^m$. One can use the following trick: the nerve functor from categories to simplicial sets obviously preserves limits. Since Δ^n is a nerve of the category $[n]$ corresponding to the totally ordered set $\{0, \dots, n\}$, $\Delta^n \times \Delta^m$ is the nerve of the poset $[n] \times [m]$. In particular, it is glued of $\binom{n+m}{n}$ $n+m$ -simplices glued along the boundary. See the case $n = m = 1$ — the square is glued of two triangles.

Step 2. Everything commutes with the colimits.

1.6.9. **Remark.** (Adjoint pairs and simplicial objects)

Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjoint pair of functors. Given an object $d \in \mathcal{D}$, one defines a simplicial object $B_\bullet(d)$ together with a map of simplicial objects

$$B_\bullet(d) \rightarrow d$$

(where d is considered as a constant simplicial object) as follows. One defines $B_0(d) = LR(d)$, and, more generally, $B_n(d) = (LR)^{n+1}(d)$. We define $s_0 : B_0(d) \rightarrow B_1(d)$ as induced by the map $\text{id} \rightarrow RL$ applied to $R(d)$. We define $d_i : B_1 \rightarrow B_0$, $i = 0, 1$ as induced by the map $LR \rightarrow \text{id}$ applied to the first

(resp., the second) pair of L, R . This easily generalizes to all face and degeneracy maps. A lot of resolutions in homological algebra (Bar-resolutions) come from this construction. The same origin has a cosimplicial object known as Čech complex connected to a covering of a topological space.

1.7. Exercises.

1.7.1. See 1.2.

1.7.2. See Exercise in 1.3.1.

1.7.3. Forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$ has both left and right adjoints. Describe them.

1.7.4. Construct a functor left adjoint to the forgetful functor from commutative algebras over a field k to the category of vector spaces over k .

1.7.5. Let \mathcal{C} be a category having finite products. For $x, y \in \mathcal{C}$ we define $\mathcal{H}om(x, y) \in \mathcal{C}$ by the property

$$\mathrm{Hom}_{\mathcal{C}}(z, \mathcal{H}om(x, y)) = \mathrm{Hom}_{\mathcal{C}}(z \times x, y).$$

Prove existence of $\mathcal{H}om$ for $\mathcal{C} = \mathbf{sSet}$.

1.7.6. The same for $\mathcal{C} = \mathbf{Cat}$. Compare to the above.

1.7.7. Let \mathcal{C} has colimits. A colimit preserving functor $F : \mathbf{sSet} \rightarrow \mathcal{C}$ is uniquely given by a functor $\Delta \rightarrow \mathcal{C}$ (a cosimplicial object in \mathcal{C}).

2. QUASICATEGORIES. SIMPLICIAL CATEGORIES

2.1. Conventional categories as simplicial sets. The functor $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$ assigning to a category C its nerve $N(C)$ whose n -simplices are the functors $[n] \rightarrow C$ (considered as a set), is fully faithful.

In fact, let C and D be two categories. A map of simplicial sets $F : N(C) \rightarrow N(D)$ is a collection of compatible maps $F_n : N_n(C) \rightarrow N_n(D)$. Let us study more carefully this compatibility. First of all, F_0 is precisely a map $\text{Ob}(C) \rightarrow \text{Ob}(D)$. Then F_1 assigns to an arrow in C an arrow in D ; compatibility with respect to $\delta^0, \delta^1 : [0] \rightarrow [1]$ means that the assignment of arrows respects the source and the target. Compatibility with respect to σ^0 means that F_1 carries identity to identity. Let us compare the rest of the data. We claim that, first of all, the rest of $F_n : N_n(C) \rightarrow N_n(D)$ are (at most) uniquely defined, and, second, that in order to define the whole F , one has to require that F_1 carries composition of arrows to a composition of arrows in D . To do so, let us mention the following property of simplicial sets $N(C)$.

Denote $\text{Sp}(n)$ (the spine of Δ^n) the simplicial subset of Δ^n spanned by non-degenerate one-simplices $[1] \rightarrow [n]$ carrying 0 to i and 1 to $i + 1$. One can easily see that

$$\text{Sp}(n) = \Delta^1 \sqcup^{\Delta^0} \Delta^1 \sqcup^{\Delta^0} \dots \sqcup^{\Delta^0} \Delta^1,$$

n copies of Δ^1 with identified consecutive ends.

2.1.1. Lemma. *The embedding $\text{Sp}(n) \rightarrow \Delta^n$ induces a bijection*

$$N_n(C) = \text{Hom}(\Delta^n, N(C)) \rightarrow \text{Hom}(\text{Sp}(n), N(C)).$$

□

Actually, the converse is also true (prove this!).

Thus, once we have a compatible pair of maps F_0, F_1 , this uniquely determines a map F_n . It remains to verify when is this map compatible with all faces and degeneracies. One can immediately see that it is always compatible with the degeneracies; and that the only condition to verify is the compatibility with $\delta^1 : [1] \rightarrow [2]$. This is the map carrying 0 to 0 and 1 to 2. Compatibility with δ^1 precisely means that the assignment of arrows is compatible with compositions.

This was a detailed explanation of the following

2.1.2. Proposition. *The functor $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$ is fully faithful.*

We actually achieved much more. We identified the essential image of N : a simplicial set X is a nerve of a category iff the canonical maps

$$X_n \rightarrow \text{Hom}(\text{Sp}(n), X) = X_1 \times_{X_0} \dots \times_{X_0} X_1$$

is a bijection. This characterization is worth remembering as it is the basis of at least two different models of infinity categories.

2.2. Advertisement. Infinity categories in this course are what is sometimes called $(\infty, 1)$ -categories. Approximately, this means that all higher arrows (arrows between arrows) are invertible, in a certain homotopy sense. Otherwise, this can be expressed by saying that, for any pair of objects x and y , the category of all arrows $x \rightarrow y$ is an infinity groupoid.

In this course we will discuss three models for such categories: quasicategories (simplicial sets satisfying certain properties), simplicial categories (that is, categories enriched over simplicial sets), and complete Segal spaces (CSS) which are bisimplicial sets satisfying certain properties.

The logic in all existing models is as follows. What is really described is a certain category with a model structure in the sense of Quillen (this will be addressed later on). One can prove (we do not intend to give complete proofs) that the model categories corresponding to different models are Quillen equivalent (the notion will be made precise later; an example of Quillen equivalence is the adjoint pair of functors between topological spaces and simplicial sets). In each approach a construction is provided, assigning an infinity category to a model category, so that Quillen equivalent model categories give rise to equivalent infinity categories (of course, the notion of equivalence between infinity categories is provided as well). All this put together should persuade us that there is an idea of infinity category of which the concrete descriptions are but realizations.

Our aim will not be to study as many models of infinity categories as possible; but to try to understand this idea of infinity category.

2.3. Quasicategory and its homotopy category. Combinatorics of simplicial sets is well-developed and well-known. Thus, the idea of quasicategory is just to slightly weaken the property which characterizes nerves of categories among the simplicial sets.

Quasicategories were first introduced by Boardman and Vogt [BV] under the name of *weak Kan complexes*, then re-introduced by A. Joyal as models for infinity categories, and vastly developed by Jacob Lurie (see [L.T], Chapter 1).

2.3.1. Nerve of a category. Reformulation. Recall some standard simplicial sets and maps in \mathbf{sSet} . Fix a pair of numbers $0 \leq k \leq n$. We define k -th horn $j_k^n : \Lambda_k^n \rightarrow \Delta^n$ as the embedding of Λ_k^n , the simplicial subset of Δ^n containing all simplices but two, $\text{id} : [n] \rightarrow [n]$ and $\partial^k : [n-1] \rightarrow [n]$, into Δ^n .

Recall that a simplicial set X is called Kan if it has lifting property with respect to all j_k^n , that is, if any map $\Lambda_k^n \rightarrow X$ extends to a map $\Delta^n \rightarrow X$.

Lemma. *A simplicial set X is a nerve of a category iff it satisfies the following property.*

Any horn $j_k^n : \Lambda_k^n \rightarrow \Delta^n$ with $k \neq 0, n$, gives rise to a bijection

$$(6) \quad X_n = \text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda_k^n, X).$$

Proof. Exercise. □

2.3.2. *Quasicategory.* A simplicial set X is called *quasicategory* if the maps j_k^n , $k \neq 0, n$, give rise to a surjective map (6).

This notion obviously generalizes both categories and Kan simplicial sets.

Remark. Let X be a quasicategory. We interpret the elements of X_0 as the objects and the elements of X_1 as the arrows with source obtained by application of d_1 and target by d_0 .

What about the composition? Commutative triangles are given by the elements of X_2 . Each one has three 1-faces: two arrows and their composition. The axiom of quasicategory described by the embedding j_1^2 , ensures that any composable pair of arrows can be composed (but in a non-unique way). The rest of the axioms should mean that composition is unique "up to higher homotopy".

Non-uniqueness of composition in quasicategory faithfully reflects our intuition we acquired trying to define an infinity version of Poincaré groupoid. Composition of the paths essentially depends of the reparametrization of the length two segment, and there is no preferred way of reparametrization — none of them leads to a strictly associative multiplication.

2.3.3. *Mapping spaces.* Let \mathcal{C} be a quasicategory. Define for $x, y \in \mathcal{C}$ a space $\text{Map}(x, y)$. There are different definitions yielding the same object up to homotopy. We will present one of possible definition, denoted $\text{Hom}^R(x, y)$ (R stands for *right*⁹).

We define $\text{Hom}^R(x, y)$ as a simplicial set whose n -simplices are $n+1$ -simplices h in \mathcal{C} such that $d_{n+1}h = x$ (more precisely, the degenerate n -simplex obtained from x), and $d_0 \dots d_n(h) = y$. Faces and degeneracies are defined in an obvious way. Otherwise, we construct a cosimplicial object in \mathbf{sSet} H_R whose n -th component is the colimit $H_R^n = \Delta^{n+1} \coprod^{\Delta^n} \Delta^0$, where the map $\Delta^n \rightarrow \Delta^{n+1}$ is given by δ^{n+1} . The simplicial set H_R^n has two objects coming from 0 and $n+1$ of Δ^{n+1} .

Now $\text{Hom}_{\mathcal{C}}^R(x, y)_n$ is the set of maps $H_R^n \rightarrow \mathcal{C}$ carrying 0 to x and $n+1$ to y .

One can define $\text{Hom}_{\mathcal{C}}^L(x, y)$ using the cosimplicial object H_L in \mathbf{sSet} , instead of H_R , defined via δ^0 instead of δ^{n+1} .

Later on we will see that one can define a composition

$$(7) \quad \text{Hom}^R(y, z) \times \text{Hom}^R(x, y) \rightarrow \text{Hom}^R(x, z),$$

defined uniquely up to homotopy. We will also see that $\text{Hom}^L(x, y)$ is homotopically equivalent to $\text{Hom}^R(x, y)$.

2.3.4. Homotopy category $\text{Ho}(\mathcal{C})$ of a quasicategory \mathcal{C} can be defined as follows. The objects are just the elements of \mathcal{C}_0 . Morphisms are defined by the formula

$$\text{Hom}_{\text{Ho}(\mathcal{C})}(x, y) = \pi_0(\text{Hom}_{\mathcal{C}}^R(x, y)).$$

⁹as opposed to *left*, not to *wrong*!

The composition is defined by the composition of Hom^R . Since we give no explicit formula for (7), it is worthwhile to give another construction of $\text{Ho } \mathcal{C}$.

Here it is. We define $\text{Ob Ho}(\mathcal{C}) = \mathcal{C}_0$ and we define $\text{Hom}_{\text{Ho}(\mathcal{C})}(x, y)$ as the set of equivalence classes of elements $f \in \mathcal{C}_1$ with $d_1 f = x$, $d_0 f = y$. Two elements $f, g \in \mathcal{C}_1$ are equivalent if there exists $u \in \mathcal{C}_2$ such that $d_0 u = f$, $d_1 u = g$, $d_2 u = s_0(d_1 d_2(u))$, see the picture below.

(8)

$$\begin{array}{ccc} & 1 & \\ \text{id} \nearrow & & \searrow f \\ 0 & \xrightarrow{g} & 2 \end{array}$$

One can verify that the above formula defines an equivalence relation if \mathcal{C} is a quasicategory. Now, if f and g is a composable pair of arrows, there exists $u \in \mathcal{C}_2$ such that $f = d_2 u$, $g = d_0 u$. Then we define the composition gf as $d_1(u)$. One can verify that the equivalence class of gf so defined depends only on equivalence classes of f and g .

The notion of homotopy category can be defined for a general simplicial set; but in general the description is less explicit.

2.3.5. *Homotopy category.* One has an adjoint pair $h : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : N$, where N is the nerve functor and h is defined by universal property. Let us study a map from a simplicial set X to the nerve of a category C . It consists of

- a map $f : X_0 \rightarrow \text{Ob}(C)$.
- an assignment for $a \in X_1$ of $f(a) \in \text{Hom}_C(f(d_1 a), d_0 a)$.
- such that for any $x \in X_2$ a cocycle condition is satisfied.

This description gives immediately a definition of the functor h : $h(X)$ is defined as the category with the set of objects X_0 , morphisms given by generators in X_1 and relations in X_2 .

Theorem. *If X is a quasicategory, $h(X)$ is canonically isomorphic to $\text{Ho}(X)$.*

Proof. We will construct a map from \mathcal{C} to the nerve of $\text{Ho}(\mathcal{C})$. This will define by adjunction a functor $h\mathcal{C} \rightarrow \text{Ho}(\mathcal{C})$.

Our map is identity on objects and it carries each $f \in \mathcal{C}_1$ to the respective equivalence class. It remains to prove that any $u \in \mathcal{C}_2$ gives rise to an equality

$$d_0(u) \circ d_2(u) = d_1(u).$$

this is obvious, once we know that the construction of $\text{Ho}(\mathcal{C})$ was sound. Let us show the constructed map is an isomorphism of categories. First, the arrows in $h(\mathcal{C})$ are generated by the elements of \mathcal{C}_1 . Any composition of elements of \mathcal{C}_1 can be clearly presented by a single arrow — this follows by induction and by lifting with respect to j_1^2 . This easily implies the claim. \square

Note that we have not really verified that the relation defining $\text{Ho}(\mathcal{C})$ is an equivalence and the composition is properly defined.

2.3.6. Definition. An arrow in a quasicategory is called an equivalence if its image in the homotopy category is an isomorphism.

In a conventional category composition of two isomorphism is an isomorphism. The quasicategorical version reads: if two edges of $u \in \mathcal{C}_2$ are equivalences, then so is the third.

2.3.7. Kan simplicial sets as infinity groupoids.

Theorem. *The following properties of a quasicategory X are equivalent.*

1. All arrows of X are equivalences.
2. $\text{Ho}(X)$ is a groupoid.
3. X satisfies RLP with respect to i -horns, $i < n$.
4. X satisfies RLP with respect to i -horns, $i > 0$.
5. X is Kan simplicial set.

Proof. One has very easy implications

$$3, 4 \Rightarrow 5 \Rightarrow 2 \Leftrightarrow 1.$$

The remaining implication $2 \Rightarrow 3$ is less trivial, requires a theory of left fibrations. It was proven by A. Joyal. □

2.3.8. Corollary. *Let \mathcal{C} be a quasicategory. Then $\text{Hom}_{\mathcal{C}}^R(x, y)$ is Kan for all x, y .*

Proof. It is clear that $\text{Hom}_{\mathcal{C}}^R(x, y)$ satisfies the lifting property with respect to j_i^n , for $i \neq 0$. This already implies the claim. □

2.3.9. Maximal subspace. Another consequence of Theorem 2.3.7 is the existence of a maximal Kan subset of any quasicategory.

Corollary. *For a quasicategory \mathcal{C} let \mathcal{C}^{eq} denote the simplicial subset consisting of simplices whose all 1-faces are equivalences. Then \mathcal{C}^{eq} is the maximal Kan subset in \mathcal{C} .*

Remark. In the quasicategorical approach to infinity categories Kan simplicial sets are seen as “infinity groupoids” or $(\infty, 0)$ -categories. Maximal Kan subset is, therefore, the $(\infty, 0)$ -category obtained from $(\infty, 1)$ category \mathcal{C} by discarding all non-invertible 1-arrows.

2.4. Simplicial categories. A simplest model for $(\infty, 1)$ categories is the one based on categories enriched over topological spaces (or, what is more or less the same), over Kan simplicial sets.

A pleasant property of this model is that composition in it is strictly associative. Another pleasant property is that a lot of interesting infinity categories can be constructed in this way — giving an explicit construction of a simplicial category. Inconvenience of this model is due to difficulty of describing functors $\mathcal{C} \rightarrow \mathcal{D}$ — they can be presented in general by a diagram $\mathcal{C} \leftarrow \tilde{\mathcal{C}} \rightarrow \mathcal{D}$.

2.4.1. Enriched categories. We will define them in the most simple case. Let M be a category with finite products. An M -enriched category C consists of a class of objects $\text{Ob}(C)$; of the objects $\text{Map}_{\mathcal{C}}(x, y) \in M$ for each pair $x, y \in \text{Ob}(C)$; of morphisms $\text{id}_x : * \rightarrow \text{Map}_{\mathcal{C}}(x, x)$ from the terminal object $* \in M$ to the endomorphism object of x for all x ; of strictly associative compositions $\text{Map}_{\mathcal{C}}(y, z) \times \text{Map}_{\mathcal{C}}(x, y) \rightarrow \text{Map}_{\mathcal{C}}(x, z)$, so that id_x is both left and right unit.

We are especially interested in simplicially enriched categories which we will call just *simplicial categories*. They form the category which we denote \mathbf{sCat} .

2.4.2. Homotopy category. Given $\mathcal{C} \in \mathbf{sCat}$, we define $\text{Ho}(\mathcal{C})$ in a way similar (but easier) to the definition for quasicategories.

$\text{Ho}(C)$ has the same objects as \mathcal{C} ; $\text{Hom}_{\text{Ho}(\mathcal{C})}(x, y) = \pi_0(\text{Map}_{\mathcal{C}}(x, y))$.

2.4.3. Simplicial groupoids. Recall that any quasicategory contains a maximal Kan simplicial subset.

We would like to better understand what is the analog of this claim in the context of simplicial categories.

The first step is clear: an arrow $f \in \text{Map}_{\mathcal{C}}(x, y)_0$ should be called equivalence if its image in the homotopy category is invertible. In other words, this means that there exists an arrow $g \in \text{Map}_{\mathcal{C}}(y, x)_0$ such that fg belongs to the same connected component of $\text{Map}_{\mathcal{C}}(y, y)$ as id_y , and similarly for gf . In the case when \mathcal{C} has Kan Map-spaces, the condition on the compositions just means that there are $u \in \text{Map}_{\mathcal{C}}(x, x)$ and $v \in \text{Map}_{\mathcal{C}}(y, y)$ such that

$$d_1 u = \text{id}_x, \quad d_0 u = gf, \quad d_1 v = \text{id}_y, \quad d_0 v = fg.$$

The simplicial category satisfying the above property is called *the simplicial groupoid*. The maximal subspace of $\mathcal{C} \in \mathbf{sCat}$ is therefore the simplicial groupoid with the same objects as \mathcal{C} and including those components of $\text{Map}_{\mathcal{C}}(x, y)$ that consist of equivalences. It is not immediately obvious why such creatures deserve to be called “spaces“.

First of all, let us restrict ourselves to connected spaces on one hand, and to connected simplicial groupoids on the other. Any connected simplicial groupoid is equivalent to a simplicial groupoid having one object, that is, to a simplicial monoid with homotopy invertible multiplication. This is slightly more general

than a simplicial group, and the difference turns out inessential. The following classical theorem of J.P. May now concludes the argument.

Theorem. *The loop space functor establishes an equivalence between connected topological spaces and simplicial groups.*

2.4.4. *Standard simplices in \mathbf{sCat} .* We are going to present a remarkable adjoint pair of functors connecting simplicial sets with simplicial categories, the one which will yield later on an equivalence between quasicategories and categories enriched over Kan simplicial sets.

As usual, in order to construct a colimit-preserving functor from simplicial sets to \mathbf{sCat} , we have to present a cosimplicial object in \mathbf{sCat} .

To each finite nonempty totally ordered set J we assign a simplicial category \mathfrak{C}^J as follows.

The objects of \mathfrak{C}^J are just the elements of J .

For $i, j \in J$ we define $\text{Map}_{\mathfrak{C}^J}(i, j)$ as the nerve of the category $P_{i,j}$ defined as follows.

- $P_{i,j} = \emptyset$ if $i > j$.
- $P_{i,j}$ is the poset of subsets of $\{i, i+1, \dots, j\}$ containing both i and j — if $i \leq j$.

It remains to define the composition in \mathfrak{C}^J . Given $I \in P_{i,j}$ and $K \in P_{j,k}$, the composition is defined by the union $I \cup K \in P_{i,k}$.

We will write \mathfrak{C}^n instead of $\mathfrak{C}^{[n]}$ for $J = [n]$.

Let us present what we got for small J .

- $J = [0]$. Then $\mathfrak{C}^0 = *$, the terminal category.
- $J = [1]$. $\mathfrak{C}^1 = [1]$ is the category with two objects and one arrow.
- $J = [2]$. \mathfrak{C}^2 has the objects $0, 1, 2$ and morphisms $f : 0 \rightarrow 1$ and $g : 1 \rightarrow 2$. The space $\text{Map}(0, 2)$ is a segment: it has two 0-simplices, gf and $h : 0 \rightarrow 2$, and a one-simplex $h \rightarrow gf$.

2.4.5. *Adjoint pair of functors.* As always, a cosimplicial object $n \mapsto \mathfrak{C}^n$ in \mathbf{sCat} determines an adjoint pair of functors

$$(9) \quad \mathfrak{C} : \mathbf{sSet} \xrightarrow{\leftarrow} : \mathbf{sCat} : \mathfrak{N}.$$

Here $\mathfrak{N}(\mathfrak{C})_n = \text{Hom}_{\mathbf{sCat}}(\mathfrak{C}^n, \mathfrak{C})$ — is an analog of the nerve functor (the official name — *homotopy coherent nerve*), and the functor \mathfrak{C} is uniquely defined by the property that $\mathfrak{C}(\Delta^n) = \mathfrak{C}^n$ and by the requirement that \mathfrak{C} preserves colimits.

Remark. To make sure the functor \mathfrak{C} exist, we use existence of colimits in \mathbf{sCat} . Colimits in \mathbf{sCat} can be easily expressed via colimits in \mathbf{Cat} . Given a functor $F : I \rightarrow \mathbf{Cat}$, its colimit $C = \text{colim } F(i)$ can be explicitly described as follows. One has $\text{Ob}(C) = \text{colim } \text{Ob}(F(i))$. The set of arrows $\text{Mor}(C)$ is defined in two steps. First of all, let $X = \text{colim } \text{Mor}(F_i)$. One has $s, t : X \rightarrow \text{Ob}(C)$ —

source and target maps coming from the source and target maps in $F(i)$. We define further Y as the set of composable paths built from the elements of X . Finally, $Mor(C)$ is the quotient of Y modulo the equivalence relation generated by commutative triangles in all $F(i)$.

2.4.6. *DK equivalences.* Dwyer and Kan suggest the following, very natural, notion of equivalence of simplicial categories.

Definition. A map $f : \mathcal{C} \rightarrow \mathcal{D}$ (that is, a simplicial functor) is called an equivalence (or Dwyer-Kan, DK, equivalence) if

1. (fully faithful) For $x, y \in \mathcal{C}$ the map $\text{Map}_{\mathcal{C}}(x, y) \rightarrow \text{Map}_{\mathcal{D}}(fx, fy)$ is a weak equivalence.
2. (essentially surjective) For any $z \in \mathcal{D}$ there exists an equivalence $f(x) \rightarrow z$ for some $x \in \mathcal{C}$.

Note that, in case Condition 1 is fulfilled, Condition 2 is equivalent to the following

3. The functor $\text{Ho}(f) : \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{D})$ is an equivalence of categories.

We will mimic the above definition to quasicategories.

Definition. A map $f : \mathcal{C} \rightarrow \mathcal{D}$ of quasicategories is called a DK equivalence if

1. (fully faithful) For $x, y \in \mathcal{C}$ the map $\text{Hom}_{\mathcal{C}}^R(x, y) \rightarrow \text{Hom}_{\mathcal{D}}^R(fx, fy)$ is a weak equivalence.
2. (essentially surjective) For any $z \in \mathcal{D}$ there exists an equivalence $f(x) \rightarrow z$ for some $x \in \mathcal{C}$.

The following theorem is quite difficult. We do not intend to prove it.

2.4.7. **Theorem.** 1. *The functor \mathfrak{N} carries simplicial categories having Kan map spaces, to quasicategories.*

2. *Given a simplicial category \mathcal{C} with Kan map spaces, the canonical map*

$$\mathcal{C} \circ \mathfrak{N}(\mathcal{C}) \rightarrow \mathcal{C}$$

is a DK equivalence.

3. *Given a quasicategory X and an equivalence $\mathcal{C}(X) \rightarrow \mathcal{C}$ where the simplicial category \mathcal{C} has fibrant map spaces, the map $X \rightarrow \mathfrak{N}(Y)$ is an equivalence of quasicategories.*

2.4.8. *Proof of the first claim.* This is an easy part, so it is worthwhile to prove it. Let \mathcal{C} be a simplicial category with Kan map spaces.

We have to verify the lifting property of $\mathfrak{N}(\mathcal{C})$ with respect to j_i^n , $i \neq 0, n$. By adjunction, this is equivalent to lifting property of \mathcal{C} with respect to $\mathcal{C}(j_i^n)$. Thus, we have to better understand how the canonical map $\mathcal{C}(\Lambda_i^n) \rightarrow \mathcal{C}^n$ looks like.

These simplicial categories have the same objects. The maps spaces $\text{Map}(i, j)$, with the only exception $(i, j) = (0, n)$, are the same. Let us compare $\text{Map}_{\mathfrak{C}^n}(0, n)$ with $\text{Map}_{\mathfrak{C}(\Delta^n)}(0, n)$. The former simplicial set is $(\Delta^1)^{n-1}$, whereas the latter is the subspace $\Pi_{i,1}^{n-1}$ we will now define. The $n-1$ -dimensional cube is the nerve of the poset $\{0, 1\}^{n-1}$ whose elements are the vectors (x_1, \dots, x_{n-1}) of numbers 0, 1. This cube has $2n-2$ faces of maximal dimension, given by the equations $x_j = 0$, $x_j = 1$, $j = 1, \dots, n-1$. The subspace $\Pi_{i,1}^{n-1}$ is obtained from the boundary of the cube by removing the face given by the equation $x_i = 1$.

The map of simplicial sets $\Pi_{i,1}^{n-1} \rightarrow (\Delta^1)^{n-1}$ is definitely a weak homotopy equivalence and an injection. According the standard homotopy theory of simplicial sets which we intend to address later, any Kan simplicial set satisfies the lifting property with respect to such map.

2.5. Examples. A lot of interesting quasicategories appear as nerves of simplicial categories. We will give three such examples.

2.5.1. Quasicategory of spaces. The category of Kan simplicial sets is a simplicial category via internal Hom.

Exercise. Prove that $\text{Fun}(X, Y)$ is Kan if Y is Kan.

Thus, we have a simplicial category whose objects correspond to Kan simplicial sets and whose map spaces are Kan. Its nerve is *the quasicategory of spaces*. It is usually denoted \mathcal{S} .

Homotopy coherent nerve was invented in 80-ies in the following context. We want to define diagrams of topological spaces up to a coherent collection of homotopies.

For instance, a functor from Δ^2 to spaces should be defined as a collection of three spaces X_i , $i = 0, 1, 2$, with maps $X_i \rightarrow X_j$ for $i < j$, and with a homotopy connecting the composition $X_0 \rightarrow X_1 \rightarrow X_2$ with the map $X_0 \rightarrow X_2$. In this way one arrives to the definition which is equivalent, in our language, to the following.

Definition. A homotopy coherent functor from a category I to the simplicial sets is a map of simplicial sets $N(I) \rightarrow \mathcal{S}$, where \mathcal{S} is defined as the (homotopy coherent) nerve of the simplicial category of Kan simplicial sets.

2.5.2. Quasicategory of quasicategories. The objects of \mathbf{Cat}_∞ are quasicategories. Note that $\text{Fun}(X, Y)$ is not Kan if X, Y are “only” quasicategories: take, for instance, $X = *$.

One has however the following

Lemma. $\text{Fun}(X, Y)$ is a quasicategory if Y is a quasicategory.

Proof. Here is the idea of the proof. Lifting property of $\text{Fun}(X, Y)$ with respect to the maps j_i^n can be rewritten as the lifting property of Y with respect to the maps $j_i^n \times X : \Delta_i^n \times X \rightarrow \Delta^n \times X$.

We define *weak saturation* of the set of arrows $\{j_i^n\}$ as the collection of arrows which can be obtained by cobase change, composition and retraction.

It is almost immediate that the lifting property with respect to a collection of morphisms extends to its weak saturation. So, it remains to verify that $j_i^n \times X$ belong to the weak saturation. Verification immediately reduces to the case $X = \Delta^m$, whose proof requires some mild combinatoric of simplicial sets. \square

The lemma allows one to define Cat_∞ as the nerve of the following simplicial category.

- Its objects are quasicategories.
- $\text{Map}(X, Y)$ is defined as the maximal Kan subcomplex of $\text{Fun}(X, Y)$.

Remark. The above definition is a curious one. Its construction seems artificial (though it is not). Theoretically, one could suggest something different instead of maximal Kan subset, for instance, a functorial Kan replacement (see later). But it turns out that the definition presented above is the correct one. For conventional categories, it corresponds to the 2-category of categories, whose 1-arrows are the functors, and 2-arrows are isomorphisms of functors.

Moreover, looking on the conventional categories teaches us that if we want to retain all of $\text{Fun}(X, Y)$ as morphisms, we have first to come out with a definition of $(\infty, 2)$ category.

2.5.3. *Functor quasicategories.* The above lemma, especially, when put together with the definition of homotopy coherent diagram of spaces, gives a hope that the assignment $I, X \mapsto \text{Fun}(I, X)$ is all we need to describe the correct notion of quasicategory of functors. This is really so, and this is a huge benefit as compared to \mathbf{sCat} which is too rigid to have enough simplicial functors $I \rightarrow \mathcal{C}$ (unless I is *cofibrant*, see later).

2.5.4. *Quasicategory of complexes.* Let \mathcal{A} be an abelian category, $C(\mathcal{A})$ the category of complexes. There are (at least) two triangulated categories associated to $C(\mathcal{A})$: the one is denoted $K(\mathcal{A})$ (it was originally called “the homotopy category of \mathcal{A} but we now find this name very misleading), and the other $D(\mathcal{A})$ — the derived category.

Both will have infinity categorical counterparts which we will construct as nerves of respective simplicial categories.

The category $C(\mathcal{A})$ is enriched over complexes of abelian groups. We will denote $\mathcal{H}om(X, Y) \in C(\mathbb{Z})$ the respective complex.

Let us now define the simplicial set $\text{Map}(X, Y)$.

- 0-simplices are the $a \in \mathcal{H}om(X, Y)^0$ such that $da = 0$.

- 1-simplices are given by triples $a, b \in \mathcal{H}om(X, Y)^0$, $c \in \mathcal{H}om(X, Y)^{-1}$, such that $da = db = 0$, $dc = b - a$. In general,
- n -simplices are morphisms of complexes $C_*(\Delta^n) \rightarrow \mathcal{H}om(X, Y)$, where C_* is the functor of *normalized chains*.

Now we can define $K_\infty(\mathcal{A})$ as the nerve of the simplicial category described above. $D_\infty^-(\mathcal{A})$ is the nerve of the full simplicial subcategory spanned by the complexes bounded above consisting of projective objects¹⁰

Remark. The passage from $\mathcal{H}om(X, Y) \in C(\mathbb{Z})$ to $\text{Map}(X, Y) \in \mathbf{sSet}$ retains an essential part of information about the complex. In fact, $\text{Map}(X, Y)$ is a simplicial abelian group, and Dold-Kan equivalence says that the functor of normalized chains establishes an equivalence between $\text{Fun}(\Delta^{\text{op}}, \mathbf{Ab})$ and $C^{\leq 0}(\mathbb{Z})$. Thus, simplicial abelian group $\text{Map}(X, Y)$ determines the nonpositive part of $\mathcal{H}om(X, Y)$, more precisely, the complex $\tau^{\leq 0}(\mathcal{H}om(X, Y))$ (one retains the negative components and zero-cocycles).

2.6. Exercises.

1. Let \mathcal{C} be a quasicategory. Prove that the relation on the set of arrows from x to y defined by (8) is an equivalence.
2. Define the functor $\text{op} : \Delta \rightarrow \Delta$ as the one carrying a totally ordered set I to its opposite I^{op} . This functor carries $[n]$ to $[n]$ but d_i to d_{n-i} and s_i to s_{n-i} . Verify that \mathcal{C} is a quasicategory iff \mathcal{C}^{op} is.
3. Prove that the equivalence relations defined by (8) on \mathcal{C} and on \mathcal{C}^{op} are the same, so that $\text{Ho}(\mathcal{C}^{\text{op}}) = \text{Ho}(\mathcal{C})^{\text{op}}$.
4. Prove that any simplicial group (in particular, simplicial abelian group) is Kan. Here *simplicial group* means just a simplicial object in the category of groups.

¹⁰One has to be slightly more careful to describe the unbounded derived category, see later.

3. MODEL CATEGORIES, 1

One of more classical approaches to deal with homotopical information is via model categories suggested by Quillen in late 60-ies.

One remains in the scope of conventional categories, but requires a choice of three types of arrows, weak equivalences, fibrations and cofibrations, satisfying some list of axioms. Importance of choosing special classes of morphisms is a result of experience coming from homological algebra and homotopy theory. Thus, in homological algebra we use resolutions. If we talk in the language of complexes, a projective resolution $P_\bullet \rightarrow M$ of a module M is a *quasiisomorphism of complexes*. Quasiisomorphisms are weak equivalences in complexes (see below). In homotopy theory, homotopy equivalences (and even better, weak homotopy equivalences) play this role. So, weak equivalences in a model category are arrows which we would like to think of as isomorphisms.

The role of fibrations and cofibrations is more subtle (and less crucial). Morally, fibrations are “good surjections” and cofibrations are “good injections”.

Details are below.

3.1. Definitions.

3.1.1. Definition. A model category \mathcal{C} is a category having small limits and small colimits (see 1.5.1)¹¹ with three collections of arrows, W, C, F satisfying a list of axioms presented below.

1. Let f, g be two composable arrows in \mathcal{C} . If two arrows among f, g, gf are in W , that the third is in W .
2. Retract of any arrow in W (resp., F, C) is in W (resp., F, C).
3. Let

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow i & \nearrow \text{dotted} & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

be a commutative diagram of solid arrows, so that $i \in C$ and $p \in F$. Then there exists a dotted arrow if

- (a) if i is in W .
 - (b) if p is in W .
4. Any map $f : X \rightarrow Y$ can be decomposed as
 - (a) $f = pi$, where $p \in F, i \in C \cap W$.
 - (b) $f = pi$, where $p \in F \cap W, i \in C$.

It takes time to grasp this notion.

¹¹Quillen requires existence of finite limits and colimits only. But existence of all (co)limits is very convenient.

The maps in $W \cap C$ are called trivial cofibrations, the maps in $W \cap F$ are trivial fibrations.

Very approximately, the last axiom allows one to construct resolutions, the previous expresses lifting properties of different types of arrows one with respect to the other.

Model category \mathcal{C} has an initial object \emptyset and a terminal object $*$. If the map $\emptyset \rightarrow X$ is a cofibration, X is called cofibrant; if $Y \rightarrow *$ is a fibration, Y is called fibrant.

3.1.2. In the commutative square as in Definition 3.1.1 we say that i satisfies LLP (left lifting property) with respect to p , or that p satisfies RLP wrt i .

3.1.3. **Remark.** One can easily prove that an arrow in \mathcal{C} is a cofibration (resp., trivial cofibration) iff it has the LLP with respect to trivial fibrations (resp., all fibrations). Dually, fibrations and trivial fibrations are defined by the RLP property. This is shown as follows. Let, for instance, $i : A \rightarrow B$ satisfy LLP with respect to all trivial fibrations. Factor $i = qj$ where j is a cofibration and q is a trivial fibration. As i has LLP wrt q , i is a retract of j . Thus, i is a cofibration.

We will discuss later the notion of homotopy category of \mathcal{C} , $\text{Ho}(\mathcal{C})$ defined either as a localization of \mathcal{C} with respect to W , or as the category whose objects are fibrant cofibrant objects of \mathcal{C} and morphisms equivalence classes of morphisms of \mathcal{C} . But we prefer to start with an easy and very meaningful example.

3.2. Example: category of complexes of modules.

3.2.1. *Projective model structure.* Fix k a ring. Let $C(k)$ be the category of unbounded complexes of k -modules. We will define an interesting model category structure on $C(k)$.

- Weak equivalences are quasiisomorphisms of complexes.
- Fibrations are the surjective maps of complexes.
- Cofibrations are defined by the LLP with respect to the trivial fibrations.

3.2.2. *Joining a variable.* Given a complex X and $x \in X^n$ such that $dx = 0$, we define $X\langle u; du = x \rangle$ as graded k -module defined as $X \oplus k \cdot u$ with the differential given by $du = x$.

Exercise. $X \rightarrow X\langle u \rangle$ is a cofibration.

3.2.3. *Standard cofibrations.* They are defined as sequential colimits of maps obtained by adding a set of variables.

Exercise. Verify that a sequential colimit of cofibrations a cofibration.

3.2.4. *Proof.* We have to prove axioms 3 and 4. Note that 3(b) is immediate.

It is also easy to prove 4(a). Given $f : X \rightarrow Y$, we should factor f as trivial cofibration followed by a surjective map. This is really easy: We define $X' = X \oplus \bigoplus T_y$ where T_y is a complex $k \xrightarrow{\text{id}} k$ concentrated in degrees $|y|$ and $|y| + 1$ and the map $T_y \rightarrow Y$ carries the generator in degree $|d|$ to y and the one in degree $|d| + 1$ to dy .

Let us prove 4(b). This is a generalization of the way we construct a free resolution of a module.

Given a map $f : X \rightarrow Y$, we want to decompose it into a standard cofibration followed by a trivial fibration.

Step 0. Using 4(a), we can assume f is already surjective.

Step 1. Adding a set of cycles, make sure the map is surjective on the set of cycles.

Step 2. We put $X_0 = X$ and $f_0 = f$. Recursively, assuming that $f_n : X_n \rightarrow Y$ is constructed, so that f_n is surjective and $Z(f_n) : Z(X_n) \rightarrow Z(Y)$ is surjective, construct a decomposition of f_n as

$$X_n \rightarrow X_{n+1} \rightarrow Y,$$

where X_{n+1} is obtained by joining to X_n variables u_α killing all cycles x_α in X_n whose image in Y is a boundary.

It is easy (but instructive) to understand that the map $\text{colim } X_n \rightarrow Y$ is a quasiisomorphism.

Thus, we presented any map of complexes as a composition of standard cofibration and trivial fibration.

It remains to prove condition 3(a), that trivial cofibrations have RLP with respect to fibrations. By the proof of 4(a) any weak equivalence f decomposes $f = pi$ where p is a trivial fibration and i is a standard trivial cofibration. If f is also a cofibration, condition 3(b) gives the existence of j such that $pj = \text{id}$, $i = jf$. This proves that any trivial cofibration is a retract of a standard one; since standard trivial cofibrations satisfy LLP wrt fibrations, we got what we needed.

- 3.2.5. **Exercise.**
1. Prove that in the above model structure any cofibration is a retract of a standard cofibration.
 2. Prove that if X is cofibrant, X^n are projective k -modules.
 3. For $k = \mathbb{Z}/4\mathbb{Z}$ let X be defined as follows.

$$X^n = k, \quad d(x) = 2x \quad \forall x \in X^n.$$

Prove X is not cofibrant (even though all X^n are free).

3.2.6. *Injective model structure.* We see that the model category structure presented above generalizes very nicely everything in homological algebra that requires projective resolution. But what about injective resolutions? They appear in another model category structure called *injective model structure*.

We will not present the full details; here are the definition.

- A map of complexes is a weak equivalence iff it is a quasiisomorphism.
- A map is a cofibration if it is componentwise injective.
- A map is a fibration if it satisfies RLP wrt all trivial cofibrations.

It is a good exercise to verify this is a model structure. See [Ho], 2.3.13, for the detailed proof.

I would like to stress that the two model structures described above are different, but “morally” define the same infinity categorical object: they have the same weak equivalences. We will discuss this later.

3.2.7. *Projective model structure: generalizations.* Model category structure described in 3.2.1 can be extended to some other contexts.

Let k be a commutative ring. Let \mathcal{C} be one of the following categories.

- The category of associative dg algebras over k .
- The category of commutative or Lie (or any other operad) dg algebras over k , when $k \supset \mathbb{Q}$.

Then \mathcal{C} has a model structure, with quasisomorphisms as weak equivalences and componentwise surjective maps as fibrations.

The proof is basically the same as for the complexes.

3.3. Homotopy category, derived functors. Homotopy category of a model category is defined by localization, the procedure well-known in commutative algebra but making sense for categories as well. Similarly to topological spaces, one has, apart of the notion of weak equivalence, a notion of homotopy.

3.3.1. *Homotopy between arrows.* First of all, the notion of “cylinder”. For $X \in \mathcal{C}$ its cylinder is a diagram

$$X \sqcup X \rightarrow C_X \rightarrow X$$

where the composition is the obvious one, the first map is a cofibration and the second map is a trivial fibration. Note that in this definition a cylinder diagram of X is not unique.

Given a pair of maps $f_0, f_1 : X \rightarrow Y$, we can convert them into one map $f : X \sqcup X \rightarrow Y$. We will say that f_0 and f_1 are left homotopic if f can be factored through a certain cylinder C_X .

These notions are not very convenient to work with; for instance, if f_i are left homotopic then the compositions gf_i are also left homotopic, but not obviously $f_i g$ are.

Also, one can (dually) define another type of homotopy, using path objects instead of cylinders: for $Y \in \mathcal{C}$ the diagram

$$Y \rightarrow P_Y \rightarrow Y \times Y$$

decomposing the diagonal map to trivial cofibration followed by a fibration, is called a path object for Y .

In general, left and right homotopy do not even define equivalence relation. Left homotopy is an equivalence relation for arrows $X \rightarrow Y$ with cofibrant X ; right homotopy is equivalence for $X \rightarrow Y$ with fibrant Y .

If both X is cofibrant and Y fibrant, left and right homotopy relations on $\text{Hom}_{\mathcal{C}}(X, Y)$ coincide.

3.3.2. Localization of a category. Let \mathcal{C} be a category, W a collection of arrows in \mathcal{C} . One defines a localization $\mathcal{C}[W^{-1}]$ by a universal property.

There are two versions, a naive one, in terms of the “naive” category of categories (the one consisting of categories and functors, without morphisms of functors), and a more sophisticated one, taking into account isomorphisms of functors.

The first definition looks like this.

Definition (naive). Let \mathcal{C} be a category and W a collection of arrows in \mathcal{C} . A localization of \mathcal{C} with respect to W is a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ carrying all arrows of W to isomorphisms, satisfying the following universal property

For any functor $f' : \mathcal{C} \rightarrow \mathcal{D}'$ carrying W to isomorphisms, there exists a unique functor $g : \mathcal{D} \rightarrow \mathcal{D}'$ such that $f' = g \circ f$.

And here is the second definition.

Definition. Let \mathcal{C} be a category and W a collection of arrows in \mathcal{C} . A localization of \mathcal{C} with respect to W is a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ carrying all arrows of W to isomorphisms, satisfying the following universal property

1. *For any functor $f' : \mathcal{C} \rightarrow \mathcal{D}'$ carrying W to isomorphisms, there exists a pair (g, θ) with a functor $g : \mathcal{D} \rightarrow \mathcal{D}'$ and an isomorphism of functors*

$$\theta : f' \rightarrow g \circ f.$$

2. *If (g', θ') is another pair as above, there is a unique isomorphism $\alpha : g \rightarrow g'$ intertwining θ and θ' .*

3.3.3. Facts to think about. Exercises.

- Localization in the first sense, if it exists, is unique up to a unique isomorphism.
- Localization in the second case, if it exists, is unique up to an equivalence that is unique up to a unique isomorphism.
- Localization in the first sense, if it exists, is also localization in the second sense.

- Localization in the first sense exists, at least if \mathcal{C} is small.

Thus, we see that localization of categories always exist, up to set-theoretical subtleties (Hom-sets may be big if we localize a category which is not small). It is worthwhile to understand that even for small categories the second notion of localization is “more correct”.

Localization of categories always exists, but it does not always have an meaningful explicit description. Similarly to localization of associative rings, it is useful to have Ore condition which allows to rewrite $s^{-1}x$ as yt^{-1} for some y, t . One has a similar Ore condition for categories ¹², but, unfortunately, it is fulfilled less often than we would expect.

3.3.4. Homotopy category of a model category. Let \mathcal{C} be a model category, with the collections W, C, F of weak equivalences, cofibrations and fibrations.

Homotopy category of \mathcal{C} , $\text{Ho } \mathcal{C}$, is defined as the localization $\mathcal{C}[W^{-1}]$.

The same localization has other presentations, up to equivalence.

Theorem. *Let \mathcal{C} be as above, \mathcal{C}^c the full subcategory of cofibrant objects in \mathcal{C} , \mathcal{C}^f the category of fibrant object, $\mathcal{C}^{cf} = \mathcal{C}^c \cap \mathcal{C}^f$.*

- *The embeddings $\mathcal{C}^* \rightarrow \mathcal{C}$, for $* = c, f, cf$, induce an equivalence of localizations $\mathcal{C}^*[(W \cap \mathcal{C}^*)^{-1}] \rightarrow \mathcal{C}[W^{-1}]$.*
- *Denote $\bar{\mathcal{C}}^*$ the category with the same objects as \mathcal{C}^* and with homotopy classes of arrows as morphisms (with respective notion of homotopy, right for \mathcal{C}^c and left for \mathcal{C}^f). Denote \bar{W}^* the image of the weak equivalences in \mathcal{C}^* . Then $\bar{\mathcal{C}}^c$ satisfies left Ore condition, $\bar{\mathcal{C}}^f$ satisfies right Ore condition, and \bar{W}^{cf} consists of isomorphisms.*

Note that the above result is very similar to the one describing the derived category (example of derived category was one of Quillen’s motivations). One can describe the derived category $D(\mathcal{A})$ of an abelian category \mathcal{A} as the localization of the category of complexes $C(\mathcal{A})$ with respect to quasiisomorphisms; but the collection of quasiisomorphisms does not satisfy Ore conditions. One can do the construction in two steps: identifying homotopic arrows (the resulting category is traditionally denoted $K(\mathcal{A})$) and then localizing. Once one identifies the homotopic arrows, Ore condition holds. Finally, derived category $D^-(\mathcal{A})$ can be also realized as a full subcategory of $K(\mathcal{A})$.

3.4. Quillen adjunction, Quillen equivalence. One does homological algebra to study derived functors. Derived functors are functors between localized categories approximating functors between the original categories.

¹²Google “calculus of fractions” which is synonymous to “Ore condition”.

3.4.1. *Generalities.* Here is, probably, the most general idea of derived functor. Let (C, W) , (C', W') be categories with weak equivalences, and let $F : C \rightarrow C'$ be a functor. It is seldom possible to complete the diagram

$$(10) \quad \begin{array}{ccc} C & \xrightarrow{F} & C' \\ \downarrow Q & & \downarrow Q \\ C[W^{-1}] & \xrightarrow{\mathbf{L}F} & C'[W'^{-1}] \end{array}$$

to be commutative. One can expect, however, to find a universal pair $(\mathbf{L}F, \alpha)$, where $\alpha : \mathbf{L}F \circ Q \rightarrow Q \circ F$ is a natural transformation (not necessarily an isomorphism). Such pair is called *left derived functor of F* . Universality means that for any pair (G, β) consisting of a functor $G : C[W^{-1}] \rightarrow C'[W'^{-1}]$ and a natural transformation $\beta : G \circ Q \rightarrow Q \circ F$ there exists a unique morphism of functors $u : G \rightarrow \mathbf{L}F$ such that $\beta = \alpha \circ (u \circ Q)$.

Right derived functors are defined similarly, as pairs $(\mathbf{R}F, \alpha)$, where $\alpha : Q \circ F \rightarrow \mathbf{R}F \circ Q$ satisfies a universal property.

Remark. Mathematics is not a deductive science. There exist different ideas of what derived functor should be. My favorite example of a derived functor, assigning to a dg algebra A the dg Lie algebra of derivations of its cofibrant replacement, is not even a functor.

3.4.2. *Quillen adjunction.* Quillen suggests the following context in which derived functors make sense and can be effectively calculated.

Definition. Left $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair of functors between two model categories. It is called a Quillen pair (or a Quillen adjunction) if the following equivalent conditions are fulfilled.

- F preserves cofibrations and trivial cofibrations.
- G preserves fibrations and trivial fibrations.

Note that for adjoint pair of model categories $F(f)$ has a LLP with respect to g iff f has a LLP with respect to $G(g)$. This proves the equivalence of the above conditions.

3.4.3. **Theorem.** *A Quillen pair*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

induces an adjoint pair of the homotopy categories

$$\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbf{R}G.$$

Proof. Let us first of all verify that F preserves weak equivalence of cofibrant objects. If $f : A \rightarrow B$ is such, decompose the map $(f, \mathrm{id}) : A \sqcup B \rightarrow B$ as

$$A \sqcup B \xrightarrow{p} C \xrightarrow{q} B,$$

a cofibration followed by a trivial fibration. The maps $i : A \rightarrow A \sqcup B$ and $j : B \rightarrow A \sqcup B$ are cofibrations, and the compositions pi and pj are weak equivalences and cofibrations, therefore, trivial cofibrations. Thus, F carries them to weak equivalences. $F(q)$ is a weak equivalence since the composition $F(qpj) = \text{id}$ is. Therefore, $F(f) = F(qpi)$ is.

Dually, G carries weak equivalences of fibrant objects in \mathcal{D} to weak equivalences.

The now idea is to realize $\text{Ho}(\mathcal{C})$ as the localization $\bar{\mathcal{C}}^c[W^{-1}]$, and $\text{Ho}(\mathcal{D})$ as $\bar{\mathcal{D}}^f[W^{-1}]$. One easily sees that for $c \in \mathcal{C}^c$ and $d \in \mathcal{D}^f$ one has an isomorphism

$$\text{Hom}_{\bar{\mathcal{C}}}(c, G(d)) = \text{Hom}_{\bar{\mathcal{D}}}(F(c), d)$$

which implies the assertion. □

3.4.4. Definition. A Quillen pair as above is called Quillen equivalence if one of the following equivalent conditions is fulfilled.

- For any cofibrant X in \mathcal{C} and fibrant Y in \mathcal{D} a map $a : X \rightarrow G(Y)$ is a weak equivalence iff $a' : F(x) \rightarrow Y$ is a weak equivalence.
- The induced adjunction of the homotopy categories is an equivalence.

Note that even though Quillen adjunction is an adjunction, Quillen equivalence is NOT an equivalence!

3.5. Exercises.

1. See 3.2.2.
2. See 3.2.3.
3. See 3.2.5.
4. See 3.3.3.
5. Prove that two mentioned above model category structures on complexes are Quillen equivalent.

The topic of the present lecture is discussed in much more detail in the books [Ho, Q] (both — Chapter 1).

4. MODEL CATEGORIES, 2: TOPOLOGICAL SPACES AND SIMPLICIAL SETS

4.1. Topological spaces. The most famous example of two Quillen equivalent model structures are provided by simplicial sets versus topological spaces. Once we describe these structures, and once we prove they are Quillen equivalent, we will be able to make sense of Kan's idea that as far as homotopy theory is concerned, topological spaces are the same as simplicial sets.

Recall the adjunction

$$| \quad | : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathbf{Sing}.$$

4.1.1. Simplicial enrichment of \mathbf{sSet} and \mathbf{Top} . For X, Y simplicial sets, one defines X^Y as the simplicial set representing the functor

$$Z \mapsto \mathrm{Hom}(Z \times Y, X).$$

Existence of X^Y is quite obvious, this is just the simplicial set whose n -simplices are the maps $\Delta^n \times Y \rightarrow X$, with the faces and the degeneracies that can be easily described. Similarly, for a simplicial set S and for a topological space X we define X^S as the topological space representing the functor

$$Z \mapsto \mathrm{Hom}(|S| \times Z, X).$$

The above functor is representable, though this fact is less obvious than the previous one. Representability follows from local compactness of the geometric realization of a simplicial set, see details in <https://ncatlab.org/nlab/show/exponential+law+for+spaces>.

The latter formula allows one to define simplicial enrichment for topological spaces, by the formula

$$\mathrm{Fun}(X, Y)_n = \mathrm{Hom}(X, Y^{\Delta^n}).$$

4.1.2. Some standard maps in \mathbf{sSet} . Let us remind that $\Delta^n \in \mathbf{sSet}$ is the simplicial set represented by $[n] \in \Delta$. Recall that the boundary $\partial\Delta^n$ is the simplicial subset of Δ^n whose k -simplices $s : \Delta^k \rightarrow \Delta^n$ are not surjective maps. In other words, $\partial\Delta^n$ is the union of the proper faces of Δ^n .

Exercise: Present $\partial\Delta^n$ as the colimit of a diagram consisting of $(n - 1)$ and $(n - 2)$ -dimensional simplices.

Another important simplicial subset of Δ^n is its k -th horn denoted Λ_k^n , $k \in \{0, \dots, n\}$. It consists of all nondegenerate simplices of Δ^n apart of $\mathrm{id}_{[n]}$ and of ∂^k . We have the maps $i^n : \partial\Delta^n \rightarrow \Delta^n$ and $j_k^n : \Lambda_k^n \rightarrow \Delta^n$, see 2.3.1.

4.1.3. Fibrations in \mathbf{sSet} . A map in \mathbf{sSet} is called fibration if it satisfies RLP with respect to all j_k^n . A map is called trivial fibration if it satisfies RLP with respect to all i^n .

Note that any trivial fibration is a fibration as any j_k^n can be presented as a composition of two maps obtained by cobase change from i^m .

Note that we have not yet defined weak equivalences for \mathbf{sSet} , so that we are not obliged to verify that trivial fibrations are precisely fibrations that are weak equivalences.

4.1.4. *Fibrations in Top.* They are called Serre fibrations: these are maps satisfying RLP with respect to all maps $D^n \rightarrow D^n \times I$, where D^n is the standard n -disc and $I = [0, 1]$.

4.1.5. *Model structure on Top.* Weak equivalences in \mathbf{Top} are, by definition, weak homotopy equivalences. Trivial fibrations are, by definition, fibrations that are weak equivalences.

Our aim is to prove

Theorem. *Top has a model structure defined by weak homotopy equivalences and Serre fibrations.*

The proof is presented below.

4.1.6. **Lemma.** *The following properties of f in \mathbf{Top} are equivalent.*

1. f is a fibration.
2. $\text{Sing}(f)$ is a fibration.
3. f has RLP with respect to $|j_k^n|$.

Proof. (2) is equivalent to (3) by adjunction.

(1) is equivalent to (3) as $|j_k^n|$ is homeomorphic to $D^{n-1} \rightarrow D^{n-1} \times I$. \square

Note also

4.1.7. **Lemma.** *In \mathbf{Top} all objects are fibrant.*

Proof. This is because the projection $D^n \times I \rightarrow D^n$ has a section. \square

We will now show how useful is the simplicial enrichment.

First of all, let us introduce a useful notation. For $f : X \rightarrow Y$ in \mathbf{Top} and $a : A \rightarrow B$ in \mathbf{sSet} we define

$$\Phi(f, a) : X^B \rightarrow X^A \times_{Y^A} Y^B,$$

the canonical map deduced from the commutative diagram

$$\begin{array}{ccc} X^B & \longrightarrow & X^A \\ \downarrow & & \downarrow \\ Y^B & \longrightarrow & Y^A \end{array} .$$

4.1.8. **Lemma.** *Let $f : X \rightarrow Y$ be a fibration in \mathbf{Top} . Then $\Phi(f, i^m)$ is a fibration.*

Proof. We have to verify RLP with respect to $D^n \times I$. This is equivalent to the RLP property of $f : X \rightarrow Y$ with respect to the map

$$(D^n \times |B|) \coprod^{D^n \times |A|} (D^n \times I \times |A|) \rightarrow D^n \times I \times |B|.$$

Let us try to imagine this map. Recall that $|B|$ is m -dimensional ball and $|A|$ is its boundary. In the special case $n = 0$ we have an embedding of “empty bucket” into “full bucket” which is homeomorphic to $D^{m-1} \rightarrow D^{m-1} \times I$. In general, direct product with D^n preserves the coproduct diagram, so the map is homeomorphic to $D^{m+n-1} \rightarrow D^{m+n-1} \times I$. □

One can easily see that if f is a fibration, $\Phi(f, a)$ is a fibration for any injective $a : A \rightarrow B$.

Let Y be path connected, $f : X \rightarrow Y$ be a fibration, $y \in Y$ and $F = f^{-1}(y)$. Choose $x \in F \subset X$. One has a long exact sequence

$$(11) \quad \pi_n(F, x) \rightarrow \pi_n(X, x) \rightarrow \pi_n(Y, y) \rightarrow \pi_{n-1}(F, x) \rightarrow \dots \rightarrow \pi_0(F) \rightarrow \pi_0(X) \rightarrow 0.$$

This immediately implies that a fibration $f : X \rightarrow Y$ is a trivial fibration (that is, a fibration and a weak homotopy equivalence) iff for any $y \in Y$ the homotopy groups of the fiber $F_y = f^{-1}(y)$ are all trivial.

As a result, we immediately get

4.1.9. Lemma. *Base change of a trivial fibration is a trivial fibration.*

Proof. Base change preserves fibrations and has the same fibers. □

4.1.10. Lemma. *In the notation of 4.1.8 Let f be a trivial fibration. Then $\Phi(f, i^m)$ is also a trivial fibration.*

Proof. Let us prove by induction that if f is a trivial fibration, f^S is a trivial fibration for finite $S \in \mathbf{sSet}$. The fiber of f^S is F^S when F is the fiber of f ; thus, we have to verify that if F has trivial homotopy groups, F^S has also trivial homotopy groups. This is easily proven by induction (S is finite!): if T is obtained from S by gluing an n -dimensional simplex, the embedding $S \rightarrow T$ induces a fibration $F^T \rightarrow F^S$ whose fiber $\Omega^n(F)$ has trivial homotopy groups.

Now the claim of the lemma follows by 2 out of 3 property of weak equivalences from the diagram

$$(12) \quad \begin{array}{ccccc} X^B & \longrightarrow & X^A \times_{Y^A} Y^B & \longrightarrow & X^A \\ & \searrow & \downarrow & & \downarrow \\ & & Y^B & \longrightarrow & Y^A \end{array}$$

□

4.1.11. **Lemma.** *The following properties of f in \mathbf{Top} are equivalent.*

1. f is a trivial fibration.
2. $\text{Sing}(f)$ is a trivial fibration.
3. f has RLP with respect to $|i^n|$.

Proof. Conditions (2) and (3) are equivalent by adjunction. Let us prove that (1) implies (3).

Right lifting property of f with respect to $|i^n|$ can be equivalently expressed by the right lifting property of $\Phi(f, i^n)$ with respect to the map $\emptyset \rightarrow *$, that is, simply surjectivity of $\Phi(f, i^n)$. It remains to prove that a trivial fibration is surjective. It is bijective on path connected components; to prove surjectivity it is sufficient to find a point $x \in X$ whose image belongs to the same component as the chosen $y \in Y$; and then to lift the path between $f(x)$ and y to X .

It remains to verify that (3) implies (1). If (3) is satisfied, then f is a fibration. Now, for any $y \in Y$ the fiber $F = f^{-1}(y)$ has trivial homotopy groups as any map $|\partial\Delta^n| \rightarrow F$ extends to a map $|\Delta^n| \rightarrow F$. This completes the proof. \square

4.1.12. *Factorization.* We define cofibrations in \mathbf{Top} as morphisms satisfying LLP with respect to trivial fibrations.

Lemma. *Any map f in \mathbf{Top} can be factored $f = p \circ i$ where i is a cofibration and p is a trivial fibration.*

Proof. Step-by-step, joining cells.

We start with a map $f : X \rightarrow Y$ and we construct a sequence of decompositions

$$X \rightarrow Z_n \rightarrow Z_{n+1} \dots \rightarrow Y$$

recursively. Look at the set of all diagrams

$$(13) \quad \begin{array}{ccc} |\partial\Delta^k| & \longrightarrow & Z_n \\ \downarrow & & \downarrow \\ |\Delta^k| & \longrightarrow & Y \end{array}$$

and define Z_{n+1} by gluing to Z_n all simplices $|\Delta^k|$ along the boundary. We get $Z = \text{colim } Z_n$ and the map $Z \rightarrow Y$ is defined. The map $Z_n \rightarrow Z_{n+1}$ satisfies LLP with respect to the trivial fibrations by Lemma 4.1.11, so these are cofibrations. To prove the map $Z \rightarrow Y$ is a trivial fibration, we use the following argument suggested by Quillen and now called *the small object argument*: Given a diagram as above, with Z instead of Z_n , we take into account that $|\partial\Delta^k|$ is compact, so¹³ its map to Z factors through a certain Z_n . The rest is clear. \square

¹³Here one uses that the maps $Z_k \rightarrow Z_{k+1}$ is “closed T^1 -embedding”, see details in [Ho], 2.4.2

4.1.13. **Lemma.** *The following properties of f in \mathbf{Top} are equivalent.*

1. f is a trivial cofibration.
2. f has LLP with respect to fibrations.
3. f is a cofibration and a strong deformation retract.

Proof. Recall that an embedding $f : A \rightarrow B$ is a strong deformation retract if there exist the maps r and h making the diagrams below commutative (we denote $I = [0, 1]$ and $s : B \rightarrow B^I$ is the obvious map).

$$(14) \quad \begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ f \downarrow & \nearrow r & \downarrow \\ B & \longrightarrow & * \end{array} \quad \begin{array}{ccc} A & \xrightarrow{sf} & B^I \\ f \downarrow & \nearrow h & \downarrow \\ B & \xrightarrow{(fr, \text{id})} & B \times B \end{array} .$$

The property (3) implies (1) as strong deformation retract is a homotopy equivalence, therefore, a weak homotopy equivalence. Let us show (1) implies (3). Thus, $f : A \rightarrow B$ is a cofibration and a weak equivalence. Present it as a composition

$$(15) \quad A \xrightarrow{g} A \times_B B^I \xrightarrow{q} B,$$

in a standard way, where $I = [0, 1]$. Here q is a fibration and g is a strong deformation retract, so weak equivalence. Therefore, q is a trivial fibration. As f is a cofibration, there is a section $u : B \rightarrow A \times_B B^I$ of q . Thus f is a retract of g , therefore, a strong deformation retract.

The property (2) implies (3): f is cofibration as any trivial fibration is a fibration. The dotted arrows in the above diagrams exist by the lifting property and Corollary 4.1.9.

The property (3) implies (2). Let $p : X \rightarrow Y$ be a fibration. Then $P : X^I \rightarrow X \times_Y Y^I$ is a fibration by 4.1.8 which is trivial by the 2 out of 3 property. A commutative diagram below (on the left)

$$(16) \quad \begin{array}{ccc} A & \xrightarrow{a} & X \\ f \downarrow & \nearrow u & \downarrow p \\ B & \xrightarrow{b} & Y \end{array} \quad \begin{array}{ccc} A & \xrightarrow{sa} & X^I \\ f \downarrow & \nearrow H & \downarrow P \\ B & \xrightarrow{(ar, b^I h)} & X \times_Y Y^I \end{array}$$

gives rise to a commutative diagram on the right, for which there exists a dotted arrow H as f is a cofibration. The one can define $u = e_1 H$ where $e_1 : X^I \rightarrow X$ is the evaluation at 1.

□

We are now ready to prove

4.1.14. **Theorem.** *The category of topological spaces has a model structure with*

- *Weak homotopy equivalences as weak equivalences.*
- *Serre fibrations as fibrations.*
- *Cofibrations defined by the LLP with respect to trivial fibrations.*

Proof. Existence of limits and colimits is standard. Two-out-of-three property for weak equivalences is obvious. Every property defined via LLP or RLP is closed under retracts. Fibrations and cofibrations are defined by RPL and LLP respectively, so are closed under retraction.

Factorization into cofibration followed by a trivial fibration is proven. To get the other factorization, we can factor $f : X \rightarrow Y$ as

$$X \xrightarrow{j} X \times_Y Y^I \xrightarrow{p} Y$$

with j weak equivalence and p fibration. Then factor j into cofibration + trivial fibration. This will give what we need.

It remains to prove axiom 3(a),(b) (lifting properties). Part (b) is definition of cofibration, and part (a) follows from Lemma 4.1.13. \square

4.2. Cofibrantly generated model categories. As we have already seen, smallness of certain objects was instrumental in proving existence of factorizations. For topological spaces this was compactness of $|\partial\Delta^n|$ which allowed one to factor a map $|\partial\Delta^n| \rightarrow Z = \operatorname{colim} Z_n$ through some $Z_n \rightarrow Z$.

We will present the simplest set of the definitions which will be sufficient for our applications.

Let I be a collection of arrows in C . A diagram

$$Z_0 \rightarrow Z_1 \rightarrow \dots$$

will be called I -diagram if all maps $Z_k \rightarrow Z_{k+1}$ are pushouts of elements of I . We assume that C has sequential colimits, so that $Z = \operatorname{colim} Z_k$ is defined.

An object X is called I -small if for any I -diagram as above the map

$$\operatorname{colim}_k \operatorname{Hom}(X, Z_k) \rightarrow \operatorname{Hom}(X, Z)$$

is bijective.

A model category \mathcal{C} is *cofibrantly generated* if there are two sets I and J of morphisms in \mathcal{C} such that

- Fibrations in \mathcal{C} are precisely the arrows satisfying RLP wrt J . Trivial fibrations are those satisfying RLP wrt I .
- For any $\alpha : X \rightarrow Y$ in I the domain X is small wrt I -diagrams.
- The same for morphisms in J and J -diagrams.

The general definition of cofibrantly generated category uses a more generous definition of smallness (κ -smallness for an arbitrary cardinal κ).

In case a model category is cofibrantly generated, one can construct decomposition of a morphism as in two cases we already know — complexes and topological spaces — as a sequential colimit of a diagram which is constructed recursively.

For more general cofibrantly generated categories (when domains are κ -small) decomposition is constructed as more sophisticated colimit (κ -filtered colimit).

If we hope our model category structure is cofibrantly generated, it is much easier to prove this.

One has

Theorem. (see [Ho], 2.1.19) *Let \mathcal{C} has small colimits and small limits. Let W be a subcategory and I, J two sets of arrows. Assume*

- *W contains all isomorphisms, satisfies two-out of three property and is closed under retracts.*
- *Domains of I and J are small wrt I and J diagrams respectively.*
- *Colimits of J -diagrams are in $W \cap LLP(RLP(I))$.*
- *$RLP(I) = RLP(J) \cap W$.*

Then \mathcal{C} has a model structure with weak equivalences defined as W and fibrations defined as $RLP(J)$.

4.3. Model structure for \mathbf{sSet} . We will construct a cofibrantly generated model structure on \mathbf{sSet} as follows.

- A map $f : X \rightarrow Y$ is a weak equivalence iff $|f|$ is a weak homotopy equivalence.
- The set I consists of $\partial\Delta^n \rightarrow \Delta^n$.
- The set J consists of $\Lambda_k^n \rightarrow \Delta^n$.

Let us verify the conditions of the theorem. Properties of W are obvious. Smallness conditions are also obvious. It remains to verify the two last conditions.

1. Geometric realization preserves colimits, so colimit of a J -diagram is in W . Since any element of J is a colimit of an I -diagram, it is in $LLP(RLP(I))$.
2. $f : X \rightarrow Y$ is \mathbf{sSet} is a trivial fibration (that is, is in $RLP(I)$) iff it is a fibration and a weak equivalence. This statement is nontrivial; it will take some time to verify it.

4.3.1. Lemma. *Geometric realization preserves finite limits.*

Proof. We know it preserves finite products. It remains to verify it preserves equalizers (kernels of a pair of maps). This is an exercise. \square

4.3.2. Lemma. *Let $f \in RLP(I)$ then $|f|$ is a fibration.*

Proof. $f : X \rightarrow Y$ has RLP with respect to all injections, in particular, with respect to $\Gamma_f : X \rightarrow X \times Y$. The lift $g : X \times Y \rightarrow X$ presents f as a retract of pr_2 . Thus, $|f|$ is a fibration. \square

4.3.3. Lemma. *If $f \in RLP(I)$ then $|f|$ is a trivial fibration.*

We have to verify that the fibers of $|f|$ at any point are contractible. It is enough to check points coming from $\Delta^0 \rightarrow Y$. By Lemma 4.3.1 the fiber is the

realization of the fiber of f . This is a simplicial set satisfying RLP with respect to I . First, this means that the fiber F is nonempty.

Let us prove that the composition $F \rightarrow * \rightarrow F$ is homotopic to identity. This follows from the following diagram

$$(17) \quad \begin{array}{ccc} F \times \partial\Delta^1 & \longrightarrow & F \\ \downarrow & & \downarrow \\ F \times \Delta^1 & \longrightarrow & \Delta^0 \end{array} .$$

Thus, F is contractible, therefore, $|F|$ is as well contractible.

To conclude the proof of the theorem, it remains to verify that $W \cap RLP(J) \subset RLP(I)$. This will follow from Proposition 4.5.2 below whose proof is based on Quillen's theory of minimal fibrations.

4.4. Homotopy groups of Kan simplicial sets.

4.4.1. Lemma. *Let $a : K \rightarrow L$ be injective and $f : X \rightarrow Y$ be a fibration of simplicial sets. Then the induced map*

$$X^L \rightarrow (X^K) \times_{Y^K} (Y^L)$$

is a fibration.

Proof. We have to verify that the above arrow satisfies RLP with respect to the elements of J . By adjunction, this amounts to proving that if $a : K \rightarrow L$ is injective and $j : C \rightarrow D$ is in J , then the map

$$(K \times D) \sqcup^{K \times C} (L \times C) \rightarrow L \times D$$

is a colimit of a J -diagram. The claim is reduced to the case $a = i^m$, $j = j_k^n$. This is a standard result which we skip. \square

Now we are ready to present an intrinsic definition of homotopy groups for fibrant (=Kan) simplicial sets.

If X is fibrant, $x, y \in X_0$ will be called equivalent if there is $h \in X_1$ such that $x = d_1(h)$, $y = d_0(h)$. This is an equivalence relation and we define $\pi_0(X)$ as the set of equivalence classes.

One easily sees $\pi_0(X) = \pi_0(|X|)$. One defines $\pi_n(X, x)$ as π_0 of the fiber of

$$X^{\Delta^n} \rightarrow X^{\partial\Delta^n}$$

at $\partial\Delta^n \rightarrow * \xrightarrow{x} X$. The fiber is fibrant by the above lemma.

4.4.2. *Fiber sequence.* Let $f : X \rightarrow Y$ be a fibration of Kan simplicial sets, $x \in X$, $y = f(x)$ and let $Z = X \times_Y \{y\}$ be the fiber. Then the usual long exact sequence of homotopy groups is defined. Let $a : \Delta^n \rightarrow Y$ represent a class in $\pi_n(Y, y)$. Then there is a dotted arrow in the commutative diagram

$$(18) \quad \begin{array}{ccc} \Lambda_n^n & \xrightarrow{x} & X \\ \downarrow & \nearrow c & \downarrow \\ \Delta^n & \xrightarrow{a} & Y \end{array}$$

with the upper horizontal arrow carrying everything to $x \in X$. The map $cd_n : \Delta^{n-1} \rightarrow \Delta^n \rightarrow X$ has an image in Z and determines a class in $\pi_{n-1}(Z)$.

We will now study Kan simplicial sets having trivial homotopy groups.

4.4.3. **Corollary.** *Let X be a Kan simplicial set having trivial homotopy groups. Then $\text{Map}(K, X)$ has also trivial homotopy groups.*

Finally, we have

4.4.4. **Lemma.** *A fibrant simplicial set with trivial homotopy groups satisfies RLP with respect to I .*

Proof. We have to extend a given map $a : \partial\Delta^n \rightarrow X$ to $\Delta^n \rightarrow X$. According to 4.4.3, $\text{Map}(\partial\Delta^n, X)$ is connected and Kan. Thus, a can be extended to a map $A : \partial\Delta^n \times \Delta^1 \rightarrow X$ such that $A_0 = a$ and A_1 maps $\partial\Delta^n$ to a point. Put

$$(19) \quad \begin{aligned} K &= (\partial\Delta^n \times \Delta^1) \sqcup^{(\partial\Delta^n \times \{1\})} *, \\ L &= (\Delta^n \times \Delta^1) \sqcup^{(\Delta^n \times \{1\})} *, \end{aligned}$$

so that A factors through $\bar{A} : K \rightarrow X$. The map $K \rightarrow L$ is a colimit of a J -diagram (this is an exercise), so \bar{A} can be extended to a map $L \rightarrow X$. The composition $\Delta^n = \Delta^n \times \{0\} \rightarrow L \rightarrow X$ yields the required lifting. \square

4.5. **Minimal fibrations.** We will now prove two results. The first generalizes 4.4.4 as follows.

4.5.1. **Proposition.** *Let $f : X \rightarrow Y$ be a fibration, such that all its fibers have trivial homotopy groups. Then f satisfies RLP with respect to I .*

The second claim concludes the verification of conditions of Theorem 4.2.

4.5.2. **Proposition.** *Let X be a Kan simplicial set. Then the natural map $\pi_n(X) \rightarrow \pi_n(|X|)$ is an isomorphism.*

Proposition 4.5.2 is easy once one knows that the geometric realization preserves fibrations. The latter result, as well as 4.5.1, requires a theory of minimal fibrations (apparently, due to Quillen).

Minimal fibrations are Kan fibrations with an extra uniqueness lifting property which we will now formulate.

4.5.3. **Definition.** Let $f : X \rightarrow Y$ be a Kan fibration.

1. Two n -simplices x, y in X are f -related if they belong to the same connected component of the fiber

$$\text{Map}(\Delta^n, X) \rightarrow \text{Map}(\Delta^n, Y) \times_{\text{Map}(\partial\Delta^n, Y)} \text{Map}(\partial\Delta^n, X).$$

2. f is called a minimal fibration if for any n any two f -related n -simplices coincide.

Minimal fibrations, on one hand, enjoy some very nice properties. One has

4.5.4. **Proposition.** *Any minimal fibration $X \rightarrow Y$ is locally trivial, that is, for any $y : \Delta^n \rightarrow Y$ the base change $X_y \rightarrow \Delta^n$ is isomorphic to the product $X_y = \Delta^n \times F$ with fibrant F .*

On the other hand, minimal fibrations can be used to describe general fibrations. One has

4.5.5. **Theorem.** *Let $f : X \rightarrow Y$ be a Kan fibration. There exists a simplicial subset $X' \subset X$ such that*

1. *The restriction $f' = f|_{X'}$ is a minimal fibration.*
2. *$f = f'r$ where $r : X \rightarrow X'$ is a retraction.*
3. *r satisfies RLP wrt I .*

We will not prove the theorem. Zorn lemma is extensively used in the proof.

4.5.6. *Proof of 4.5.1.* Theorem 4.5.5 reduces the claim to the case the fibration is minimal. The latter is locally trivial, so Lemma 4.4.4 concludes the proof.

4.5.7. It remains to prove that homotopy groups of fibrant X are isomorphic to homotopy groups of $|X|$. We can use the path space fibration to shift homotopy groups: For a simplicial set X and $x \in X$ we define the path space as the fiber of $\text{Map}(\Delta^1, X) \rightarrow \text{Map}(\{0\}, X)$ at x . One has a fibration $P(X) \rightarrow X$ whose fiber at x is the loop space whose homotopy groups are the homotopy groups of X shifted by 1. This allows one to deduce the assertion about homotopy groups from the following Quillen's result.

4.5.8. **Proposition.** *The functor of geometric realization preserves fibrations.*

Proof. The claim is proven as follows. Using Theorem 4.5.5, one deduces the claim to the case of minimal fibrations. Minimal fibrations are locally trivial, and realization preserves locally trivial fibrations. \square

4.6. Quillen equivalence of Top and sSet. This is already easy. Geometric realization carries I to cofibrations and J to trivial cofibrations. So, one has a Quillen pair.

It remains to prove this is a Quillen equivalence. Let S be a simplicial set and X a topological space (S is automatically cofibrant and X is automatically fibrant). We have to prove that $|S| \rightarrow X$ is weak equivalence iff $S \rightarrow \text{Sing } X$ is. The latter in turn also means that $|S| \rightarrow |\text{Sing } X|$ is a weak equivalence. Therefore, it remains to verify that the natural map $|\text{Sing}(X)| \rightarrow X$ is a weak homotopy equivalence.

It is quite obvious that $\pi_n(\text{Sing}(X)) = \pi_n(X)$. Taking into account our comparison of homotopy groups of Kan simplicial sets and their realizations, we get the result.

4.7. Exercise. 0. Verify that a retract of a strong deformation retract is itself a strong deformation retract.

1. Let $f : X \rightarrow Y$ be a map of topological spaces. Prove that the composition $X \times_Y Y^I \rightarrow Y^I \xrightarrow{\text{ev}_1} Y$ is a Serre fibration.
2. Prove 4.3.1 (showing that the realization preserves equalizers).
3. Minimal fibrations (of simplicial sets) are closed under base change.
4. Verify that the connecting homomorphism $\pi_n(Y) \rightarrow \pi_{n-1}(Z)$ of the exact sequence of fibration is correctly defined.
5. Prove that $f : X \rightarrow Y$ is a minimal fibration iff for any $\Delta^n \rightarrow Y$ the base change is a trivial fibration whose fiber is a minimal Kan simplicial set.

5. MODELS FOR ∞ -CATEGORIES AND DWYER-KAN LOCALIZATION.

First of all, we discuss Quillen equivalence between two model categories: simplicial sets with the Joyal model structure, and simplicial categories with Bergner model structure. Then we present Dwyer-Kan localization which is a derived version of localization of categories. Dwyer-Kan localization of model categories produces a simplicial category which is the “underlying ∞ -category” of a model category.

5.1. Simplicial categories. Simplicial categories, that is, simplicially enriched categories, form a meaningful approach to the theory of ∞ -categories. Thus, it is pleasant to know that the category \mathbf{sCat} of (small) simplicial categories has a nice cofibrantly generated model structure.

5.1.1. Weak equivalences. Recall that any simplicial category \mathcal{C} defines a conventional category $\mathrm{Ho}(\mathcal{C})$.

A map $f : \mathcal{C} \rightarrow \mathcal{D}$ (a simplicial functor) in \mathbf{sCat} is a weak equivalence (often called DK equivalence) if

1. f induces a weak homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Map}_{\mathcal{D}}(fx, fy)$ for any $x, y \in \mathcal{C}$.
2. f is essentially surjective, that is, any object of \mathcal{D} is equivalent to an object in the image of f .

Note that if Condition 1 is satisfied, Condition 2 is equivalent to the following.

- 2' $\mathrm{Ho}(f)$ is an equivalence of (conventional) categories.

5.1.2. Fibrations. A map $f : \mathcal{C} \rightarrow \mathcal{D}$ is called a fibration if

1. f induces a Kan fibration $\mathrm{Map}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Map}_{\mathcal{D}}(fx, fy)$ for any $x, y \in \mathcal{C}$.
2. Any equivalence $\alpha : f(c) \rightarrow d$ in \mathcal{D} can be lifted to an equivalence $a : c \rightarrow c'$ in \mathcal{C} (that is, so that $\alpha = f(a)$).

5.1.3. Simplicial categories with a fixed set of objects. Let $\mathbf{sCat}_{\mathcal{O}}$ denote the category whose objects are simplicial categories with a fixed set of objects \mathcal{O} .

This category has a model structure defined by Dwyer-Kan. Weak equivalences and fibrations are defined as above ¹⁴

It is very easy to prove these definitions define a model category structure on $\mathbf{sCat}_{\mathcal{O}}$.

Note that if \mathcal{O} is one-element set, this is the category of simplicial monoids. The general case is only slightly more general than that for simplicial monoids.

The reason for fixing \mathcal{O} is that, on one hand, the model structure is much easier to deduce, and, on the other hand, this is already good enough in some applications (DK localization).

¹⁴Property 2 is, of course, automatically fulfilled.

Note that cofibrant objects in $\mathbf{sCat}_\mathcal{O}$ are obtained by gluing to the discrete category \mathcal{O} of a sequence of arrows (of various simplicial dimensions). Let $x, y \in \mathcal{O}$. To glue an n -arrow from x to y to a simplicial category \mathcal{C} we have to choose a map $\phi : \partial\Delta^n \rightarrow \text{Map}_\mathcal{C}(x, y)$, to consider \mathcal{C} as a (simplicially enriched) quiver, glue an n -simplex to $\text{Map}_\mathcal{C}(x, y)$ along the boundary given by ϕ , produce a free simplicial category generated by the obtained quiver, and mod out by some obvious relations.

5.1.4. *Generating cofibrations for \mathbf{sCat} .* To define generating cofibrations, we will use the following notation. For a simplicial set S we denote $\mathbf{1}_S$ the simplicial category having two objects 0 and 1, with the morphisms given by

$$\text{Map}(0, 0) = \{\text{id}\} = \text{Map}(1, 1), \quad \text{Map}(1, 0) = \emptyset, \quad \text{Map}(0, 1) = S.$$

Here is the list of generating cofibrations for \mathbf{sCat} .

- $\mathbf{1}_{\partial\Delta^n} \rightarrow \mathbf{1}_{\Delta^n}$.
- $\emptyset \rightarrow *$, the embedding of the empty category into the discrete category consisting of one object.

And here is a list of generating trivial cofibrations.

- The maps $\mathbf{1}_{\Delta_k^n} \rightarrow \mathbf{1}_{\Delta^n}$, $0 \leq k \leq n$.
- $* \rightarrow \mathcal{J}$ where \mathcal{J} belongs to a set of representatives of isomorphism classes of simplicial categories with two objects 0 and 1, and countably many simplices in each map simplicial set which are all required to be weakly contractible. Additionally, the map $\{0, 1\} \rightarrow \mathcal{J}$ is assumed to be a cofibration in $\mathbf{sCat}_{\{0,1\}}$.

The following result was proven by Bergner

5.1.5. **Theorem.** *The category \mathbf{sCat} has a model structure with weak equivalences defined as above and cofibrations generated by the above list.*

We will prove only a part of the theorem.

5.2. **Parts of the proof.** Obviously $RLP(I) \subset RLP(J)$. Let us show $RLP(I) \subset W$. Let $f : C \rightarrow D$ satisfies RLP with respect to I . We immediately deduce that for $x, y \in C$ the map $\text{Map}_C(x, y) \rightarrow \text{Map}_D(fx, fy)$ is a trivial fibration. Thus, it remains to verify essential surjectivity. This follows from the lifting property with respect to $\emptyset \rightarrow *$.

Let us verify $RLP(J) \cap W \subset RLP(I)$. If $f : \mathcal{C} \rightarrow \mathcal{D}$ satisfies RLP with respect to J and is a weak equivalence, for any pair $x, y \in \mathcal{C}$ the map $\text{Map}_\mathcal{C}(x, y) \rightarrow \text{Map}_\mathcal{D}(fx, fy)$ is a trivial fibration, so we have RLP with respect to $\mathbf{1}_{\partial\Delta^n} \rightarrow \mathbf{1}_{\Delta^n}$. It remains to verify RLP with respect to $\emptyset \rightarrow *$, that is, surjectivity on objects. Since f is weak equivalence, it is essentially surjective, that is, for any $d \in \mathcal{D}$ there exists $c \in \mathcal{C}$ and an equivalence $f(c) \rightarrow d$. The following result which we do not intend to prove, completes the proof.

5.2.1. Proposition. *Let \mathcal{C} be a simplicial category and $a : x \rightarrow y$ an equivalence in \mathcal{C} . Then there exists a simplicial category \mathcal{J} as defined in 5.1.4 with an arrow $j : 0 \rightarrow 1$ and a functor $\mathcal{J} \rightarrow \mathcal{C}$ carrying j to a .*

5.2.2. Exercise. Verify that the fibrations in the Bergner model structure are precisely defined by RLP with respect to trivial cofibrations.

5.2.3. Exercise. Prove that the functor \mathfrak{C} defined in 2.4.5 carries injective maps of simplicial sets to cofibrations. Note that it is enough to verify that $\mathfrak{C}(\partial\Delta^n) \rightarrow \mathfrak{C}(\Delta^n)$ is a cofibration in \mathbf{sCat} .

5.3. Quillen equivalence between quasicategories and simplicial categories. Recall 2.4.5 that the adjoint pair of functors

$$(20) \quad \mathfrak{C} : \mathbf{sSet} \overset{\rightarrow}{\underset{\leftarrow}{\rightleftarrows}} \mathbf{sCat} : \mathfrak{N}$$

was constructed, based on a cosimplicial object $n \mapsto \mathfrak{C}^n$ in \mathbf{sCat} .

Recall that \mathfrak{C}^n is a cofibrant simplicial category whose homotopy category is $[n]$.

The adjoint pair $(\mathfrak{C}, \mathfrak{N})$ can be upgraded to a Quillen equivalence.

One has

5.3.1. Theorem. *There exists a model category structure on \mathbf{sSet} (Joyal model structure) defined by the following properties.*

- *Cofibrations are the injective maps.*
- *Weak equivalences are the maps of simplicial sets carried by \mathfrak{C} to DK equivalences of simplicial categories.*
- *Fibrant objects are precisely the quasicategories, that is simplicial sets satisfying RLP with respect to the inner horns.*

Furthermore, the adjoint pair $(\mathfrak{C}, \mathfrak{N})$ is a Quillen equivalence.

Fibrations and equivalences in the Joyal model structure are called *categorical fibrations* and *categorical equivalences*.

Note that this is a difficult theorem. Not only because the proof is longer than we saw earlier, but also because the notions of categorical fibration and of categorical equivalence in \mathbf{sSet} are not easily expressible.

Note once more that fibrant objects in Joyal model structure are quasicategories, that is simplicial sets with RLP with respect to the inner horns. All categorical fibrations satisfy RLP with respect to the inner horns; but not vice versa.

5.3.2. Remarks. Cofibrations are just injective maps of simplicial sets. Two other types of maps have no easy description. However, if we are talking about quasicategories, the situation is much better. One has

Theorem. *A map $f : X \rightarrow Y$ of quasicategories is a weak equivalence iff it is a DK equivalence (that is, is essentially surjective and fully faithful).*

A map of quasicategories is a categorical fibration iff it is an inner fibration (that is satisfies RLP wrt inner horns) and admits a lifting of equivalences.

5.3.3. Exercise. Categorical fibration of conventional categories is a functor with lifting of isomorphisms.

It is probably worthwhile to define weak equivalence in Joyal model structure independently of the functor \mathfrak{C} .

One has

5.3.4. Proposition. *Let $f : X \rightarrow Y$ be an arrow in \mathbf{sSet} . The following conditions on f are equivalent.*

1. *f is an equivalence in Joyal model structure, that is, $\mathfrak{C}(f)$ is a DK equivalence.*
2. *For any quasicategory \mathfrak{C} the induced map*

$$\mathrm{Fun}(Y, \mathfrak{C}) \rightarrow \mathrm{Fun}(X, \mathfrak{C})$$

is a DK equivalence of quasicategories.

3. *For any quasicategory \mathfrak{C} the induced map*

$$\mathrm{Map}(Y, \mathfrak{C}) \rightarrow \mathrm{Map}(X, \mathfrak{C}),$$

with $\mathrm{Map}(X, \mathfrak{C})$ defined as the maximal Kan subspace of $\mathrm{Fun}(X, \mathfrak{C})$, is a homotopy equivalence.

5.4. DK localization.

5.4.1. Introduction. In the first lecture of the course we discussed that usage of localization in the construction of the derived (or homotopic) category results in losing an important information about the category.

Here is a very simple manifestation of what happens when we localize a category. Let C be a category. If we invert all arrows in C , we get a groupoid G whose nerve $N(G)$ is an Eilenberg-MacLane space. It is easy to prove that $N(G)$ describes the first floor of the Postnikov tower for the nerve $N(C)$.

Dwyer-Kan localization is the procedure that allows one a more clever, “up to homotopy”, way of inverting arrows. Of course, this is just a version of derived functor.

The most important homotopical information of model category is contained in the collection of its weak equivalences. This has been quite early understood by Dan Kan and he spent a lot of time¹⁵ in trying to express this. One of the most important constructions connected to this is a construction of simplicial category from a pair (a category, a collection of arrows in it). This is called DK localization.

¹⁵Successfully!

5.4.2. *As derived functor.* Let C be a simplicial category and W a subcategory (one could consider any map $W \rightarrow C$ of simplicial categories). We describe DK localization of the pair as the left derived functor of the “usual” localization functor assigning to the arrow $W \rightarrow C$ the localization $C[W^{-1}]$ whose category of n -simplices is, by definition, the localization $C_n[W_n^{-1}]$.

This derived functor is calculated as follows. First of all, we construct a commutative square (see below) with p and q trivial fibrations, \widetilde{W} cofibrant and \widetilde{f} cofibration. Then we define $L(C, W)$ as $\widetilde{C}[\widetilde{W}^{-1}]$.

$$(21) \quad \begin{array}{ccc} \widetilde{W} & \xrightarrow{\widetilde{f}} & \widetilde{C} & \longrightarrow & L(C, W) \\ \downarrow p & & \downarrow q & & \\ W & \xrightarrow{f} & C & & \end{array}$$

Note that in the most interesting case, when W and C have the same objects, everything can be done in \mathbf{sCat}_\emptyset , $\emptyset = \text{Ob}(C)$.

5.4.3. *Explicit resolution.* Given a category C , one can forget a composition and get a quiver GC ; then generate a free category FGC . We will have a functor $FG(C) \rightarrow C$. Applying once more the composition FG , we get two possible ways to map $(FG)^2C$ to $FG(C)$, as well as a degeneracy $GF(C) \rightarrow (FG)^2(C)$ induced by the unit $\text{id} \rightarrow GF$. This leads to a simplicial object, in this case in the category of categories; they have all the same set of objects, so we end up with a canonical cofibrant replacement of a category C as an object of \mathbf{sCat} (this is a bar resolution). In case W is a subcategory of C (having the same objects), we get a cofibration of their bar resolutions. Thus, $L(C, W)$ can be defined as localization of the resolution \widetilde{C} with respect to the arrows coming from W .

5.4.4. *DK localization (hammock version).* Even more explicit construction ($W \subset C$): One defines hammock localization $L^H(C, W)$ as simplicial category having the same objects as C , such that n -simplices of $\text{Map}_{L^H}(x, y)$ are the following

5.4.9. Definition. Let \mathcal{C} be a model category having a simplicial structure. We call it a *weak simplicial model category* if it admits weak path functors and satisfies the condition

If $i : A \rightarrow B$ is a cofibration in \mathcal{C} and $p : X \rightarrow Y$ is a fibration in \mathcal{C} , then the map of simplicial sets

$$\mathrm{Map}(B, X) \rightarrow \mathrm{Map}(A, X) \times_{\mathrm{Map}(A, Y)} \mathrm{Map}(B, Y)$$

is a fibration which is a trivial fibration if either i or p is a weak equivalence.

5.4.10. Theorem. *Assume \mathcal{C} is a weak simplicial model category.*

Then one has equivalences

$$\mathcal{C}_*^{cf} \rightarrow L^H(\mathcal{C}_*^{cf}) \rightarrow L^H(\mathcal{C}_*) \leftarrow L^H(\mathcal{C}).$$

The theorem is proven by Dwyer-Kan [DK1, DK2, DK3] for simplicial model categories.

Here is the key result which allows one to prove all claims of this type.

Let \mathcal{C}, \mathcal{D} be categories, $f : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. For $x \in \mathcal{D}$ we denote by \mathcal{C}_x the “homomopy” fiber $\{(c, \theta) | c \in \mathcal{C}, \theta : f(c) \xrightarrow{\sim} x\}$.

More generally, for n -simplex $\sigma \in N_n(\mathcal{D})$ we denote by \mathcal{C}_σ the homotopy fiber of the functor $f^{[n]} : \mathcal{C}^{[n]} \rightarrow \mathcal{D}^{[n]}$ at σ .

Here we denote $\mathcal{C}^{[n]}$ the category of functors $[n] \rightarrow \mathcal{C}$ where $[n]$ is the category consisting of n consecutive arrows.

5.4.11. Lemma. *Let \mathcal{C}, \mathcal{D} be categories and $f : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Assume that for any $\sigma \in N(\mathcal{D})$ the fiber \mathcal{C}_σ has a weakly contractible nerve. Then the functor f induces a DK equivalence*

$$\hat{f} : L^H(\mathcal{C}, W) \rightarrow \mathcal{D},$$

where $W = \{a | f(a) \text{ is an isomorphism}\}$.

The lemma is easily proven in yet another model for infinity categories which we will discuss next time.

In a typical application \mathcal{D} is a model category and \mathcal{C} is the category of weak equivalences $\tilde{X} \rightarrow X$ with cofibrant \tilde{X} , with the functor f forgetting \tilde{X} .

The lemma conceptualizes the idea that, since resolutions are “homotopically unique”, we can replace the model category with the category consisting of cofibrant objects only.

Details of the proof, as well as the proof of Theorem 5.4.10, can be found in [H.L].

Dwyer-Kan localization of a model category provides a simplicial category whose homotopy category is what we called the homotopy category of the model

category. Thus, DK localization can be thought of as the infinity category underlying a model category — at least if one accepts simplicial categories as models for infinity categories. We prefer working with quasicategories — so we will define the quasicategory underlying a model category as $\mathfrak{N}(L^H(\mathcal{C})^f)$ where superscript f denotes a (Bergner) fibrant replacement and \mathfrak{N} denotes the homotopy coherent nerve.

Important result of Dwyer-Kan: one has

5.4.12. **Theorem.** *Quillen equivalence of model categories gives rise to an equivalence of DK localizations.*

5.5. **Examples: quasicategory of spaces, of quasicategories, of complexes.** We have already discussed examples of quasicategories which can be assigned to spaces, quasicategories, complexes.

We will see now that all these examples are actually quasicategories underlying certain model categories.

We will also present some more model categories and respective quasicategories.

5.5.1. *Spaces.* DK localization of the category \mathbf{sSet} with respect to weak homotopy equivalences is equivalent, according to the general theorem above, to the simplicial category whose objects are Kan simplicial sets and morphism spaces are inner Hom's. This is precisely what we denoted \mathcal{S} in Section 2.

5.5.2. *Infinity categories.* We have already two Quillen equivalent model categories (\mathbf{sSet} with the Joyal model structure and \mathbf{sCat} with the Bergner model structure) to apply DK localization. It is, however, not easy to understand the answer, as none of them is weak simplicial model category.

Fortunately, there are simplicial model categories Quillen equivalent to already mentioned ones. One of them, the model category of marked simplicial sets (see [L.T], 3.1) yields the answer described in Part 2.

Recall that the respective simplicial category is the following. Its objects are quasicategories. For X, Y we define $\text{Map}(X, Y)$ to be the maximal Kan subset (=maximal subspace) of the quasicategory $\text{Fun}(X, Y)$. Then the quasicategory Cat_∞ is defined as the nerve \mathfrak{N} of the fibrant simplicial category described above.

5.5.3. *Complexes.* The category $C(k)$ of complexes over a ring k has internal Hom: for a pair of complexes $X, Y \in C(k)$ one defines a complex $\mathcal{H}om(X, Y)$ of abelian groups as follows.

$$\mathcal{H}om(X, Y)^n = \prod \text{Hom}(X^k, Y^{n+k}); (df)(x) = df(x) - (-1)^{|f|} f(dx),$$

where $|y|$ denotes the degree of the element y . This leads to the structure of simplicial category on $C(k)$: for a pair of complexes X and Y we define the simplicial space $\text{Map}(X, Y)$ by the formulas

$$\text{Map}(X, Y)_n = \text{Hom}_{C(\mathbb{Z})}(C_*(\Delta^n), \mathcal{H}om(X, Y)),$$

where $C_*(\Delta^n)$ is the complex of normalized chains of the n -simplex.

Exercise. Verify that a canonical composition

$$\text{Map}(Y, Z) \times \text{Map}(X, Y) \rightarrow \text{Map}(X, Z)$$

is defined.

One can see that the simplicial enrichment defined above is compatible with the projective model structure — they form a weak simplicial model category. This implies

Proposition. *DK localization of the category of complexes with respect to the quasiisomorphisms is equivalent to the simplicial category whose objects are cofibrant complexes, and simplicial Hom sets are defined as above.*

5.5.4. *DG algebras.* Let k be commutative and $k \supset \mathbb{Q}$. The category $\mathcal{C} = \mathbf{Com}(k)$ of commutative DG algebras over k has a model structure with quasiisomorphisms as weak equivalences and surjective maps as fibrations. It has a simplicial structure defined as follows.

Define

$$\Omega_n = k[x_0, \dots, x_n, d_0, \dots, d_n] / (\sum x_i - 1, \sum d_i),$$

with the variable x_i having degree 0 and d_i having degree 1. With the differential given by $dx_i = d_i$, Ω_n becomes a commutative DG algebra (this is the algebra of polynomial differential forms on the standard n -simplex).

For a commutative DG algebra A we can now define A^{Δ^n} to be just $\Omega_n \otimes A$. This is a weak simplicial structure on the model category of commutative DG algebras. Therefore, the quasicategory underlying \mathcal{C} can be described as the nerve \mathfrak{N} of the simplicial category of cofibrant commutative DG algebras.

The same construction makes sense for algebras of “any type” (that is, algebras over any operad).

5.5.5. *Complexes of sheaves.* A common wisdom says that the category of sheaves has no projective objects, so one should look for injectives.

Here is another approach.

Let (X, τ) be a site (X a category, τ a topology), k a ring, and let $C(\widehat{X}_k)$ be the category of complexes of presheaves of k -modules in X .

We say that a map $M \rightarrow N$ is a weak equivalence if it induces a quasiisomorphism of the respective sheafifications.

Cofibrations are generated by maps $M \rightarrow M\langle x; dx = z \rangle$ where $z \in M(U)$ is a cycle.

Fibrations are defined by RLP with respect to trivial cofibrations.

This is a model category; cofibrations are described as retractions of colimits of I -diagrams where I is described by joining a variable to kill a cycle as above.

Generating acyclic cofibrations can be explicitly described in terms of hypercoverings. Let us remind this notion.

Let, as above, (X, τ) be a site and let \hat{X} denote the category of presheaves (of sets) on X .

- Definition.**
1. A presheaf $P \in \hat{X}$ is called semirepresentable if it is isomorphic to a coproduct of representable presheaves.
 2. A map of presheaves $V \rightarrow U$ is called a covering of presheaves if its sheafification is surjective.
 3. A simplicial object V_\bullet in \hat{X} is called a hypercovering if all V_n are semirepresentable and for any map the map $V_n \rightarrow V(\partial\Delta^n)$ is a covering. (For $n = 0$ we assume $V(\emptyset) = *$).
 4. An augmented simplicial presheaf $V_\bullet \rightarrow U$ is a hypercovering if it defined a hypercovering in \hat{X}/U .

Here is the meaning of definition in case X is the category of open subsets. A hypercovering of an open set U is the following. V_0 is a collection of open subsets covering U ; V_1 is a collection of open subsets covering the pairwise intersections of the components of V_0 , etc.

The generating trivial cofibration corresponds to a pair (ϕ, n) where $\phi : V_\bullet \rightarrow U$ is a hypercovering and n is an integer. It has form $K \rightarrow L$ where L is the cone of id_K , and K is the cone of the map $C_*(V_\bullet) \rightarrow k \cdot U$, C_* being the complex of normalized chains, shifted by n .

If M is an abelian presheaf and $V_\bullet \rightarrow U$ is a hypercovering, one has a map $M(U) \rightarrow \check{C}(V_\bullet, M)$, where $\check{C}(V_\bullet, M)$ is the Čech complex defined as the total complex of the cosimplicial object $\text{Hom}(V_\bullet, M)$.

In these terms fibrations are described very easily.

A map $f : M \rightarrow N$ is a fibration iff

- for any U the map $M(U) \rightarrow N(U)$ is surjective.
- for any hypercovering $V_\bullet \rightarrow U$ the diagram

$$(22) \quad \begin{array}{ccc} M(U) & \longrightarrow & \check{C}(V_\bullet, M) \\ \downarrow & & \downarrow \\ N(U) & \longrightarrow & \check{C}(V_\bullet, N) \end{array}$$

is homotopy cartesian.

The model structure described above is weakly simplicial. Thus, we can define infinity-version of the derived category of sheaves applying the functor \mathfrak{N} to the simplicial category spanned by fibrant cofibrant objects of $C(\hat{X}_k, \tau)$.

5.5.6. *Topology as localization.* Note that the category $C(\widehat{X}_k, \tau)$ does not depend on the topology on X . Cofibrations in $C(\widehat{X}_k, \tau)$ also know nothing about the topology. Weak equivalence is the datum depending on τ . For the coarse topology τ_0 (all presheaves are sheaves) weak equivalence is just a map $f : M \rightarrow N$ inducing a quasiisomorphism $f(U) : M(U) \rightarrow N(U)$ for all U . In general one has more weak equivalences and, respectively, less fibrations.

Thus, identity functor yields a Quillen adjunction

$$C(\widehat{X}_k, \tau_0) \rightleftarrows C(\widehat{X}_k, \tau).$$

The right adjoint functor identifies $\text{Ho}(C(\widehat{X}_k, \tau))$ with a full subcategory of $\text{Ho}(C(\widehat{X}_k, \tau_0))$.

This is a general phenomenon called *left Bousfield localization*. Given a model category, one can sometimes enlarge the collection of weak equivalences retaining the same cofibrations. We will see later more examples of Bousfield localization.

6. SIMPLICIAL SPACES. SEGAL SPACES. COMPLETE SEGAL SPACES

Among various models of infinity categories, that of complete Segal spaces is especially pleasant.

6.1. Why Segal spaces? Recall that the nerve functor $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$ is fully faithful, and its essential image can be described by any of two equivalent properties of a simplicial set X (see Part 2).

1. The maps $X_n \rightarrow X_1 \times_{X_0} \dots \times_{X_0} X_1$, induced by the embeddings $\mathrm{Sp}(n) \rightarrow \Delta^n$, are bijections. Here $\mathrm{Sp}(n)$ is the simplicial subset of Δ^n spanned by the 1-simplices $\{i, i+1\}$, $i = 0, \dots, n-1$ (the “spine” of Δ^n).
2. The maps $X_n \rightarrow \mathrm{Hom}(\Lambda_i^n, X)$ are bijective for all $1 \leq i \leq n-1$.

A slight generalization of condition 2 gave us the notion of quasicategory. The notion of Segal space is based on a version of the first condition. The idea is that an infinity category should have a space of objects, a space of morphisms, a space of commutative triangles, and so on. Thus, we have to replace simplicial sets with simplicial spaces; then the condition 1 can be replaced with a meaningful weakening — saying that the respective map is a weak equivalence.

By definition, a CSS is a simplicial space (more precisely, simplicial object in \mathbf{sSet} , that is bisimplicial set) satisfying some special properties which we will describe later.

6.1.1. Some history. Back 1960-ies topologists were interested in describing a condition on topological space X which would ensure it is homotopy equivalent to a loop space. Loop space has a (sort of) associative composition, so it turned out that the problem reduces to describing composition laws, associative up to (higher and higher) homotopies. Here is G. Segal’s suggestion.

A Segal monoid is a simplicial object X_\bullet such that

- The map $X_n \rightarrow (X_1)^n$ (induced by n embeddings $\Delta^1 \rightarrow \Delta^n$ carrying 0 and 1 to i and $i+1$ respectively, is a weak equivalence.
- X_0 is a point.

Segal monoid defines a loose structure of associative monoid on X_1 : if X_n were actually X_1^n , the associative operation would be given by $d_1 : X_1^2 \rightarrow X_1$.

We know that associative monoids are just categories with one object. So the following modification seems to be very natural.

Let \mathcal{T} be a category with products ($\mathcal{T} = \mathbf{Top}$ or \mathbf{sSet} especially important for us) and with a notion of weak equivalence. A Segal object in \mathcal{T} is a simplicial object X_\bullet such that the canonical map

$$(23) \quad X_n \rightarrow X_1 \times_{X_0} \dots \times_{X_0} X_1.$$

is a weak equivalence.

Complete Segal spaces are Segal objects in the category of spaces (technically: simplicial sets) satisfying an extra *completeness* condition which will be explained later.

6.2. Bisimplicial sets. Our new way to model infinity categories will be via bisimplicial sets satisfying certain properties. Let us introduce an appropriate notation.

\mathbf{ssSet} is the category of bisimplicial sets $X_{\bullet\bullet} \in \text{Fun}(\Delta^{\text{op}} \times \Delta^{\text{op}}, \mathbf{Set})$. We have to get accustomed to look at them as at model for higher categories; in particular, the roles of the indices will be very different.

A bisimplicial set should be seen as a simplicial object in “spaces”. So, if $X \in \mathbf{ssSet}$, $X_n = X_{n\bullet}$ denotes the “space” of n -simplices.

Correspondingly, there are two functors from simplicial sets to bisimplicial sets, corresponding to two projections $\Delta \times \Delta \rightarrow \Delta$.

The first one, $c : \mathbf{sSet} \rightarrow \mathbf{ssSet}$ is defined by the formula $c(X)_n = X$. (“c” stands for “constant”). The second one, $d : \mathbf{sSet} \rightarrow \mathbf{ssSet}$, is defined as $d(X)_n = X_n$, “d” meaning “discrete”. The latter means that the space of n -simplices in $d(X)$ a discrete simplicial space X_n .

Denote $\Delta^{m,n} = d(\Delta^m) \times c(\Delta^n)$. This is a presheaf on $\Delta \times \Delta$ represented by the pair $([m], [n])$. Thus, one has

$$\text{Hom}(\Delta^{m,n}, X) = X_{m,n}.$$

6.2.1. Internal Hom. Direct product in \mathbf{ssSet} has a right adjoint, so that $\text{Fun}(X, Y) \in \mathbf{ssSet}$ is defined for $X, Y \in \mathbf{ssSet}$. This is always so since \mathbf{ssSet} is a category of presheaves (of sets, in the conventional sense). Explicitly,

$$\text{Fun}(X, Y)_{m,n} = \text{Hom}(X \times \Delta^{m,n}, Y).$$

Forgetting a part of the structure, we can get simplicial enrichment. Actually, we can get two of them, but we are now more interested in this one:

$$\text{Map}(X, Y)_n = \text{Fun}(X, Y)_{0,n} = \text{Hom}(X \times c(\Delta^n), Y).$$

This simplicial structure will be compatible with the model structure.

6.2.2. One has $X_n = \text{Map}(d(\Delta^n), X)$. Taking this into account, we use the notation $X(S) = \text{Map}(d(S), X)$ for a simplicial set S . Thus, $X(\Delta^n) = X_n$.

6.3. Review: Model structure on functor categories. Let I be a category and M a model category. Does the category $\text{Fun}(I, M)$ admit a canonical, or “standard” model structure?

Let us try to imagine what we would like. Here is an obvious choice.

6.3.1. Definition. A map of functors $f : F \rightarrow G$ is a weak equivalence if for all $i \in I$ $f(i)$ is a weak equivalence in M .

The next step is less obvious. We could copy the above definition for cofibrations and for fibrations. Unfortunately, we cannot do this simultaneously.

So, we have to make a choice.

6.3.2. Definition. A map of functors $f : F \rightarrow G$ is a *projective fibration* if for all $i \in I$ $f(i)$ is a fibration in M .

Then we will have to say

6.3.3. Definition. A map of functors $f : F \rightarrow G$ is a *projective cofibration* if it satisfies LLP with respect to trivial projective fibrations.

Alternatively, we can define

6.3.4. Definition. A map of functors $f : F \rightarrow G$ is an *injective cofibration* if for all $i \in I$ $f(i)$ is a cofibration in M .

Then we have to define

6.3.5. Definition. A map of functors $f : F \rightarrow G$ is an *injective fibration* if it satisfies RLP with respect to trivial injective cofibrations.

This is only beginning of the story. It turns out that both model structures exist only if one makes some additional assumptions. These are requirements on M . One has

- 6.3.6. Theorem.**
1. *Assume M is cofibrantly generated. Then $\text{Fun}(I, M)$ admits a projective model structure.*
 2. *Assume M is combinatorial¹⁶ model category. Then $\text{Fun}(I, M)$ admits an injective model structure.*

For a special class of categories I (Reedy categories) there is one more model category structure — this one for arbitrary model category M . This model structure has the same weak equivalences, but (in general) less fibrations than the projective model structure, and (in general) less cofibrations than the injective model structure. It is convenient that both Reedy fibrations and Reedy cofibrations have an explicit description. The example we are especially interested in, is when $I = \Delta^{\text{op}}$ and $M = \mathbf{sSet}$ with the standard (Kan) model structure. In this case Reedy model structure coincides with the injective one.

6.4. Reedy model structure.

6.4.1. Theorem. *The category $\mathbf{ssSet} = \text{Fun}(\Delta^{\text{op}}, \mathbf{sSet})$ has a model structure with the classes of arrows defined as follows.*

- *A map $f : X \rightarrow Y$ is a weak equivalence iff $f_n : X_n \rightarrow Y_n$ is a weak homotopy equivalence of simplicial sets.*

¹⁶That is, cofibrantly generated and (locally) presentable, a property that we do not explain in this course.

- A map $f : X \rightarrow Y$ is a cofibration iff $f_n : X_n \rightarrow Y_n$ are cofibrations, that is, are injective maps of simplicial sets.
- A map $f : X \rightarrow Y$ is a (trivial) (Reedy) fibration iff $X_0 \rightarrow Y_0$ is a (trivial) Kan fibration and for each $n > 0$ the map

$$(24) \quad X_n \rightarrow Y_n \times_{Y(\partial\Delta^n)} X(\partial\Delta^n)$$

is a (trivial) Kan fibration of simplicial sets.

Proof. This is a relatively easy result. We will only verify the lifting properties.

Let, for instance, $i : A \rightarrow B$ be componentwise trivial cofibration, and $p : X \rightarrow Y$ be a Reedy fibration. We will construct a lifting $g : B \rightarrow X$ step by step. First of all, we construct g_0 as f_0 satisfies RLP with respect to i_0 . Let, by induction, g_i be constructed for $i < n$.

The collection of g_i extends to a map $B(\partial\Delta^n) \rightarrow X(\partial\Delta^n)$ which allows one to construct a commutative diagram

$$(25) \quad \begin{array}{ccccc} A_n & \longrightarrow & X_n & & \\ \downarrow & & \downarrow & \searrow & \\ B_n & \longrightarrow & Y_n & & X(\partial\Delta^n) \\ \downarrow & & \downarrow & \searrow & \downarrow \\ B(\partial\Delta^n) & \longrightarrow & Y(\partial\Delta^n) & & \end{array}$$

(Note: A dotted arrow also points from $B(\partial\Delta^n)$ to $X(\partial\Delta^n)$ in the original diagram.)

which yields a commutative diagram

$$(26) \quad \begin{array}{ccc} A_n & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ B_n & \longrightarrow & Y_n \times_{Y(\partial\Delta^n)} X(\partial\Delta^n) \end{array}$$

(Note: A dotted arrow also points from B_n to X_n in the original diagram.)

having a lifting by the assumption on $f : X \rightarrow Y$.

The other lifting property is verified in essentially the same manner. □

6.4.2. Properness. A model category M is said to be *left proper* if all pushouts of weak equivalences along cofibrations are weak equivalences.

It is *right proper* if all pullbacks of weak equivalences along fibrations are weak equivalences. It is *proper* if it is both left and right proper.

Exercise. A pushout of a weak equivalence between cofibrant objects along a cofibration is always a weak equivalence. Thus, any model category whose all objects are cofibrant is left proper.¹⁷

Dually, if all objects are fibrant, the model category is right proper.

¹⁷This is [Ho], 13.1.2.

6.4.3. Lemma. *Reedy model structure on \mathbf{ssSet} is proper.*

Proof. Note first that \mathbf{sSet} (with the standard model structure) is left proper and \mathbf{Top} is right proper. Let us show that \mathbf{sSet} is also right proper. The functor of geometric realization preserves finite limits. It also preserves fibrations, see 4.5.8. This allows one to deduce right properness of \mathbf{sSet} from the right properness of \mathbf{Top} .

In the Reedy model structure any cofibration is a componentwise cofibration and any fibration is a componentwise fibration. This implies that Reedy model structure on \mathbf{ssSet} is proper. \square

6.5. Rezk nerve of a (conventional) category. The following easy construction is in the base of intuition about CSS.

Let C be a category. As zeroth approximation to C we can study “moduli space of objects” of C . This is the maximal subgroupoid of C , denoted C^{iso} . This gives a full information about the objects (including automorphisms) but no information about non-invertible morphisms. To catch it, consider $C^{[1]}$, the category of arrows of C , and once more take the maximal subgroupoid.

In this way we end up with a simplicial object

$$n \mapsto (C^{[n]})^{\text{iso}}$$

in groupoids. Applying the nerve functor N , we get a simplicial simplicial set, that is, a bisimplicial set which we denote $B(C)$. This is the Rezk nerve of a (conventional) category C .

Note that two simplicial dimensions play a very different role here!

6.6. Segal spaces. Segal condition says that the space of n -simplices can be reconstructed, up to homotopy, from its 1-simplex components. Here is a formal definition we will use.

6.6.1. Definition. A bisimplicial set $X \in \mathbf{ssSet}$ is called a *Segal space* if

1. X is Reedy fibrant.
2. The map $X_n \rightarrow X_1 \times_{X_0} \dots \times_{X_0} X_1$ is a weak equivalence.

Taking into account Reedy fibrantness, the second condition actually means that the map is a trivial Kan fibration.

6.6.2. Simplicial categories. Note that any simplicial category \mathcal{C} gives rise to a simplicial object $\mathcal{C}_* = \{\mathcal{C}_n\}$ in categories; all \mathcal{C}_n have the same objects, and $\text{Hom}_{\mathcal{C}_n}(x, y)$ is just the set of n -simplices of $\text{Map}_{\mathcal{C}}(x, y)$.

This defines a bisimplicial set by the formula

$$\tilde{\mathcal{C}}_{m,n} = N_m(\mathcal{C}_n),$$

the nerve of the corresponding category.

One can easily see (exercise) that for any $\mathcal{C} \in \mathbf{sCat}$ the Reedy fibrant replacement $\tilde{\mathcal{C}}^f$ of $\tilde{\mathcal{C}}$ is Segal. Furthermore, one can assume that $\tilde{\mathcal{C}}_0^f = \text{Ob}(\mathcal{C})$.

6.6.3. Remark. The bisimplicial set $\tilde{\mathcal{C}}$ defined above is fibrant in *projective model structure* on \mathbf{ssSet} . It satisfies the (second) Segal condition and has discrete zero component. Such bisimplicial spaces are called *Segal categories*. One can think of Segal categories as a weakened version of the notion of simplicial category. There should be no doubt: there is an approach to infinity categories based on Segal categories.

Some standard infinity-categorical notions make sense for general Segal spaces.

6.6.4. Objects. The set of objects of a Segal space X is X_{00} .

6.6.5. Space of arrows. By definition, the map

$$(27) \quad X_1 \rightarrow X_0 \times X_0$$

is a fibration. In particular, for any $x, y \in X_0$ the fiber of (27) at (x, y) denoted as $\text{Map}(x, y)$ is a Kan simplicial set.

6.6.6. Composition. The map $X_2 \rightarrow X_1 \times_{X_0} X_1$ is a trivial fibration, so admits a section. Composing it with $d_1 : X_2 \rightarrow X_1$, we get a (non-unique) composition.

6.6.7. Homotopy category. The set of objects of the homotopy category $\text{Ho}(X)$ of a Segal space X is X_{00} . For $x, y \in X_{00}$ we define $\text{Hom}_{\text{Ho}(X)}(x, y)$ as $\pi_0(\text{Map}_X(x, y))$. Non-unique composition defined above induces a unique (and associative) composition in the homotopy category (exercise).

An arrow $f \in \text{Map}_X(x, y)$ is an equivalence if its image in $\text{Ho}(X)$ is invertible. By definition, an arrow f is a point (0-simplex) in X_1 . We will show below (see 6.6.10) that the property of being equivalence, as defined above, depends on the connected component of X_1 only.

The fundamental groupoid $\Pi_1(X_0)$ has the same objects as $\text{Ho}(X)$. Actually, one has a canonical functor $\Pi_1(X_0) \rightarrow \text{Ho}(X)$ which is identity on objects. In fact, the sequence of maps

$$X_0 \xrightarrow{s_0} X_1 \xrightarrow{(d_1, d_0)} X_0 \times X_0$$

induces for any pair $(x, y) \in X_0 \times X_0$ a map of homotopy fibers

$$\text{Map}_{X_0}(x, y) \rightarrow \text{Map}_X(x, y).$$

6.6.8. *DK equivalence.* A map $f : X \rightarrow Y$ of Segal spaces is a DK equivalence if it is fully faithful and essentially surjective, that is

1. For any pair $x, y \in X$ the map $\text{Map}_X(x, y) \rightarrow \text{Map}_Y(fx, fy)$ is an equivalence.
2. For any $y \in Y$ there exists $x \in X$ and an equivalence $f(x) \rightarrow y$.

It is easy to see that any Reedy equivalence of Segal spaces is a DK equivalence. The converse is not true.

Example. Take, for example, a conventional category C . Let NC be the nerve of C (this is a simplicial set) and let us compare it to the Rezk nerve $B(C)$. One has an obvious map $d(NC) \rightarrow B(C)$. It is a DK equivalence but is not a Reedy equivalence.

This is the reason Segal spaces do not fit to be models for infinity categories: they have nonequivalent presentations to the same object.

There are two ways to correct this: one (probably, less interesting) is to work with Segal categories. The other one is to work with complete Segal spaces, CSS for short, see Definition 6.7.1 below.

6.6.9. *Equivalences.* Recall that an arrow f in a Segal space X is an equivalence if its image in $\text{Ho}(X)$ is invertible.

Let us reformulate this condition in terms of a lifting property. Since X is Reedy fibrant, $f \in \text{Map}_X(x, y)$ is an equivalence iff there exist $g, h \in \text{Map}_X(y, x)$ and homotopies $\text{id}_x \rightarrow hf$, $\text{id}_y \rightarrow fg$.

This can be encoded as follows. Let $Z \subset \Delta^3$ be the simplicial subset spanned by one-simplices $\{0, 2\}$, $\{1, 2\}$, $\{1, 3\}$. Any arrow $f \in \text{Map}_X(x, y)$ defines a map $z_f : d(Z) \rightarrow X$ carrying $\{1, 2\}$ to f , $\{1, 3\}$ to id_x and $\{0, 2\}$ to id_y . We claim that the arrow f is an equivalence iff the map $z_f : d(Z) \rightarrow X$ lifts to a map $d(\Delta^3) \rightarrow X$. The “if” implication is clear; let f be equivalence. This means z_f lifts to a map $y_f : d(D) \rightarrow X$ where D is the union of two faces of Δ^3 adjacent to the edge $\{1, 2\}$. In particular, left and right inverts g and h exist. Since X is Segal space, the triple (g, f, h) can be lifted to some $u \in X_3$. The image of u in $X(d(D))$ being homotopic to y_f , the homotopy can be lifted to X_3 .

6.6.10. **Lemma.** *If f and g belong to the same connected component of X_1 and if f is an equivalence, then g is also equivalence.*

Proof. The map $X_3 = \text{Map}(d(\Delta^3), X) \rightarrow \text{Map}(d(Z), X)$ is a fibration. If f and g belong to the same connected component of X_1 , there is a map $\Delta^1 \rightarrow \text{Map}(d(Z), X)$ connecting z_f with z_g . If z_f lifts to X_3 , we get a lifting for z_g from

the diagram

$$(28) \quad \begin{array}{ccc} [0] & \longrightarrow & X_3 \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ [1] & \longrightarrow & \text{Map}(d(Z), X) \end{array}$$

as the map $[0] \rightarrow [1]$ is a trivial cofibration. \square

6.6.11. *The space of equivalences.* Taking into account Lemma 6.6.10, we define the space of equivalences X^{eq} of a Segal space X as the subspace of X_1 spanned by the equivalences.

The degeneracy $s_0 : X_0 \rightarrow X_1$ carries any object to an equivalence. Thus, s_0 factors through $s : X_0 \rightarrow X^{eq}$.

6.7. Completeness.

6.7.1. **Definition.** A Segal space X is *complete* if $s : X_0 \rightarrow X^{eq}$ is an equivalence.

Complete Segal spaces (CSS) will be now our models for infinity categories. If a Segal space X is complete, the fundamental groupoid $\Pi_1(X_0)$ identifies with the maximal subgroupoid of $\text{Ho}(X)$.

Note that, for X complete, the space of equivalences $\text{Map}_X^{eq}(x, y)$ defined as the fiber of $X^{eq} \rightarrow X_1 \rightarrow X_0 \times X_0$ at (x, y) , is equivalent to the space of paths from x to y in X_0 . In fact, since $X_0 \rightarrow X^{eq}$ is an equivalence, the space $\text{Map}_X^{eq}(x, y)$ can be also described as the homotopy fiber of the diagonal $X_0 \rightarrow X^{eq} \rightarrow X_0 \times X_0$.

6.7.2. *DK equivalence.* If $f : X \rightarrow Y$ is a Reedy equivalence of Segal spaces, then it is obviously a DK equivalence. We will now show that the converse holds if X and Y are CSS.

Let us, first of all, prove that $f_0 : X_0 \rightarrow Y_0$ is a homotopy equivalence. Since f induces an equivalence of the homotopy categories, one has a bijection $\pi_0(X_0) \rightarrow \pi_0(Y_0)$.

Furthermore, for any pair x, x' one has an homotopy equivalence $\text{Map}_X(x, x') \rightarrow \text{Map}_Y(fx, fx')$ which has to preserve the components describing equivalences. By completeness, these are precisely the spaces of paths from x to x' in X_0 and from fx to fx' in Y_0 respectively. This proves that f_0 is an equivalence.

Finally, in the commutative diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow & & \downarrow \\ X_0^{n+1} & \xrightarrow{f_0^{n+1}} & Y_0^{n+1} \end{array}$$

the lower horizontal arrow, as well as the fibers are weakly homotopy equivalent. This implies that f_n is also a weak equivalence.

6.8. Classification diagram of a relative category. DK localization assigns to a category \mathcal{C} endowed with a subcategory W of “weak equivalences”, a simplicially enriched category denoted $L^H(\mathcal{C}, W)$. There is a similar construction in the framework of complete Segal spaces. We will now describe it.

6.8.1. Here is the construction. Recall that for a category C we assign its Rezk nerve BC whose n -simplices form the Kan simplicial space $N(\text{Fun}([n], C)^{\text{iso}})$.

More generally, for a pair (C, W) , with W subcategory of C containing all isomorphisms, we define $B(C, W)$ as the bisimplicial space defined by the formula $B(C, W)_n = N(\text{Fun}([n], (C, W)))$. Here $\text{Fun}([n], (C, W))$ is the category whose objects are the functors $[n] \rightarrow C$ and whose arrows are the pointwise weak equivalences of the functors. Thus, $B(C, W)_{n,m}$ is the set of commutative diagrams of rectangular shape $n \times m$, with vertical arrows belonging to W .

We cannot expect $B(C, W)$ to be a Segal space as the 0-th component is $N(W)$ which is not Kan, and so $B(C, W)$ is not even Reedy fibrant. On the other hand, it is easy to see (exercise) that $X = B(C, W)$ satisfies Condition 2 of the definition of Segal space.¹⁸

Exercise. Verify condition 2.

One has (see [R], Theorem 8.3, [B2], 6.2).

6.8.2. Theorem. *Let (\mathcal{C}, W) be a model category with weak equivalences W . Then $B(\mathcal{C}, W)^f$, a Reedy fibrant replacement of $B(\mathcal{C}, W)$, is a CSS.*

The result was proven by Rezk in the case of simplicial model category, and by Bergner for general model categories.

6.9. CSS model structure. We defined CSS as Reedy fibrant bisimplicial sets satisfying some extra properties. It is worth mentioning that there is a model category structure on \mathbf{ssSet} (called CSS model structure) for which complete Segal spaces are precisely the fibrant objects.

Here is the result (see [R], Thm. 7.2).

6.9.1. Theorem. *There exists a simplicial cartesian (see below) model category structure on \mathbf{ssSet} such that*

1. *Cofibrations are monomorphisms.*
2. *Fibrant objects are precisely the complete Segal spaces.*
3. *A map f is a weak equivalence iff $\text{Map}(f, X)$ is a weak equivalence of simplicial sets for any complete Segal space X .*
4. *Reedy weak equivalence in \mathbf{ssSet} is a weak equivalence in CSS structure; the converse holds for a map between two CSS.*

¹⁸It is, however, not necessarily true that the property will persist if one makes a Reedy fibrant replacement.

A model category M is called *cartesian* if it admits internal Hom (that is, direct product has right adjoint), and the following axiom is fulfilled.

Let $i : A \rightarrow B$ be a cofibration and $f : X \rightarrow Y$ be a fibration. Then the map

$$\Phi(i, f) : X^B \rightarrow X^A \times_{Y^A} Y^B$$

is a fibration, trivial if one of the maps i, f is a weak equivalence.

The model structure described in the theorem is constructed as *Bousfield localization* of the Reedy model structure. We present below some details about Bousfield localization. Cartesian structure is important and its proof is a separate story (see below).

6.9.2. *Bousfield localization — generalities.* Let M be a model category. Its left Bousfield localization is another model structure on M , denoted M_{loc} , such that one has a Quillen pair

$$\text{id} : M \rightleftarrows M_{\text{loc}} : \text{id}.$$

This immediately implies that M and M_{loc} have the same cofibrations (Exercise). Furthermore, M_{loc} has more weak equivalences and less fibrations.

We present a very general setup where such localization exists. The theorem below is usually attributed to Jeff Smith. A proof can be found in [L.T], A.3.7. An earlier version for cellular model categories is in [Hir].

Theorem. *Let \mathcal{C} be a left proper simplicial combinatorial model category. Let S be a set of cofibrations with cofibrant source. The following sequence of definitions determines a new simplicial model category structure on \mathcal{C} (denoted $L_S(\mathcal{C})$), in which S is added to a collection of trivial cofibrations.*

1. *Fibrant objects in $L_S(\mathcal{C})$ are the fibrant objects of \mathcal{C} satisfying the RLP with respect to S .*
2. *A map $f : X \rightarrow Y$ between cofibrant objects is a weak equivalence in $L_S(\mathcal{C})$ if $\text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$ is a homotopy equivalence for all Z fibrant in $L_S(\mathcal{C})$.*

6.9.3. *Two Bousfield localizations.* We start with the category of bisimplicial sets endowed with the Reedy model structure.

Choose the set S of maps to consist of the maps $d(\text{Sp}(n)) \rightarrow d(\Delta^n)$, $n \geq 2$. The Bousfield localization of the Reedy model structure with respect to S will have Segal spaces as fibrant objects.

We will now add one more arrow to S .

Recall the definition of embedding $Z \rightarrow \Delta^3$. Here Z is the simplicial subset of Δ^3 spanned by the edges $\{0, 2\}, \{1, 2\}, \{1, 3\}$. Define \bar{Z} by contracting the edges $(0, 2)$ and $(1, 3)$. Similarly, define $\bar{\Delta}^3$ by contracting the edges $(0, 2)$ and $(1, 3)$. Then one has a trivial cofibration $i : \Delta^1 = \bar{Z} \rightarrow \bar{\Delta}^3$. The induced map $\text{Map}(d(\bar{\Delta}^3), X) \rightarrow X_1$ is a fibration with the image X^{eq} , so one has a fibration $\text{Map}(d(\bar{\Delta}^3), X) \rightarrow X^{eq}$. One has

Lemma. *This is a trivial fibration.*

We omit the proof; it can be found in [L.G], 1.1.13. The above lemma implies that a Segal space X is complete iff the embedding $\Delta^0 = \{1\} \rightarrow \bar{\Delta}^3$ induces an equivalence $X(\bar{\Delta}^3) \rightarrow X_0$.

Thus, we add to S the map $d([0]) \rightarrow d(\bar{\Delta}^3)$ ¹⁹

The CSS model structure is obtained by Bousfield localization along S .

6.9.4. *Cartesian structure.* The category \mathbf{ssSet} has internal Hom. Cartesian property of a model category M , having internal Hom, can be reformulated as follows.

Given $i : A \rightarrow B$ and $j : C \rightarrow D$ two cofibrations. Then the map

$$(A \times D) \prod^{A \times C} (B \times C) \rightarrow B \times D$$

is a cofibration; it is trivial if i or j is trivial.

Exercise. Verify the equivalence of this reformulation with the original definition of cartesian model category.

The Reedy model category structure is cartesian as \mathbf{sSet} is cartesian and weak equivalences in the Reedy structure are defined componentwise.

The CSS model structure is also compatible with the internal Homs. This is not at all obvious, but true — for the proof see [R], 7.2, 9.2, 12.1. An immediate consequence, we have

Theorem. *Let X be a CSS. Then for each $K \in \mathbf{ssSet}$ the bisimplicial set X^K is a CSS.*

6.9.5. *DK equivalence of Segal spaces.* We know that DK equivalence of CSS coincides with Reedy equivalence. We also know that this is not true for general Segal spaces: Rezk nerve $B(C)$ of a category is DK equivalent to the discrete nerve $d(N(C))$, but they are not Reedy equivalent. It turns out that a DK equivalence of Segal spaces is CSS equivalence. This result follows from an explicit construction of “completion” of a Segal space.

One has (see [R], Sect. 14)

Theorem. *Let X be a Segal space. There is a map $i : X \rightarrow \widehat{X}$ such that*

1. \widehat{X} is a CSS.
2. i is a weak equivalence in CSS model structure.
3. i is a DK equivalence.

¹⁹Rezk [R] uses the discrete nerve of the contractible groupoid with two objects instead of $\bar{\Delta}^3$.

The result is nontrivial. The “completion” construction is a Reedy fibrant replacement of an explicitly defined bisimplicial set \tilde{X} . Here is the formula

$$\tilde{X}_{m,n} = \text{Hom}(d(\Delta^m) \times d(E^n) \times c(\Delta^n), X),$$

where E^n is the (discrete) nerve of the contractible groupoid on objects $0, \dots, n$.

Note that if $X = d(N(C))$ is the discrete nerve of a conventional category C , $\tilde{X}_{m,n} = \text{Hom}(d(\Delta^m \times E^n), X)$ coincides with the Rezk nerve of C . Thus, the above construction is a generalization of the construction of Rezk nerve.

6.10. Equivalence of various models for ∞ -categories. The category of simplicial sets with Joyal model structure is Quillen equivalent to simplicial categories with Bergner model structure, see 5.3.1. Joyal model structure is also Quillen equivalent to \mathbf{ssSet} with CSS model structure (see Joyal-Tierney [JT]): the right Quillen functor carries a bisimplicial set $X \in \mathbf{ssSet}$ to $X_{\bullet,0}$.

As we already know, a Quillen equivalence gives rise to an equivalence of the underlying infinity categories.

This is true regardless of the model we are using. Thus, different models for infinity categories have the same underlying infinity category, regardless of the models used.

It is worthwhile to note that the graph of Quillen equivalences between different models infinity categories is very far from being a tree, so that one has more than one way to connect two models with a sequence of Quillen equivalences. Fortunately, Toen’s result [T] says that different Quillen equivalences define basically the same equivalence of infinity categories.

Here is how it is done. Toen calculates the (homotopy) automorphisms of $L^H(\mathbf{sCat})$ considered as an object of $L^H(\mathbf{sCat})$ (it is worthwhile here to take care of the universes). The resulting simplicial group is equivalent to \mathbb{Z}_2 , with the nontrivial automorphism describing $\mathcal{C} \mapsto \mathcal{C}^{\text{op}}$.

6.10.1. Rezk nerve for relative categories. It turns out that, in general, CSS fibrant replacement of $B(C, W)$ is equivalent to $L^H(C, W)$ (composed with a fibrant replacement to complete Segal spaces).

Unfortunately, we do not know a direct proof of this.

In the series of papers by Barwick and Kan (see [BK1, BK2]) a model structure is introduced on the category of “relative categories”. Barwick and Kan also prove that both Rezk’s nerve functor and hammock localization are Quillen equivalences.

The result then follows once more from the Toen’s work [T].

6.11. Exercise.

1. See 6.4.2.
2. See 6.6.2.
3. See 6.6.7.

4. See 6.8.1.
5. Verify that the Rezk nerve $B(C)$ of a conventional category C is a CSS.
6. When the “standard” nerve $d(NC)$ is a CSS?

7. LEFT FIBRATIONS. GROTHENDIECK CONSTRUCTION.

7.1. Conventional notions. The notion of fibered category was suggested by Grothendieck in 1959 and developed in SGA1. A typical example: assignment of the category of quasicoherent sheaves to a scheme.

Morally, one has a functor $QC : Sch^{op} \rightarrow \mathbf{Cat}$ assigning to any scheme X the category $QC(X)$ of quasicoherent sheaves on X and to any map $f : X \rightarrow Y$ the inverse image functor $f^* : QC(Y) \rightarrow QC(X)$. But if one looks carefully, this assignment does not preserve composition; there is a natural equivalence $(fg)^* \sim g^*f^*$ but no equality.

This teaches us that one should be careful talking about functors to \mathbf{Cat} : the latter is a two-category and even when we are talking about functors from a conventional category C to \mathbf{Cat} , we should allow weakened notion of functor, preserving compositions up to a natural equivalence.

Another (but similar) notion formalizes the idea of functor to the category of groupoids. This is especially important in deformation theory. It can be described as follows.

Assume we want to talk about formal deformations of a scheme X over a field k . This means that, for each artinian local ring (A, \mathfrak{m}) such that $A/\mathfrak{m} = k$ (*Artinian* means here that \mathfrak{m} is finitely generated and nilpotent) we have the groupoid of A -schemes \tilde{X} endowed with an isomorphism $\tilde{X} \otimes_A k \rightarrow X$. This is a covariant functor from artinian algebras to groupoids, with the same *caveat* as one had for quasicoherent sheaves: there is no full compatibility with the compositions.

The first example (QC sheaves) leads to the notion of *cocartesian fibration* (Grothendieck's term: *catégorie cofibrée*), while the deformation functor is an example of *left fibration* (Grothendieck's *catégorie cofibrée en groupoides*).

Left fibrations are a special case of cocartesian fibrations. We will study first left fibrations, and, later on, cocartesian fibrations.

7.1.1. Formal definition. A functor $f : C \rightarrow D$ between categories is a left fibration ("category cofibered in groupoids") if (the corresponding nerve) satisfies RLP with respect to $\{0\} \rightarrow \Delta^1$ and to $\Lambda_0^2 \rightarrow \Delta^2$. In other words, if for any $x \in C$ any arrow $f(x) \rightarrow d$ has a lifting to an arrows $x \rightarrow y$ in C , and for any pair $a : x \rightarrow y$, $b : x \rightarrow z$ of arrows in C there is a bijection between $c : y \rightarrow z$ such that $ca = b$, and $\bar{c} : f(y) \rightarrow f(z)$ such that $\bar{c}f(a) = f(b)$.

7.1.2. Grothendieck construction in the conventional context. Given $f : C \rightarrow D$ as above, we can construct $F : D \rightarrow \mathbf{Grp}$ as follows. We assign to $d \in D$ the fiber $F(d) = f^{-1}(d)$. Let us verify that the fiber of a left fibration is a groupoid. This follows from definition where we choose $x = z$ and y having the same image in D and $b = \text{id}$. Now, for any arrow $d \rightarrow d'$ we have to present a functor from $F(d)$ to $F(d')$. This is done as follows. Given $c \in C$, $d = f(c)$, and $\phi : d \rightarrow d'$, we lift ϕ to an arrow $\tilde{\phi}$ and define $\phi_*(x)$ as the target of $\tilde{\phi}$. This uniquely extends to a

functor $\phi_* : F(d) \rightarrow F(d')$ which is unique up to unique isomorphism. This sort of uniqueness leads to the minor lack of associativity: instead of equality

$$\phi_* \circ \psi_* = (\phi \circ \psi)_*$$

we have a canonical isomorphism

$$(29) \quad \theta_{\phi, \psi} : \phi_* \circ \psi_* \rightarrow (\phi \circ \psi)_*$$

satisfying (as a result of canonicity) some standard compatibility.

Grothendieck construction has an inverse. In the opposite direction the construction assigns to each pseudofunctor $F : D \rightarrow \mathbf{Grp}$ (consisting of groupoids $F(d)$ for all $d \in D$, functors $\phi_* : F(d) \rightarrow F(d')$ for all $\phi : d \rightarrow d'$ and isomorphisms of functors $\theta_{\phi, \psi}$ satisfying compatibility), a left fibration $f : C \rightarrow D$. We will present this construction only in the case $F : D \rightarrow \mathbf{Grp}$ is a functor, that is when $\theta_{\phi, \psi} = \text{id}$ for all ϕ, ψ . The respective left fibrations are called *split*.

Here is the construction. The objects of C are the disjoint union $\sqcup \text{Ob}(F(d))$. Morphisms from $x \in F(d)$ to $y \in F(d')$ are pairs (ϕ, α) where $\phi : d \rightarrow d'$ in D and $\alpha : \phi_*(x) \rightarrow y$ an arrow in $F(d')$.

7.1.3. Exercise. Let $f : G \rightarrow H$ be a surjective homomorphism of groups which has no splitting. Then $Bf : BG \rightarrow BH$ is a left fibration which is not equivalent to a split fibration.

7.1.4. Why do we care about left fibrations? One of the most important constructions for the development of classical category theory is the functor

$$(30) \quad C^{\text{op}} \times C \rightarrow \mathbf{Set},$$

assigning to a pair of objects (x, y) the set $\text{Hom}_C(x, y)$. This functor can be interpreted as Yoneda map whose properties (Yoneda lemma) allow one to use universal constructions.

Existence of universal constructions is the most important feature of category theory. Thus, there cannot be understanding of infinity categories without infinity categorical version of Yoneda lemma.

In the infinity categorical version of (30) \mathbf{Set} should be replaced with \mathcal{S} , but also defining a functor now means a lot of coherence data; it turns that one can avoid explicit description of the functor.

It turns out that it is very easy (in any model) to construct a left fibration $X \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$ with fiber $\text{Map}(x, y)$ at $(x, y) \in \mathcal{C}^{\text{op}} \times \mathcal{C}$, for any infinity category \mathcal{C} . This means that, if we have a way to convert left fibrations into functors to \mathcal{S} , we can get Yoneda embedding for free.

Thus, we need left fibrations for two (interconnected) reasons:

- To have an infinity-categorical version of Grothendieck's notion of category cofibered in groupoids.
- To describe Yoneda embedding.

There is a notion of right fibration corresponding to Grothendick's categories *fibered* in groupoids.

7.2. Left fibrations for CSS. We will define now left fibrations in the language of CSS and try to formulate the Grothendieck construction in this setting.

7.2.1. Definition. A Reedy fibration $f : X \rightarrow Y$ of bisimplicial sets is a left fibration if the induced map

$$\mathrm{Fun}(d(\Delta^1), X) \rightarrow X \times_Y \mathrm{Fun}(d(\Delta^1), Y)$$

is a trivial fibration.

Here are some basic properties of left fibrations.

7.2.2. Lemma. *Base change of a left fibration is a left fibration.*

□

7.2.3. Lemma. *If $f : X \rightarrow Y$ is a left fibration, $f^Z : X^Z \rightarrow Y^Z$ is also a left fibration.*

□

7.2.4. Lemma. *The following conditions on $f : X \rightarrow Y$ are equivalent.*

1. f is a left fibration.
2. for any n the map

$$\mathrm{Fun}(d(\Delta^n), X) \rightarrow X \times_Y \mathrm{Fun}(d(\Delta^n), Y)$$

is a trivial fibration.

3. For any n the map $X_n \rightarrow X_0 \times_{Y_0} Y_n$ induced by the embedding $[0] = \{0\} \in [n]$, is a trivial fibration.

7.2.5. Corollary. *A map $X \rightarrow *$ is a left fibrations iff X is fibrant and $X_0 \rightarrow X_n$ are trivial cofibrations. In other words, X is equivalent to $c(X_0)$ where X_0 is Kan.*

7.2.6. Exercise. Using Lemma 7.2.4 (3), prove that

- If $f : X \rightarrow Y$ is a left fibration and Y is a CSS then X is a CSS.
- $f : X \rightarrow Y$ is a left fibration iff any base change $X \times_Y \Delta^{m,n} \rightarrow \Delta^{m,n}$ is a left fibration.

7.2.7. Lemma. *A morphism $f : X \rightarrow X'$ of left fibrations over Y is a weak equivalence iff the respective map of fibers $f_y : (X_y)_0 \rightarrow (X'_y)_0$ is a weak equivalence for each $y \in Y$.*

About the proofs: all easy except the implication $3 \Rightarrow 1$ of 7.2.4 which is first proven for fibrant Y ; the general case is deduced with the help of the following result on extension of fibrations.

7.2.8. **Lemma.** *Given a commutative diagram*

$$(31) \quad \begin{array}{ccc} Y_A & \xrightarrow{i} & X_B \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

where the vertical arrows are fibrations, the horizontal arrows are cofibrations, such that the induced map

$$Y_A \rightarrow X_A = A \times_B X_B$$

is a trivial cofibration. Then there exists a largest simplicial subspace Y_B of X_B such that i factors through $Y_B \rightarrow X_B$ and induces an isomorphism $Y_A = A \times_B Y_B$. Moreover, $Y_B \rightarrow X_B$ is a strong deformation retract over B , so that $Y_B \rightarrow B$ is a fibration and $Y_B \rightarrow X_B$ is a trivial cofibration.

Proof. One defines Y_B as the biggest simplicial subspace of X_B such that its intersection with X_A is in Y_A . Then one verifies the properties of Y_B . \square

7.3. **Grothendieck construction.** Our aim is to compare totality of left fibrations with a given base B and totality of functors $B \rightarrow \mathcal{S}$, where \mathcal{S} is an infinity category of “spaces”.

The approach we are going to take (due to [KV, KV1]) is as follows. First of all, we will represent the functor $B \mapsto \{\text{left fibrations over } B\}$ as the functor with the values in sets. This seems very naive, but easy. Then we will study the representing object (denoting it \mathcal{S}) and will understand that it actually solves a much more meaningful problem.

The notion of “set of left fibrations based on B ” sounds very weird, because of set-theoretical difficulties involved. The difficulties are not more serious than when we write $N(\mathbf{Set})$ and can be completely avoided working with a pair of universes, one an element of the another. We will present an approach which shows that one can stay completely in the framework of naive set theory, simply restricting the size of some sets involved.

Fix a set \mathcal{U} of infinite cardinality α .

A \mathcal{U} -marking of a map $f : Z \rightarrow X$ in \mathbf{ssSet} is an assignment, for each $x : \Delta^{m,n} \rightarrow X$, of an injective map of the set $f^{-1}(x) \in Z_{m,n}$ into \mathcal{U} .

7.3.1. *Construction of \mathcal{S}^α .* This is a bisimplicial set. Define a set $\mathcal{S}_{m,n}^\alpha$ as follows. This is π_0 of the following groupoid (all components are contractible; the idea of taking π_0 is that the objects of the groupoid form a class, but π_0 is a set).

The objects of the groupoid are \mathcal{U} -marked left fibrations $Z \rightarrow \Delta^{m,n}$. Isomorphisms between two such fibrations are isomorphisms compatible with the \mathcal{U} -marking. Functoriality with respect to (m, n) is obvious as the \mathcal{U} -marking of a fiber product $Z' = Z \times_{\Delta^{m,n}} \Delta^{m',n'}$ is inherited from the \mathcal{U} -marking of $Z \rightarrow \Delta^{m,n}$.

The bisimplicial set \mathcal{S}^α constructed as above admits a left fibration $\mathcal{E}^\alpha \rightarrow \mathcal{S}^\alpha$ glued from the left fibrations $E \rightarrow \Delta^{m,n}$ that are the elements of $\mathcal{S}_{m,n}^\alpha$. This is a *universal left fibration* in the sense of the following (obvious) lemma.

7.3.2. Lemma. *For any bisimplicial set B the base change determines a bijection between the set of \mathcal{U} -marked left fibrations on B and the set $\text{Hom}(B, \mathcal{S}^\alpha)$.*

Of course, we would like to have equivalence of infinity-categories instead of bijection of sets. The right-hand side is the $(0,0)$ -set of a complete Segal space, at least once we verify that \mathcal{S}_α is a CSS (see 7.4.1). The left-hand side is yet to be defined as an infinity-category.

7.3.3. Remark. Note that the bisimplicial set \mathcal{S}^α defined above has (formally) nothing to do with the infinity category of spaces \mathcal{S} we gave earlier as an example. We will be able to prove they are equivalent once we have the equivalence between the category of left fibrations on B and the category of functors $B \rightarrow \mathcal{S}^\alpha$ (see 7.4.2) — as the special case of the equivalence for $B = *$.

7.4. Category of left fibrations. For $B \in \mathbf{ssSet}$ we will define the CSS $\text{Left}_\alpha(B)$ as the CSS version of the simplicial subcategory $\mathfrak{L}(B)$ of $\mathbf{ssSet}/_B$ spanned by the left fibrations $X \rightarrow B$ with fibers of cardinality $\leq \alpha$. This means the following. Let $L(B)$ be the bisimplicial set corresponding to $\mathfrak{L}(B)$. Then $L(B)$ is a Segal space and $\text{Left}_\alpha(B)$ is its completion in the sense of 6.9.5.

Theorem 7.4.2 below claims that \mathcal{S}^α represents the functor $B \mapsto \text{Left}_\alpha(B)$.

First of all, it is good to know the following.

7.4.1. Proposition. *\mathcal{S}^α is a complete Segal space.*

This will imply that $\text{Fun}(B, \mathcal{S}^\alpha)$ is a CSS. Finally, one has

7.4.2. Theorem. *For any B one has an equivalence of CSS*

$$\text{Left}_\alpha(B) \rightarrow \text{Fun}(B, \mathcal{S}^\alpha).$$

In what follows we will suppress α from the notation.

The right-hand side is a bisimplicial set R given explicitly by the formula $R_{m,n} = \text{Hom}(B \times \Delta^{m,n}, \mathcal{S})$. According to Lemma 7.3.2, this is the set of left fibrations on $B \times \Delta^{m,n}$.

We will have to compare this CSS with another one, $\text{Left}(B)$. One cannot expect them to be literally isomorphic — so it is a bad idea to simply compare the sets of (m, n) -simplices.

Taking this into account, it makes sense to define some more general bisimplicial spaces $\mathcal{S}^{(k)}$ whose set of (m, n) -simplices is the set of diagrams $E_0 \rightarrow \dots \rightarrow E_k$ of left fibrations over $\Delta^{m,n}$ ²⁰.

²⁰We understand the diagram as a commutative diagram in \mathbf{ssSet} .

7.4.3. *Reedy fibrantness of \mathcal{S} .* It is enough to verify that for a generating set of trivial cofibrations $A \rightarrow B$ any left fibration on A extends to a left fibration on B . We can think $B = \Delta^{m,n}$. Let $X \rightarrow A$ be a left fibration.

We can decompose the composition $X \rightarrow A \rightarrow B$ to a trivial cofibration followed by a fibration $X \rightarrow Y \rightarrow B$. The map $X \rightarrow Y_A = X \times_A Y$ is injective. It is also weak equivalence since Reedy model structure is right proper. Thus, it is trivial cofibration. Now, we cannot expect $Y \rightarrow B$ to be automatically a left fibration. Instead, we use Lemma 7.2.8 to find a subobject X_B of Y satisfying the following properties.

- $X = A \times_B X_B$.
- $X \rightarrow X_B$ is a trivial cofibration.
- $X_B \rightarrow B$ is a fibration.

It remains to prove $X_B \rightarrow B$ is a left fibration. This can be seen from the following commutative diagram.

$$(32) \quad \begin{array}{ccc} X_n & \longrightarrow & (X_B)_n \\ \downarrow & & \downarrow \\ X_0 \times_{A_0} A_n & \longrightarrow & (X_B)_0 \times_{B_0} B_n \end{array} .$$

Since $X \rightarrow A$ and $X_B \rightarrow B$ are fibrations, the upper and the lower horizontal arrows are weak equivalences. Since the left vertical arrow is a trivial fibration, so is the right vertical arrow.

The following construction is very instrumental in studying further properties of \mathcal{S} .

7.4.4. *Cylinder.* Let \mathcal{C} be a simplicial category such that the simplicial functor

$$y \mapsto \text{Map}(K, \text{Map}_{\mathcal{C}}(x, y))$$

is corepresentable for all $K \in \mathbf{sSet}, x \in \mathcal{C}$; the corepresenting object will be denoted $K \otimes x$. We will also use the notation $d(K) = K \otimes *$ where $*$ is a terminal object of \mathcal{C} .

We will apply the construction below to the category \mathbf{ssSet} , with the simplicial structure given by $K \otimes X = d(K) \times X$. In this way two meanings of the notation d (as a functor $\mathbf{sSet} \rightarrow \mathcal{C}$ and as the already defined functor $\mathbf{sSet} \rightarrow \mathbf{ssSet}$) coincide.

Given a sequence of maps

$$s : \mathcal{E}_0 \rightarrow \dots \rightarrow \mathcal{E}_n$$

we will construct a new object $\text{Cyl}(s)$ endowed with canonical maps

$$\Delta^{n-k} \otimes \mathcal{E}_k \rightarrow \text{Cyl}(s)$$

. Here is the definition.

- If $n = 0$, $\text{Cyl}(s) = \mathcal{E}_0$.
- If $n = 1$, $\text{Cyl}(s) = (\Delta^1 \otimes \mathcal{E}_0) \amalg^{\mathcal{E}_0} \mathcal{E}_1$, where the map $\mathcal{E}_0 \rightarrow \Delta^1 \otimes \mathcal{E}_0$ is induced by $\{1\} \rightarrow [1]$.
- In general, by induction,

$$\text{Cyl}(s) = (\Delta^n \otimes \mathcal{E}_0) \amalg^{\Delta^{n-1} \otimes \mathcal{E}_0} \amalg \text{Cyl}(s_{\geq 1}),$$

where the map $[n-1] \rightarrow [n]$ is d_0 and the map $\Delta^{n-1} \otimes \mathcal{E}_0 \rightarrow \text{Cyl}(s_{\geq 1})$ is induced by the map $\mathcal{E}_0 \rightarrow \mathcal{E}_1$.

The construction of $\text{Cyl}(s)$ is functorial in s in two senses; first of all, for fixed n , it is functorial with respect to maps of sequences. In particular, for $s_0 : * \rightarrow \dots \rightarrow *$, we get $\text{Cyl}(s_0) = d(\Delta^n)$, so we have a canonical map $\text{Cyl}(s) \rightarrow d(\Delta^n)$.

The second type of functoriality is with respect to the simplicial operations $a : [m] \rightarrow [n]$. If $s : \mathcal{E}_0 \rightarrow \dots \mathcal{E}_n$ is a sequence of maps, one has $a^*(s) : \mathcal{E}_{a(0)} \rightarrow \dots \rightarrow \mathcal{E}_{a(m)}$ and this leads to a natural isomorphism

$$(33) \quad \text{Cyl}(a^*(s)) \rightarrow d(\Delta^m) \times_{d(\Delta^n)} \text{Cyl}(s).$$

7.4.5. The map $\text{Cyl}(s) \rightarrow d(\Delta^n)$ defined above is very nice homotopically, for instance, if \mathcal{E}_i are spaces, $\text{Cyl}(s)$ becomes a left fibration after Reedy fibrant replacement. Thus, its only drawback is that it is not Reedy fibration. Fortunately, it is a quasifibration in the sense that we will now define, and quasifibrations are almost as good as fibrations.

7.5. **Quasifibrations.** Let \mathcal{C} be a right proper model category (as \mathbf{sSet} or \mathbf{ssSet} with Reedy model structure). A map $f : X \rightarrow Y$ is called *quasifibration* if for any weak equivalence $Z \rightarrow T$ over Y the base change $X_Z \rightarrow X_T$ is a weak equivalence. Note that this is not the standard notion of quasifibration; we took it from [KV].

Since \mathcal{C} is right proper, fibrations are quasifibrations. An example of quasifibration which is not a fibration: the projection $X = Y \times Z$ with Z not fibrant.

One can easily prove

7.5.1. **Lemma.** $f : X \rightarrow Y$ in \mathbf{ssSet} is a quasifibration iff $f_n : X_n \rightarrow Y_n$ are quasifibrations in \mathbf{sSet} .

We will now define left quasifibration as a quasifibration $f : X \rightarrow Y$ for which the embedding $[0] = \{0\} \rightarrow [n]$ induces, for each n , a weak equivalence

$$X_n \rightarrow X_0 \times_{Y_0} Y_n.$$

7.5.2. **Lemma.** *Given a commutative diagram*

$$(34) \quad \begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow f & & \downarrow f' \\ Y & \xrightarrow{h} & Y' \end{array}$$

with g, h weak equivalences and f, f' quasifibrations. Then f is a left quasifibration iff f' is a left quasifibration.

Proof. Exercise. □

7.5.3. **Lemma.** 1. *Let $s : E_0 \rightarrow \dots \rightarrow E_n$ be a sequence of left fibrations over B . Then the cylinder $\text{Cyl}(s) \rightarrow B \times d(\Delta^n)$ is a left quasifibration. Thus, its Reedy fibrant replacement is a left fibration.*

2. *Conversely, any left fibration $E \rightarrow B \times d(\Delta^n)$, such that E_i is the fiber of E at $\{i\} \in [n]$, is equivalent to $\text{Cyl}(s)$ for some sequence of maps $s : E_0 \rightarrow \dots \rightarrow E_n$ of left fibrations over B .*

3. *Moreover, the maps $E_i \rightarrow E_{i+1}$ are defined uniquely up to homotopy.*

Proof. Here is a key observation. Given a left fibration $E \rightarrow B \times d(\Delta^1)$ and a map $X \rightarrow B$, the restriction

$$\text{Map}_{B \times d(\Delta^1)}(X \times d(\Delta^1), E) \rightarrow \text{Map}_{B \times d(\Delta^1)}(X, E)$$

induced by embedding $\{0\} \rightarrow \Delta^1$, is a trivial fibration. Thus, if E_i is the fiber of E at $i = 0, 1$, the embedding $E_0 \rightarrow E$ extends, essentially uniquely, to $E_0 \times d(\Delta^1) \rightarrow E$ which yields, in particular, $E_0 \rightarrow E_1$. □

7.6. Highlights of the proof.

7.6.1. Recall that the bisimplicial sets \mathcal{S} and $\mathcal{S}^{(n)}$ are defined explicitly by description of the sets of their (p, q) -simplices as sets of left fibrations (resp., sequences $E_0 \rightarrow \dots \rightarrow E_n$ of left fibrations) on $\Delta^{p,q}$.

We will now use the construction of $\text{Cyl}(s)$ to construct an (essentially unique) map $\psi^{(n)} : \mathcal{S}^{(n)} \rightarrow \text{Fun}(d(\Delta^n), \mathcal{S})$.

Equivalently, we have to present a map $\mathcal{S}^{(n)} \times d(\Delta^n) \rightarrow \mathcal{S}$, or, in other words, a left fibration on $\mathcal{S}^{(n)} \times d(\Delta^n)$.

One has a universal sequence of left fibrations on $\mathcal{S}^{(n)}$,

$$(35) \quad \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \dots \rightarrow \mathcal{E}_n,$$

classified by the identity map on $\mathcal{S}^{(n)}$. Applying the cylinder construction to the category $\mathbf{ssSet}_{/\mathcal{S}^{(n)}}$ and to the sequence (35) in it, we get a left quasifibration on $\mathcal{S}^{(n)} \times d(\Delta^n)$. Its Reedy fibrant replacement \mathcal{E} can be chosen so that the restriction of \mathcal{E} to $\mathcal{S}^{(n)} \times \{k\}$ is precisely \mathcal{E}_k .

The maps $\psi^{(n)}$ are essentially unique. So, it is natural that any map $a : [m] \rightarrow [n]$ gives rise to a homotopy commutative diagram

$$(36) \quad \begin{array}{ccc} \mathcal{S}^{(n)} & \xrightarrow{\psi^{(n)}} & \text{Fun}(d(\Delta^n), \mathcal{S}) \\ \downarrow & & \downarrow \\ \mathcal{S}^{(m)} & \xrightarrow{\psi^{(m)}} & \text{Fun}(d(\Delta^m), \mathcal{S}) \end{array}$$

7.6.2. Let us explain how to verify that $\psi^{(n)} : \mathcal{S}^{(n)} \rightarrow \text{Fun}(d(\Delta^n), \mathcal{S})$ is a homotopy equivalence over \mathcal{S}^{n+1} . It is sufficient to verify that, for each $\eta : K \rightarrow \mathcal{S}^{n+1}$, the map

$$(37) \quad \pi_0(\text{Map}_{\mathcal{S}^{n+1}}(K, \mathcal{S}^{(n)})) \rightarrow \pi_0(\text{Map}_{\mathcal{S}^{n+1}}(K, \text{Fun}(d(\Delta^n), \mathcal{S}))),$$

induced by $\psi^{(n)}$, is a bijection.

Now, all objects involved have a modular interpretation. The map η is given by a collection of left fibrations H_0, \dots, H_n on K . An element of the right-hand side of (37) is given by a sequence

$$H_0 \rightarrow \dots \rightarrow H_n$$

of maps between these left fibrations, whereas an element of the left-hand side corresponds to a left fibration $H \rightarrow K \times d(\Delta^n)$. Bijectivity of (37) now follows from Lemma 7.5.3.

7.6.3. *Proof of 7.4.1.* Let us prove \mathcal{S} is a Segal space. We have to verify that

$$\mathcal{S}_n \rightarrow \mathcal{S}_1 \times_{\mathcal{S}_0} \dots \times_{\mathcal{S}_0} \mathcal{S}_1$$

is a weak equivalence. We will verify an even stronger claim — that the map

$$\mathcal{S}^{d(\Delta^n)} \rightarrow \mathcal{S}^{d(\Delta^1)} \times_{\mathcal{S}} \dots \times_{\mathcal{S}} \mathcal{S}^{d(\Delta^1)}$$

is a weak equivalence. We use the homotopy equivalences $\psi^{(n)}$. We have a homotopy commutative diagram

$$(38) \quad \begin{array}{ccc} \mathcal{S}^{(n)} & \xrightarrow{\quad} & \mathcal{S}^{d(\Delta^n)} \\ \downarrow & & \downarrow \\ \mathcal{S}^{(1)} \times_{\mathcal{S}} \dots \times_{\mathcal{S}} \mathcal{S}^{(1)} & \xrightarrow{\quad} & \mathcal{S}^{d(\Delta^1)} \times_{\mathcal{S}} \dots \times_{\mathcal{S}} \mathcal{S}^{d(\Delta^1)} \end{array} .$$

Since the horizontal arrows are equivalences, the right vertical map is a fibration, and the left vertical map is a bijection, the claim follows.

In order to verify completeness of \mathcal{S} , one defines $\mathcal{S}^{we} \subset \mathcal{S}^{(1)}$ classifying weak equivalences $\mathcal{E}_0 \rightarrow \mathcal{E}_1$ of left fibrations. Then completeness follows from the fact that $\psi^{(1)}$ induces an equivalence of \mathcal{S}^{we} with \mathcal{S}^{Δ^1} .

7.7. Proof of 7.4.2. Recall that $\text{Left}(B)$ is defined as CSS fibrant replacement of the Segal space $L(B)$ defined as the nerve of the simplicial category $\mathfrak{L}(B)$ of left fibrations $X \rightarrow B$.

Note that we have a bijection $\text{Ob}(\mathfrak{L}(B)) \rightarrow \text{Hom}(B, \mathcal{S})$. Moreover, homotopy equivalence $\psi^{(1)}$ defines, for any pair of left fibrations $E_i \rightarrow B$, $i = 0, 1$, represented by $e_i : K \rightarrow \mathcal{S}$, an equivalence $\text{Map}_B(E_0, E_1) \rightarrow \text{Map}_{\mathcal{S}^B}(e_0, e_1)$.

Thus, we more or less know that 7.4.2 has to be true; we only need to construct a canonical map from $\text{Left}(B)$ to $\text{Fun}(B, \mathcal{S})$. To construct this map, it suffices to present a left fibration \mathcal{E} on $L(B) \times B$.

One has

$$(39) \quad \begin{aligned} L(B)_0 &= \text{Ob}(\mathfrak{L}(B)), \\ L(B)_n &= \coprod_{E_0, \dots, E_n \in \text{Ob}(\mathfrak{L}(B))} \text{Map}(E_0, E_1) \times \dots \times \text{Map}(E_{n-1}, E_n). \end{aligned}$$

We will define \mathcal{E} as a fibrant replacement of the left quasifibration $\mathcal{E}' \rightarrow L(B) \times B$ explicitly given by the formula

$$(40) \quad \mathcal{E}'_n = \coprod_{E_0, \dots, E_n \in \text{Ob}(\mathfrak{L}(B))} \text{Map}(E_0, E_1) \times \dots \times \text{Map}(E_{n-1}, E_n) \times (E_0)_n.$$

The above formula defines a simplicial space: a map $a : [m] \rightarrow [n]$ defines $a^* : \mathcal{E}'_n \rightarrow \mathcal{E}'_m$ induced by the map

$$(41) \quad \text{Map}(E_0, E_1) \times \dots \times \text{Map}(E_{a(0)-1}, E_{a(0)}) \times E_0 \rightarrow E_{a(0)}$$

(identity of $a(0) = 0$). The projection $\mathcal{E}' \rightarrow L(B) \times B$ is induced by $E_0 \rightarrow B$. It can be easily proven to be a left quasifibration.

A fibrant replacement \mathcal{E} of \mathcal{E}' is automatically a left fibration, so it defines a map $\psi : \text{Left}(B) \rightarrow \text{Fun}(B, \mathcal{S})$. The maps is easily verified to be DK equivalence.

7.7.1. Corollary. *The CSS \mathcal{S} defined as in 7.3.1 is the infinity category of spaces.*

Proof. Apply Theorem 7.4.2 to $B = *$. □

8. YONEDA LEMMA. APPLICATIONS

In the presentation of Yoneda lemma we follow [KV].

8.1. Presheaves and Yoneda lemma.

8.1.1. *The opposite ∞ -category.* The functor op on totally ordered finite sets carries any such set to the same set with the opposite order. Considered as a functor $\text{op} : \Delta \rightarrow \Delta$, it carries $[n]$ to itself, but carries d_i to d_{n-i} and s_i to s_{n-i} . Let us decide that, given $\mathcal{C} \in \mathbf{ssSet}$, its opposite \mathcal{C}^{op} will be the bisimplicial set obtained by precomposing it with $\text{op} \times \text{id} : \Delta \times \Delta \rightarrow \Delta \times \Delta$. It is clear that this operation carries CSS to CSS.

This means that the “spaces” $\mathcal{C}_n^{\text{op}}$ and \mathcal{C}_n coincide; only the faces and the degeneracies between them reshuffle.

Remark. This is not an obvious choice. For instance, it does not commute with the construction of classifying CSS of a category $C \mapsto B(C)$. The other choice could be to precompose a bisimplicial set with $\text{op} \times \text{op}$. The difference between two definitions is in the choice of the opposite X^{op} for $X = c(K)$, where K is a Kan simplicial set. When K is the nerve of a groupoid, there is a canonical isomorphism between K and K^{op} ; a weaker form of equivalence between K and K^{op} exists also for Kan simplicial sets.

8.1.2. *Presheaves.* Given a CSS \mathcal{C} , we define a CSS $P(\mathcal{C})$ as $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$.

Our aim is to construct a fully faithful functor $Y : \mathcal{C} \rightarrow P(\mathcal{C})$ called Yoneda embedding.

8.1.3. *Twisted arrows.* We now combine the functor op with the identity to get a new functor defined below.

Recall that for two conventional categories C and D their join $C \star D$ is defined by the formulas

$$(42) \quad \text{Ob}(C \star D) = \text{Ob}(C) \sqcup \text{Ob}(D).$$

$$(43) \quad \text{Hom}_{C \star D}(c, c') = \text{Hom}_C(c, c'); \quad \text{Hom}_{C \star D}(d, d') = \text{Hom}_D(d, d').$$

$$(44) \quad \text{Hom}_{C \star D}(c, d) = \{*\}; \quad \text{Hom}_{C \star D}(d, c) = \emptyset.$$

Let I be a finite totally ordered set. We define $\tau(I) = I^{\text{op}} \star I$. This defines a functor $\tau : \Delta \rightarrow \Delta$ and a pair of natural transformations $\text{id} \rightarrow \tau$, $\text{op} \rightarrow \tau$ defined by the obvious embeddings $I \rightarrow \tau(I)$, $I^{\text{op}} \rightarrow \tau(I)$.

For a simplicial object X in \mathbf{sSet} we define a new simplicial object $\text{Tw}(X)$ as the composition

$$\Delta^{\text{op}} \xrightarrow{\tau} \Delta^{\text{op}} \xrightarrow{X} \mathbf{sSet}.$$

As a result, we have a simplicial object $\text{Tw}(X)$ endowed with a canonical map $p : \text{Tw}(X) \rightarrow X \times X^{\text{op}}$.

8.1.4. Proposition. *Let \mathcal{C} be a Segal space. Then the map $p : \mathrm{Tw}(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a left fibration.*

Proof. We skip the verification of the fact that p is a Reedy fibration.

We have to check that for any n the map

$$(45) \quad \mathrm{Tw}(\mathcal{C})_n \rightarrow \mathrm{Tw}(\mathcal{C})_0 \times_{\mathcal{C}_0 \times \mathcal{C}_0^{\mathrm{op}}} (\mathcal{C}_n \times \mathcal{C}_n^{\mathrm{op}})$$

is a trivial fibration. The map (45) can be rewritten as the map

$$\mathrm{Map}(B, \mathcal{C}) \rightarrow \mathrm{Map}(A, \mathcal{C})$$

where $A = d\left((\Delta^0 \star \Delta^0) \coprod^{\Delta^0 \sqcup \Delta^0} (\Delta^n \sqcup (\Delta^n)^{\mathrm{op}})\right)$ and $B = d(\Delta^n \star \Delta^n)$. In other words, $A = d(\Delta^n \sqcup^{\Delta^0} \Delta^1 \sqcup^{\Delta^0} \Delta^n)$ and $B = d(\Delta^{2n+1})$. Such map is a trivial fibration for any Segal space \mathcal{C} . \square

Now, given $x \in \mathcal{C}$, we denote $Y(x)$ the left fibration over $\mathcal{C}^{\mathrm{op}}$ obtained from $\mathrm{Tw}(\mathcal{C})$ via the base change with respect to the map

$$\mathcal{C}^{\mathrm{op}} = \{x\} \times \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

8.1.5. Let fibration $\mathrm{Tw}(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ gives rise to a map

$$\tilde{Y} : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{S}$$

which can be rewritten as a map

$$Y : \mathcal{C} \rightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) = P(\mathcal{C}).$$

This is Yoneda embedding. The image of $x \in \mathcal{C}$ is precisely the functor $\mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{S}$ corresponding to the left fibration $Y(x)$.

If C is a conventional category, the left fibration $Y(x) \rightarrow C^{\mathrm{op}}$ has another description — this is the category opposite to the overcategory $C_{/x}$. Here is the definition of the corresponding ∞ -categorical notion.

8.1.6. Definition. Given a Reedy fibrant \mathcal{C} and $x \in \mathcal{C}$, the “overcategory” $\mathcal{C}_{/x}$ is defined as the fiber of the map

$$\mathrm{Fun}(d(\Delta^1), \mathcal{C}) \rightarrow \mathcal{C}$$

induced by $\{1\} \rightarrow [1]$, at x .

8.1.7. Exercise. The map $\mathcal{C}_{/x} \rightarrow \mathcal{C}$ induced by the restriction along $\{0\} \rightarrow [1]$, is a right fibration.

In particular, if \mathcal{C} is a CSS, so is $\mathcal{C}_{/x}$.

It turns out $Y(x)$ is equivalent to the left fibration

$$(\mathcal{C}_{/x})^{\mathrm{op}} \rightarrow \mathcal{C}^{\mathrm{op}}.$$

8.1.8. Lemma. *There is a natural homotopy equivalence $Y(x) \rightarrow (\mathcal{C}_{/x})^{\mathrm{op}}$ of left fibrations over $\mathcal{C}^{\mathrm{op}}$ carrying $\mathrm{id}_x \in Y(x)$ to $\mathrm{id}_x \in (\mathcal{C}_{/x})^{\mathrm{op}}$.*

Proof. Each one of the functors involved can be represented by a cosimplicial object in pointed spaces as follows.

We define $A^n = \Delta^0 \sqcup^{\Delta^n} \tau(\Delta^n)$. This is a pointed space (simplicial set) with the marked point Δ^0 . We also define $B^n = \Delta^0 \sqcup^{\Delta^n} \text{Fun}(\Delta^1, \Delta^n)$, also considered as a pointed space.

One has

$$Y(x)_n = \text{Map}_x(d(A^n), \mathcal{C}) \text{ and } (\mathcal{C}/x)_n^{\text{op}} = \text{Map}_x(d(B^n), \mathcal{C})$$

functorially in n , where Map_x denotes the space of maps carrying the base point to x . One does not have an obvious map between A^n and B^n ; instead one has the functorial maps $A^n \rightarrow C^n \leftarrow B^n$ inducing homotopy equivalences

$$\text{Map}_x(d(A^n), \mathcal{C}) \leftarrow \text{Map}_x(d(C^n), \mathcal{C}) \rightarrow \text{Map}_x(d(B^n), \mathcal{C}).$$

The pointed spaces C^n are defined here as Δ^{n+1} , with the marked point $\{0\} \in [n+1]$. \square

One has

8.1.9. Proposition. *Given $F \in P(\mathcal{C})$ and $x \in \mathcal{C}$, the natural map (evaluation)*

$$\text{Map}_{P(\mathcal{C})}(Y(x), F) \rightarrow F(x)$$

is an equivalence.

Proof. Lemma 8.1.8 allows one to replace $Y(x)$ in the claim with the left fibration $(\mathcal{C}/x)^{\text{op}}$.

Thus, we have to verify that the evaluation map induces an equivalence

$$\text{Map}_{\text{Left}(\mathcal{C}^{\text{op}})}((\mathcal{C}/x)^{\text{op}}, F) \rightarrow F(x).$$

We know that $\text{Left}(\mathcal{C}^{\text{op}})$ is a full subcategory of the CSS underlying $\mathbf{ssSet}/_{\mathcal{C}^{\text{op}}}$. Thus, the map space on the right can be calculated in $\mathbf{ssSet}/_{\mathcal{C}^{\text{op}}}$.

Since \mathbf{ssSet} with Reedy model structure is a cartesian model category, and since the map $\{x\} \rightarrow (\mathcal{C}/x)^{\text{op}}$ is a cofibration, we deduce that the map

$$\text{Map}_{\mathcal{C}^{\text{op}}}((\mathcal{C}/x)^{\text{op}}, F) \rightarrow F(x)$$

is a fibration. We will now prove it is a trivial fibration. We only have to verify that the fibers of the map are contractible. Let $f \in F(x)$. F is a left fibration, so the map

$$\text{Fun}(d(\Delta^1), F) \rightarrow F \times_{\mathcal{C}^{\text{op}}} \text{Fun}(d(\Delta^1), \mathcal{C}^{\text{op}})$$

is a trivial fibration. Its fiber at $f \in F$ is

$$F_{f/} \rightarrow (\mathcal{C}/x)^{\text{op}}$$

and it is also a trivial fibration. Our fiber is just the space of sections of this trivial fibration. \square

8.2. Limits and colimits. We will discuss colimits. Limits are obtained by passing to the opposite categories.

The most basic notion is that of colimit of an empty diagram.

8.2.1. Definition. An object $x \in \mathcal{C}$ is *initial* if $\text{Map}(x, y)$ is contractible for any $y \in \mathcal{C}$.

8.2.2. Proposition. *The full subcategory of initial objects in \mathcal{C} is either empty or contractible space.*

Proof. First of all, the projection of this category to its homotopy category is DK equivalence. Furthermore, the respective homotopy category has a unique isomorphism between any two objects. This implies the claim. \square

More general colimits are defined using a general notion of undercategory.

8.2.3. Undercategories. Let $f : K \rightarrow \mathcal{C}$ be a functor. We define $\mathcal{C}_{f/}$ as the fiber product

$$(46) \quad \{f\} \times_{\text{Fun}(K, \mathcal{C})} \text{Fun}(d(\Delta^1) \times K, \mathcal{C}) \times_{\text{Fun}(K, \mathcal{C})} \mathcal{C},$$

where the projections $\text{Fun}(d(\Delta^1) \times K, \mathcal{C}) \rightarrow \text{Fun}(K, \mathcal{C})$ are given by embeddings $\{0\} \rightarrow [1]$ and $\{1\} \rightarrow [1]$, and the map $\mathcal{C} \rightarrow \text{Fun}(K, \mathcal{C})$ is given by $K \rightarrow *$.

The special case $K = *$ played an important role in the theory of left fibrations and in Yoneda lemma. One has in general the following.

8.2.4. Lemma. *For \mathcal{C} Segal space the map $\mathcal{C}_{f/} \rightarrow \mathcal{C}$ is a left fibration. In particular, if \mathcal{C} is CSS, $\mathcal{C}_{f/}$ is also CSS.*

Proof. The map $\mathcal{C}_{f/} \rightarrow \mathcal{C}$ is obtained by base change from $\mathcal{D}_{f/} \rightarrow \mathcal{D}$ where $f \in \mathcal{D} = \text{Fun}(K, \mathcal{C})$. Thus, the result follows from Exercise 8.1.7. \square

8.2.5. Colimits. Given a functor $f : K \rightarrow \mathcal{C}$, its colimit is an initial object of the category $\mathcal{C}_{f/}$.

8.2.6. Exercise. Prove that f admits a colimit iff the presheaf on \mathcal{C}^{op} determined by $\mathcal{C}_{f/}$ is representable.

8.3. Adjoint functors. The notion of adjoint pair of functors between conventional categories does not automatically translate into the language of $(\infty, 1)$, as it includes morphisms of functors (unit and counit) which are not isomorphisms, so that adjointness is a priori 2-categorical notion. Fortunately, Yoneda lemma allows one to avoid usage of 2-categorical notions.

8.3.1. For conventional categories an adjoint pair $F : C \rightleftarrows D : G$ is uniquely defined by a bifunctor $C^{\text{op}} \times D \rightarrow \mathbf{Set}$ carrying a pair (c, d) to the set $\text{Hom}_D(F(c), d) = \text{Hom}_C(c, G(d))$. So, it is natural to expect that an adjunction between two infinity categories can be defined as a left fibration on $C^{\text{op}} \times D$ satisfying some representability conditions. Details are explained below for CSS.

8.3.2. Given a pair of CSS \mathcal{C}, \mathcal{D} , a correspondence from \mathcal{C} to \mathcal{D} is a left fibration $p : \mathcal{E} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{D}$. Such correspondence is called left-representable if for each $x \in \mathcal{C}$ the base change of p with respect to $\mathcal{D} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{D}$ determined by x , defines a representable presheaf on \mathcal{D}^{op} .

A correspondence $p : \mathcal{E} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{D}$ is called right-representable if for each $y \in \mathcal{D}$ the base change of p with respect to morphism $\mathcal{C}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{D}$, determined by y , corresponds to a representable presheaf on \mathcal{C} .

8.3.3. **Definition.** A left fibration $\mathcal{E} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{D}$ determines an adjoint pair between \mathcal{C} and \mathcal{D} if p is both left and right representable.

A correspondence $p : \mathcal{E} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{D}$ can be interpreted as a functor $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{S}$ or as $p_{\mathcal{C}} : \mathcal{C}^{\text{op}} \rightarrow P(\mathcal{D}^{\text{op}})$ or even as $p_{\mathcal{D}} : \mathcal{D} \rightarrow P(\mathcal{C})$. A left fibration $p : \mathcal{E} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{D}$ is left representable if the functor $p_{\mathcal{C}}$ factors through \mathcal{D}^{op} . It is right representable if $p_{\mathcal{D}}$ factors through \mathcal{C} .

8.3.4. *Existence of adjoint.* A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ gives rise to a composition $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}} \rightarrow P(\mathcal{D}^{\text{op}})$, that is, to a left-representable left fibration $p : \mathcal{E} \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{D}$. We say that F admits right adjoint if p is also right-representable. In this case the functor $\mathcal{D} \rightarrow P(\mathcal{C})$ corresponding to p , can be factored (uniquely up to contractible space of choices) through a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ called *the functor right adjoint to F* .

By definition, in order to verify that F admits right adjoint, it is sufficient to verify that for any $d \in \mathcal{D}$ the composition $\mathcal{C} \rightarrow P(\mathcal{D}) \xrightarrow{ev_d} \mathcal{S}$, ev_d being the evaluation at d functor, is (co)representable.

8.3.5. *Colimits and adjoint functors.* One has the following interpretation of colimits in terms of adjoint functors.

Let \mathcal{C} be a CSS and let $K \in \mathbf{ssSet}$.

Proposition. *The following conditions are equivalent.*

1. Any functor $f : K \rightarrow \mathcal{C}$ has a colimit.
2. The functor $G : \mathcal{C} \rightarrow \text{Fun}(K, \mathcal{C})$, induced by the map $K \rightarrow *$, has a left adjoint.

Proof. Existence of adjoint is verified objectwise. That is, to verify the existence of adjoint, one has to verify that for any $f : K \rightarrow \mathcal{C}$ the respective left fibration on \mathcal{C}^{op} is representable.

The functor $G : \mathcal{C} \rightarrow \text{Fun}(K, \mathcal{C})$ yields a left fibration over $\text{Fun}(K, \mathcal{C})^{\text{op}} \times \mathcal{C}$ which we will denote as \mathcal{E} . The base change \mathcal{E}_f of \mathcal{E} defined by $f \in \text{Fun}(K, \mathcal{C})$ is a left fibration on \mathcal{C} . It is homotopy equivalent to $\mathcal{C}_{f/}$ (this is a version of 8.1.8), so its representability is equivalent to existence of initial object in $\mathcal{C}_{f/}$. \square

8.3.6. *A source of correspondences.* Let $f : \mathcal{C} \rightarrow d(\Delta^1)$ be a Reedy fibration. Denote \mathcal{C}_0 and \mathcal{C}_1 the fibers of f at 0 and 1 respectively. Yoneda embedding gives rise to a map

$$\mathcal{C}_1 \rightarrow \mathcal{C} \xrightarrow{Y} P(\mathcal{C}) \rightarrow P(\mathcal{C}_0),$$

where the last arrow is the restriction of a presheaf to a subcategory. In other words, any functor as above gives rise to a correspondence. Later we will see that the opposite is also true.

8.4. **Quillen adjunction.** Given a model category \mathcal{C} with a collection of weak equivalences W , we assigned a CSS as fibrant replacement $B^f(\mathcal{C}, W)$ of Rezk nerve of the relative category (\mathcal{C}, W) . In what follows we will denote this CSS model (or any other ∞ -categorical model) $L(\mathcal{C})$.

We will show now that a Quillen adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

gives rise to an adjunction of the respective CSS. For understandable reasons, we will denote the respective functors

$$\mathbf{L}F : L(\mathcal{C}) \rightleftarrows L(\mathcal{D}) : \mathbf{R}G.$$

We proceed as follows. We realize $L(\mathcal{C})$ as a simplicial localization $L^H(\mathcal{C}^c, W)$ and $L(\mathcal{D})$ as $L^H(\mathcal{D}^f)$ ²¹. Then we construct a simplicial category \mathcal{M} with a functor $\mathcal{M} \rightarrow [1]$ as follows.

The fiber at 0 is $L^H(\mathcal{C}^c, W)^f$ (the functorial fibrant replacement in \mathbf{sCat}) and the fiber at 1 is $L^H(\mathcal{D}^f, W)^f$. Finally, for $c \in \mathcal{C}$ and $d \in \mathcal{D}$ we define $\text{Map}_{\mathcal{M}}(c, d) = \text{Map}_{L^H(\mathcal{D}, W)^f}(F(c), d)$. Compositions are defined and are strictly associative by functoriality of construction of hammock localization. Passing to CSS, we get a CSS over $d(\Delta^1)$ and we claim that it defines a required adjoint pair. To do so, we have to verify left and right representability of the respective correspondence.

Details can be found in [H.L].

8.5. **DK localization as ∞ -localization.** Using the ∞ -categorical notion of adjoint functor, we will be able to define localization of infinity categories. Then we will find out that DK localization calculates this ∞ -version of localization.

8.5.1. *Spaces and categories.* Recall that \mathbf{Cat}_{∞} is the infinity category of (small) infinity categories (e.g., of CSS), and \mathcal{S} is the full subcategory of spaces (realized, for instance, as essentially constant CSS). We will see that the embedding $\mathcal{S} \rightarrow \mathbf{Cat}_{\infty}$ admits both left and right adjoint.

The easiest way to present an adjoint is via Quillen adjunction. One has the Quillen adjunctions

²¹Recall that \mathcal{C}^c (resp., \mathcal{C}^f) is the full subcategory of cofibrant (resp., fibrant) objects in \mathcal{C} .

$$(47) \quad \mathbf{sSet} \xleftrightarrow{\quad} (\mathbf{ssSet}, R) \xleftrightarrow{\quad} (\mathbf{ssSet}, CSS),$$

with the first right adjoint functor carrying $X \in \mathbf{ssSet}$ to X_0 and the second Quillen adjunction being Bousfield localization.

Thus, the embedding of \mathcal{S} into \mathbf{Cat}_∞ has a right adjoint assigning to an infinity category \mathcal{C} its maximal subspace \mathcal{C}^{eq} , the space of its objects (or, equivalently, the infinity category obtained by discarding non-equivalences).

We will now describe the left adjoint functor. Here is the easiest way.

The standard model structure on the simplicial sets can be considered as a Bousfield localization of the Joyal model structure. This yields a Quillen pair

$$\mathrm{id} : (\mathbf{sSet}, J) \xleftrightarrow{\quad} (\mathbf{sSet}, Q) : \mathrm{id}$$

with the right adjoint functor representing the embedding $\mathcal{S} \rightarrow \mathbf{Cat}_\infty$. Thus, the left adjoint functor is the respective left derived functor of id . In this model it consists of replacing a quasicategory with its Kan fibrant replacement.

The above construction is equivalent to Dwyer-Kan “total localization”. In fact, let \mathcal{C} be a fibrant simplicial category. Its total localization is a Kan fibrant replacement of the homotopy coherent nerve $\mathfrak{N}(\mathcal{C})$. Total DK localization can be described as the total localization of a cofibrant replacement $\tilde{\mathcal{C}}$ of \mathcal{C} . According to Dwyer-Kan, their localization does not alter the homotopy type of the nerve; since total localization of $L(\tilde{\mathcal{C}}, \tilde{\mathcal{C}})$ is a simplicial groupoid, it becomes Kan after the application of \mathfrak{N} .

Thus, total ∞ -localization is represented by total DK localization.

8.5.2. ∞ -localization. The functor of maximal subspace defined above yields a functor $K : \mathbf{Cat}_\infty \rightarrow \mathrm{Fun}(\Delta^1, \mathbf{Cat}_\infty)$ carrying \mathcal{C} to the embedding of the maximal subspace \mathcal{C}^{eq} of \mathcal{C} into \mathcal{C} .

We define the general localization as the functor

$$L : \mathrm{Fun}(\Delta^1, \mathbf{Cat}_\infty) \rightarrow \mathbf{Cat}_\infty$$

left adjoint to K . One can easily see that L carries a morphism $\mathcal{W} \rightarrow \mathcal{C}$ to the pushout $L(\mathcal{W}) \sqcup^{\mathcal{W}} \mathcal{C}$.

This implies that DK localization represents also this more general version of localization.

8.6. Functor categories. Let \mathcal{C} be a combinatorial model category. Let I be a conventional category. By the universal property, we get a canonical map

$$(48) \quad L(\mathrm{Fun}(I, \mathcal{C})) \rightarrow \mathrm{Fun}(I, L(\mathcal{C})).$$

8.6.1. Theorem. *This map is an equivalence.*

Proof. The proof is based on the following very important result of D. Dugger [D]. Let J be a small category. Dugger endows the category of simplicial presheaves $U(J) := \text{Fun}(J^{\text{op}}, \mathbf{sSet})$ with the projective model structure; then he proves that for any combinatorial model category \mathcal{C} there exists a Quillen pair $F : U(J) \rightleftarrows \mathcal{C} : G$ and an isomorphism of functors $\mathbf{L}F \circ \mathbf{R}G \rightarrow \text{id}_{\mathcal{C}}$. This allows him to construct a Quillen equivalence between a certain Bousfield localization of $U(J)$ and \mathcal{C} .

Now the proof goes as follows. First of all, a Quillen equivalence $\mathcal{C} \rightleftarrows \mathcal{D}$ gives rise to an equivalence of $L(\mathcal{C})$ and $L(\mathcal{D})$. Thus, we can assume that \mathcal{C} is a Bousfield localization of $U(J)$. Next, a Bousfield localization $M \rightleftarrows M_{\text{loc}}$ identifies the underlying infinity category $L(M_{\text{loc}})$ with the full subcategory of $L(M)$ spanned by S -local objects. Thus, both $\text{Fun}(I, L(\mathcal{C}))$ and $L(\text{Fun}(I, \mathcal{C}))$ are full subcategories of $\text{Fun}(I, L(U(J)))$ and $L(\text{Fun}(I, U(J)))$ respectively. This reduces the claim to the case $\mathcal{C} = \mathbf{sSet}$.

It remains to verify the claim for $\mathcal{C} = \mathbf{sSet}$.

In this case $L(\mathcal{C})$ can be represented by the simplicial category \mathbf{Kan}_* of Kan simplicial sets and the left-hand side is the simplicial category of cofibrant functors $I \rightarrow \mathbf{Kan}_*$. The right-hand side $\text{Fun}(I, L(\mathcal{C}))$ is an infinity category having as objects the simplicial functors $\mathfrak{C}(I) \rightarrow \mathbf{Kan}_*$. The fact that any simplicial functor $\mathfrak{C}(I) \rightarrow \mathbf{Kan}_*$ is equivalent to a genuine functor, is classical. Thus, our functor is essentially surjective. The more precise statement we need is proven in [L.T], A.3.4, for any simplicial combinatorial model category \mathcal{C} . □

8.7. (Co)limits. Let I be a category and let \mathcal{C} be a combinatorial model category. We want to explain that I -indexed limits and colimits in $L(\mathcal{C})$ can be expressed in terms of derived limits and colimits in \mathcal{C} .

The functor $c : \mathcal{C} \rightarrow \text{Fun}(I, \mathcal{C})$ induced by the projection $I \rightarrow *$, gives rise to two Quillen pairs.

The first one,

$$(49) \quad \text{colim} : \text{Fun}(I, \mathcal{C}) \rightleftarrows \mathcal{C} : c,$$

identifies $\mathbf{L} \text{colim}$ with the functor left adjoint to the constant functor $L(\mathcal{C}) \rightarrow L(\text{Fun}(I, \mathcal{C})) = \text{Fun}(I, L(\mathcal{C}))$, that is with the colimit functor between the respective infinity categories.

The second,

$$(50) \quad c : \mathcal{C} \rightleftarrows \text{Fun}(I, \mathcal{C}) : \text{lim},$$

identifies $\mathbf{R} \text{lim}$ with the limit functor between the respective infinity categories.

8.7.1. Exercise. 1. Prove that an object $x \in \mathcal{C}$ is initial iff the left fibration $\mathcal{C}_{x/} \rightarrow \mathcal{C}$ is an equivalence.

2. See 8.2.6.

9. COCARTESIAN FIBRATIONS

Treatment of cocartesian fibrations in our course will be different from that of Lurie [L.T], Section 3. In order to get Grothendieck construction for cocartesian fibrations, Lurie describes a new model category, that of marked simplicial sets over a given simplicial set B . In case B is a point, this gives a new simplicial model category modeling infinity categories. For general B this models the infinity category of cocartesian fibrations over B .

Our approach is an attempt to deduce Grothendieck construction for cocartesian fibrations from the special case we studied earlier, see Parts 7, 8, as well as [KV].

9.1. Cocartesian fibrations for conventional categories.

9.1.1. *Classical definitions.* The definitions below are classical and belong to Grothendieck; the terminology is changed to coincide with the infinity-categorical one.

Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between two conventional categories. An arrow $\alpha : x \rightarrow y$ in \mathcal{C} with image $\bar{\alpha} = f(\alpha) : \bar{x} \rightarrow \bar{y}$ is called f -cocartesian if for any $z \in \mathcal{C}$ the following commutative diagram

$$(51) \quad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(y, z) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(x, z) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}}(\bar{y}, \bar{z}) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(\bar{x}, \bar{z}) \end{array}$$

is cartesian. An arrow α is called locally f -cocartesian if it defines a f' -cocartesian arrow, where f' is the base change of f with respect to $f \circ \alpha : [1] \rightarrow \mathcal{D}$.

Any f -cocartesian arrow is f -locally cocartesian.

Given $x \in \mathcal{C}$ and $\bar{\alpha} : \bar{x} \rightarrow \bar{y}$, α as defined above is called a cocartesian (resp., a locally cocartesian) lifting of $\bar{\alpha}$. A locally cocartesian lifting, if exists, is unique up to unique isomorphism in the following sense: if both $\alpha : x \rightarrow y$ and $\alpha' : x \rightarrow y'$ are locally cocartesian liftings of $\bar{\alpha}$, there exists a unique isomorphism $\theta : y \rightarrow y'$ over $\mathrm{id}_{\bar{y}}$ such that $\alpha' = \theta \circ \alpha$.

A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is called a *cocartesian fibration* if for any $x \in \mathcal{C}$ and any $\bar{\alpha} : \bar{x} \rightarrow \bar{y}$ in \mathcal{D} there is a cocartesian lifting α of $\bar{\alpha}$ with the source x .

A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is called a *locally cocartesian fibration* if for any $x \in \mathcal{C}$ and any $\bar{\alpha} : \bar{x} \rightarrow \bar{y}$ in \mathcal{D} there is a locally cocartesian lifting α of $\bar{\alpha}$ with the source x .

A locally cocartesian fibration is cocartesian iff composition of locally cocartesian arrows in \mathcal{C} is locally cocartesian ²².

²²Here are the original Grothendieck's terminology: cocartesian fibrations are called *catégories cofibrées*, cartesian fibrations are called *catégories fibrées*.

9.1.2. Here is an example of a locally cocartesian fibration which is not cocartesian. We put $\mathcal{D} = [2]$, the category with three objects d_0, d_1, d_2 , and a unique map from d_i to d_j for $i \leq j$. The category \mathcal{C} is $[1] \times [1]$, with the objects $c_{i,j}$, $i, j = 0, 1$. The functor $f : \mathcal{C} \rightarrow \mathcal{D}$ carries $c_{0,0}$ to d_0 , $c_{0,1}$ to d_1 , $c_{1,0}$ and $c_{1,1}$ to d_2 .

The arrows $c_{0,0} \rightarrow c_{0,1}$, $c_{0,0} \rightarrow c_{1,0}$ and $c_{0,1} \rightarrow c_{1,1}$ are locally f -cocartesian.

9.1.3. *Grothendieck construction.* Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be a locally cocartesian fibration. We assign to any $d \in \mathcal{D}$ the fiber $F(d) = f^{-1}(d)$. For any $a : d \rightarrow d'$ we define a functor $a_! : F(d) \rightarrow F(d')$ as follows. For any $c \in F(d)$ let $\alpha : c \rightarrow c'$ be a cocartesian lifting of a . Then we put $a_!(c) = c'$. The property of cocartesian lifting allow a unique extension of $a_!$ to a functor $a_! : F(d) \rightarrow F(d')$. Furthermore, given $b : d' \rightarrow d''$, one has a canonical morphism of functors $\theta_{a,b} : (b \circ a)_! \rightarrow b_! \circ a_!$ from $F(d)$ to $F(d'')$.

In case f is a cocartesian fibration, the morphisms $\theta_{a,b}$ are isomorphisms of functors. This means that cocartesian fibrations over \mathcal{D} correspond to “pseudo-functors” $\mathcal{D} \rightarrow \mathbf{Cat}$.

9.2. **A language of ∞ -categories.** In this subsection we will introduce a language of infinity categories which will not explicitly mention any concrete model. We will cease using any explicit constructions in a model category, replacing them with notions invariant under weak equivalences.

In what follows we will use the following notation.

\mathbf{Cat}_∞ will denote the (infinity) category of (infinity) categories. This category has products, and has internal Hom denoted $\mathbf{Fun}(C, D)$ or D^C . Spaces form a full subcategory \mathcal{S} of \mathbf{Cat}_∞ . The embedding $\mathcal{S} \rightarrow \mathbf{Cat}_\infty$ has a right adjoint functor (of maximal subspace) and a left adjoint functor (of total localization).

For a category \mathcal{C} and $x, y \in \mathcal{C}$ a space $\mathbf{Map}_{\mathcal{C}}(x, y)$ is defined, canonically “up to a contractible space of choices”.

The conventional categories form a full subcategory \mathbf{Cat} of \mathbf{Cat}_∞ . The functor $\mathbf{Ho} : \mathbf{Cat}_\infty \rightarrow \mathbf{Cat}$ assigns to any ∞ -category C the conventional category $\mathbf{Ho}(C)$ with the same objects, and with morphisms defined by the formula

$$\mathbf{Hom}_{\mathbf{Ho}(C)}(x, y) = \pi_0(\mathbf{Map}_C(x, y)).$$

An arrow in C is called equivalence if its image in $\mathbf{Ho}(C)$ is invertible. An arrow $f : \mathcal{C} \rightarrow \mathcal{D}$ in \mathbf{Cat}_∞ is an equivalence iff it is a DK equivalence, that is

- For any $x, y \in \mathcal{C}$ the map $\mathbf{Map}_{\mathcal{C}}(x, y) \rightarrow \mathbf{Map}_{\mathcal{D}}(f(x), f(y))$ is an equivalence of spaces.
- The induced map $\mathbf{Ho}(\mathcal{C}) \rightarrow \mathbf{Ho}(\mathcal{D})$ is an equivalence of conventional categories.

9.2.1. *Subspace. Subobject. Subcategory.* The ∞ -category \mathcal{S} is cartesian closed: it has products and internal Hom which is right adjoint to product. A map

$Y \rightarrow X$ is called injective (we will say Y is a subspace of X) if the induced map $\pi_0(Y) \rightarrow \pi_0(X)$ is injective and the diagram

$$(52) \quad \begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \pi_0(Y) & \longrightarrow & \pi_0(X) \end{array}$$

is cartesian. In other words, Y is defined, up to equivalence, by a subset of the set of connected components of X . Note that $Y \rightarrow X$ is a subspace iff for any Z the map $\text{Map}_s(Z, Y) \rightarrow \text{Map}_s(Z, X)$ is a subspace.

Let \mathcal{C} be an ∞ -category. An arrow $a : x \rightarrow y$ is injective (or defines x as a subobject of y) if for each z the map of spaces $\text{Map}_{\mathcal{C}}(z, x) \rightarrow \text{Map}_{\mathcal{C}}(z, y)$ (defined uniquely up to homotopy) is injective.

This definition makes a lot of sense. For instance, a functor $f : X \rightarrow Y$ defines a subcategory if for any $Z \in \mathbf{Cat}_{\infty}$ the map $\text{Map}_{\mathbf{Cat}_{\infty}}(Z, X) \rightarrow \text{Map}_{\mathbf{Cat}_{\infty}}(Z, Y)$ is a subspace. This means that, first of all, the map $X^{eq} \rightarrow Y^{eq}$ of maximal subspaces is injective, and for any $x, x' \in X$ the map

$$\text{Map}_X(x, x') \rightarrow \text{Map}_Y(f(x), f(x'))$$

is injective as well.

9.2.2. A map $f : \mathcal{C} \rightarrow B$ in \mathbf{Cat}_{∞} will be called a left fibration if the map

$$\mathcal{C}^{[1]} \rightarrow B^{[1]} \times_B \mathcal{C}$$

induced by the embedding $[0] = \{0\} \rightarrow [1]$ is an equivalence. It is clear that if f is represented by a Reedy fibration in \mathbf{ssSet} , this coincides with the definition presented in Section 7.

Given $B \in \mathbf{Cat}_{\infty}$, the full subcategory of $\mathbf{Cat}_{\infty/B}$ spanned by left fibrations identifies with the infinity category $\text{Left}(B)$ defined in Section 7. According to Theorem 7.4.2, Grothendieck construction for left fibrations provides an equivalence $\text{Left}(B) \rightarrow \text{Fun}(B, \mathcal{S})$.

9.2.3. *Subfunctors.* It is sometimes difficult to explicitly define a functor in ∞ -category setting. It is not enough, even for the conventional categories, to say what does a functor do with the objects; but for the ∞ -categories it is also not enough to describe how does a functor act on the arrows — there are higher homotopies which should also be taken into account. Still, sometimes this is possible.

Let $F : B \rightarrow \mathcal{S}$ be a functor to spaces. Let, for each $x \in B$, a subspace G_x of $F(x)$ be given, so that for each arrow $a : x \rightarrow y$ the map $F(a) : F(x) \rightarrow F(y)$ carries G_x to G_y . We claim that in this case the collection of G_x defines a canonical subfunctor $G : B \rightarrow \mathcal{S}$ of F . In fact, let $f : X_F \rightarrow B$ be a left fibration corresponding to F . We define X_G as the full subcategory of X_F spanned by the

objects of G_x , $x \in B$. It is easy to see that the composition $X_G \rightarrow X_F \rightarrow B$ is a left fibration. This defines a functor $G : B \rightarrow \mathcal{S}$.

One can replace in the above reasoning \mathcal{S} with arbitrary ∞ -category.

Proposition. *Let $F : B \rightarrow \mathcal{C}$ be a functor between ∞ -categories. Let, for each $x \in B$, a subobject G_x of $F(x)$ be given, so that for each $a : x \rightarrow y$ the composition $G_x \rightarrow F(x) \rightarrow F(y)$ factors through G_y . Then the collection of subobjects G_y uniquely glues into a subfunctor $G : B \rightarrow \mathcal{C}$.*

Proof. The functor $F : B \rightarrow \mathcal{C}$ gives rise to $Y \circ F : B \rightarrow \mathcal{C} \rightarrow P(\mathcal{C})$ which can be rewritten as a functor $F' : B \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}$. We define a subfunctor G' of F' by the subobjects $G'_{(x,c)} = \text{Map}_{\mathcal{C}}(c, G_x)$ of $\text{Map}_{\mathcal{C}}(c, F(x))$. It remains to notice that the functor G' factors through \mathcal{C} — as this fact is verified objectwise. \square

9.2.4. *Yoneda.* An important feature of the assignment $(x, y) \mapsto \text{Map}_{\mathcal{C}}(x, y)$ is that it is functorial, that is it gives rise to a functor $Y : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{S}$. A nice way to visualize it is to construct a left fibration

$$\text{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$$

corresponding, via Grothendieck construction, to Y .

Here $\text{Tw}(\mathcal{C})$ is the category of twisted arrows. In the formalism of CSS we defined $\text{Tw}(\mathcal{C})$ via the endofunctor $\tau : I \mapsto I^{\text{op}} \star I$ on Δ .

In fact, as it is shown in 9.2.5, assuming Yoneda embedding for \mathbf{Cat}_{∞} , we can identify \mathbf{Cat}_{∞} with a full subcategory of $\text{Fun}(\Delta^{\text{op}}, \mathcal{S})$ spanned by *complete Segal objects* in the sense of 9.4 below, which provides an ∞ -categorical explanation of the CSS model. This allow to define $\text{Tw}(\mathcal{C})$ using the functor τ , and define Yoneda embedding $\mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$ as the map corresponding to the left fibration $\text{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$.

9.2.5. One has an embedding $\Delta \rightarrow \mathbf{Cat}_{\infty}$; we will denote $[n] \in \mathbf{Cat}_{\infty}$ the image of the respective ordered set in \mathbf{Cat}_{∞} . This embedding induces the composition

$$\mathbf{Cat}_{\infty} \xrightarrow{Y} P(\mathbf{Cat}_{\infty}) \rightarrow P(\Delta) = \text{Fun}(\Delta^{\text{op}}, \mathcal{S}).$$

We claim that this functor is fully faithful. This means that any infinity category is determined by a simplicial object in \mathcal{S} . Moreover, the essential image of this functor consists of simplicial spaces satisfying two properties: they are *complete and Segal* (see 9.4.4).

9.3. **Overcategories.** If $C \in \mathbf{Cat}_{\infty}$ and $x \in \mathcal{C}$, we denote, naturally, $C_{/x} = C^{[1]} \times_C \{x\}$. It is worthwhile to compare this to another natural construction, in case C is the ∞ -category underlying a model category.

Let \mathcal{C} be a model category and $x \in \mathcal{C}$. We will denote $L(\mathcal{C})$ the ∞ -category underlying \mathcal{C} . One has a localization map $\mathcal{C} \rightarrow L(\mathcal{C})$ in \mathbf{Cat}_{∞} ; Therefore, one has a map $\mathcal{C}_{/x} \rightarrow L(\mathcal{C})_{/x}$ carrying weak equivalences in $\mathcal{C}_{/x}$ to equivalences.

This gives us a canonical map

$$(53) \quad L(\mathcal{C}/x) \rightarrow L(\mathcal{C})/x.$$

The source of this map is the ∞ -category underlying the model category \mathcal{C}/x ; the target is the ∞ -overcategory. We will now show how one can prove that this is an equivalence, provided x is fibrant.

Exercise.

- Prove that $\mathrm{Ho}(\mathcal{C}/x)$ is equivalent to $\mathrm{Ho}(\mathcal{C})/x$.
- Deduce that (53) becomes an equivalence after the passage to homotopy categories; in particular, it is essentially surjective.
- Prove that for $a : y \rightarrow x$ and $b : z \rightarrow x$, two objects in \mathcal{C}/x , one has an equivalence

$$(54) \quad \mathrm{Map}_{\mathcal{C}/x}(a, b) = \mathrm{Map}_{\mathcal{C}}(y, z) \times_{\mathrm{Map}_{\mathcal{C}}(y, x)} \{a\}.$$

Once we did the exercise, the proof goes as follows. The map (53) is essentially surjective, so it remains to prove it is fully faithful. The map spaces in the Dwyer-Kan localization of a model category were calculated by Dwyer-Kan in [DK3]. For a model category \mathcal{C} and for $y, z \in \mathcal{C}$, so that z is fibrant, the map space $\mathrm{Map}_{L(\mathcal{C})}(y, z)$ can be calculated, up to homotopy, using a cosimplicial resolution of y defined as follows.

A cosimplicial resolution for y is a map of cosimplicial objects $Y^\bullet \rightarrow y$, where y is considered as a constant cosimplicial object in \mathcal{C} , satisfying the properties (1)–(3) listed below.

For a cosimplicial object Y^\bullet and for a simplicial set K we will denote as $K \otimes Y^\bullet$ the object of \mathcal{C} defined by the following properties:

- $K \otimes Y^\bullet$ preserves colimits in the first argument.
- $\Delta^n \otimes Y^\bullet = Y^n$.

We require that the map $Y^\bullet \rightarrow y$ satisfies the following properties.

1. $Y_0 \rightarrow y$ is a weak equivalence and Y_0 is cofibrant.
2. coface maps in Y^\bullet are trivial cofibrations.
3. for any $n \geq 1$ the map $\partial\Delta^n \otimes Y^\bullet \rightarrow Y^n$ is a cofibration.

According to [DK3], $\mathrm{Map}_{L(\mathcal{C}/x)}(y, z)$ is calculated by the simplicial set $\mathrm{Hom}_{\mathcal{C}/x}(Y^\bullet, z)$ which is the fiber of the map

$$(55) \quad \mathrm{Hom}_{\mathcal{C}}(Y^\bullet, z) \rightarrow \mathrm{Hom}_{\mathcal{C}}(Y^\bullet, x)$$

at $Y^\bullet \rightarrow y \rightarrow x$. Since $\mathrm{Map}_{L(\mathcal{C})}(y, z)$ is calculated by the simplicial set $\mathrm{Hom}_{\mathcal{C}}(Y^\bullet, z)$, it remains to verify that the map (55) is a fibration. This follows from the following property of resolutions: if $K \rightarrow L$ is a trivial cofibration of simplicial sets, the map $K \otimes Y^\bullet \rightarrow L \otimes Y^\bullet$ is a trivial cofibration (see [Hir], 16.4.11).

9.4. Complete Segal objects. Let \mathcal{C} be an ∞ -category with finite limits. A simplicial object in \mathcal{C} is, by definition, an object in $\text{Fun}(\Delta^{\text{op}}, \mathcal{C})$. A simplicial object X is called Segal if the canonical map $X_n \rightarrow X(\mathbf{Sp}(n))$ is an equivalence in \mathcal{C} for all n . We denote $\text{Seg}(\mathcal{S})$ the full subcategory of $\text{Fun}(\Delta^{\text{op}}, \mathcal{C})$ spanned by Segal objects.

A simplicial object X is called *groupoid object* if for any presentation $[n] = S \cup T$ such that $S \cap T = \{s\}$, the commutative diagram

$$(56) \quad \begin{array}{ccc} X_n & \longrightarrow & X(S) \\ \downarrow & & \downarrow \\ X(T) & \longrightarrow & X(\{s\}) \end{array}$$

is cartesian. We denote $\mathbf{Grp}(\mathcal{C})$ the full subcategory of $\text{Fun}(\Delta^{\text{op}}, \mathcal{C})$ spanned by groupoid objects. It is clear that $\mathbf{Grp}(\mathcal{C}) \subset \text{Seg}(\mathcal{C})$.

Let us first study the case $\mathcal{C} = \mathcal{S}$.

In this case any Segal object X in \mathcal{S} defines a (conventional) homotopy category $\text{Ho}(X)$ in a usual way. Moreover, X is a groupoid object iff $\text{Ho}(X)$ is a groupoid.

9.4.1. Proposition. *The embedding $\mathbf{Grp}(\mathcal{S}) \rightarrow \text{Seg}(\mathcal{S})$ admits a right adjoint functor $X \mapsto X^{eq}$. The space X_n^{eq} is defined as a subspace of X_n consisting of components whose image in the homotopy category is a sequence of isomorphisms.*

Proof. This looks almost obvious after all the time we spent working with complete Segal spaces.

The collection $\{X_n^{eq}\}$ defines a simplicial object in \mathcal{S} as this is a subfunctor of X . □

Exercise. Prove that the construction $X \mapsto X^{eq}$ defines a functor right adjoint to the embedding $\mathbf{Grp}(\mathcal{S}) \rightarrow \text{Seg}(\mathcal{S})$.

Keep in mind that adjoint functors can be constructed objectwise.

The result extends to any ∞ -category \mathcal{C} having finite limits. Recall the notation $\Delta^1 = \bar{Z} \subset \bar{\Delta}^3$ we used in Section 6.

9.4.2. Proposition. 1. *Let \mathcal{C} have finite limits. The embedding $\mathbf{Grp}(\mathcal{C}) \rightarrow \text{Seg}(\mathcal{C})$ has right adjoint $X \mapsto X^{eq}$.*

2. *For any $X \in \text{Seg}(\mathcal{C})$ one has natural equivalences*

$$X_0^{eq} \rightarrow X_0, \quad X_1^{eq} = X^{eq}(\bar{Z}) \leftarrow X^{eq}(\bar{\Delta}^3) \rightarrow X(\bar{\Delta}^3).$$

Proof. First of all, verify the claim for $\mathcal{C} = \mathcal{S}$. Then deduce it for $P(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$. Finally, use Yoneda lemma to embed \mathcal{C} into $P(\mathcal{C})$. Here we need to know that the Yoneda embedding preserves all limits existing in \mathcal{C} . □

We are now ready to define the completeness condition.

9.4.3. Definition. A Segal object $X \in \text{Fun}(\Delta^{\text{op}}, \mathcal{C})$ is called complete if the groupoid $X^{e\mathfrak{a}}$ is essentially constant.

The category of complete Segal objects in \mathcal{C} will be denoted $\text{CS}(\mathcal{C})$.

This notion is ∞ -categorical version of complete Segal spaces. One has

9.4.4. Lemma. *The Yoneda embedding*

$$\text{Cat}_\infty \rightarrow \text{Fun}(\Delta^{\text{op}}, \mathcal{S})$$

identifies Cat_∞ with $\text{CS}(\mathcal{S})$.

Proof. The claim results from the following general observation. Let $Y : C \rightarrow P(C)$ the Yoneda embedding and let D be a full subcategory of $P(C)$ containing the essential image of Y . Then the composition $C^{\text{op}} \rightarrow D^{\text{op}} \rightarrow P(D^{\text{op}})$ and the embedding $D \rightarrow P(C)$ define equivalent functors $C^{\text{op}} \times D \rightarrow \mathcal{S}$ (see 9.12.2 below). We apply this observation to $C = \Delta$. Then $P(C) = \text{Fun}(\Delta^{\text{op}}, \mathcal{S})$ can be realized as the simplicial category of Reedy fibrant bisimplicial sets, and $D = \text{Cat}_\infty$ is realized as the full subcategory of $P(C)$ spanned by the CSS. According to the observation above, the embedding of Cat_∞ into $P(\Delta)$ is equivalent to the one defined by the Yoneda embedding. \square

We will now present a relative version of 9.4.4.

9.4.5. Lemma. *Let \mathcal{C} be an ∞ -category having small limits and let $B \in \text{Cat}_\infty$. The equivalence*

$$\text{Fun}(B, \text{Fun}(\Delta^{\text{op}}, \mathcal{C})) = \text{Fun}(\Delta^{\text{op}}, \text{Fun}(B, \mathcal{C}))$$

identifies $\text{Fun}(B, \text{CS}(\mathcal{C}))$ with $\text{CS}(\text{Fun}(B, \mathcal{C}))$. In particular, $\text{Fun}(B, \text{Cat}_\infty)$ identifies with $\text{CS}(\text{Fun}(B, \mathcal{S})) = \text{CS}(\text{Left}(B))$.

Proof. First of all, the category $\text{Fun}(B, \mathcal{C})$ has limits which are calculated pointwise (see Exercise 9.12.3). Thus, $\text{CS}(\text{Fun}(B, \mathcal{C}))$ makes sense.

A simplicial object $E : \Delta^{\text{op}} \rightarrow \text{Fun}(B, \mathcal{C})$ is a complete Segal object if the maps $E(\Delta^n) \rightarrow E(\text{Sp}(n))$ and $E(\bar{\Delta}^3) \rightarrow E(\Delta^0)$ are equivalences. All objects of $\text{Fun}(B, \mathcal{S})$ involved are presented as finite limits of E_n . Since the limits are calculated componentwise, the claim follows. \square

In this section we will identify the category $\text{CS}(\text{Left}(B))$ with a certain subcategory of $\text{Cat}_{/B}$ which will be called *the category of cocartesian fibrations over B* .

9.5. Cocartesian fibrations.

9.5.1. Let $f : X \rightarrow B$ be a morphism in \mathbf{Cat}_∞ . The construction $X \mapsto \mathrm{Tw}(X)$ is functorial, so that f induces a commutative diagram

$$(57) \quad \begin{array}{ccc} \mathrm{Tw}(X) & \longrightarrow & \mathrm{Tw}(B) \\ \downarrow & & \downarrow \\ X^{\mathrm{op}} \times X & \longrightarrow & B^{\mathrm{op}} \times B \end{array}$$

of left fibrations. Choose now $a : [1] \rightarrow X$ with $a(0) = x$ and $a(1) = y$. Base change of $\mathrm{Tw}(X)$ with respect to $(a^{\mathrm{op}}, \mathrm{id}_X) : [1] \times X \rightarrow X^{\mathrm{op}} \times X$ gives a left fibration over $[1] \times X$ which, by Grothendieck construction (or by the results of Section 8), can be replaced with a map of left fibrations $X_{y/} \rightarrow X_{x/}$ over X .

Applying the same base change to the whole diagram (57), we get a commutative diagram

$$(58) \quad \begin{array}{ccc} X_{y/} & \longrightarrow & X_{x/} \\ \downarrow & & \downarrow \\ X \times_B B_{f(y)/} & \longrightarrow & X \times_B B_{f(x)/} \end{array}$$

of left fibrations over X .

9.5.2. **Definition.** An arrow $a : x \rightarrow y$ in X is called f -cocartesian if the diagram (58) is cartesian.

Note that (58) is cartesian if and only if the (slightly simpler) diagram

$$(59) \quad \begin{array}{ccc} X_{y/} & \longrightarrow & X_{x/} \\ \downarrow & & \downarrow \\ B_{f(y)/} & \longrightarrow & B_{f(x)/} \end{array}$$

is cartesian.

Equivalences in X are obviously f -cocartesian. If a, a' lie in the same connected component of $\mathrm{Map}(x, y)$, a is f -cocartesian if and only if a' is f -cocartesian.

9.5.3. **Definition.** An arrow $\bar{a} : b \rightarrow c$ in B is said to admit a cocartesian lifting if for any $x \in X$ such that $f(x) = b$ there exists a cocartesian arrow $a : x \rightarrow y$ in X such that $f(a) = \bar{a}$.

9.5.4. *Uniqueness.* Let us show that a cocartesian lifting, if it exists, is unique up to equivalence. In fact, if the diagram (58) is cartesian, $X_{y/} = B_{f(y)/} \times_{B_{f(x)/}} X_{x/}$ has an initial object whose image in $X_{x/}$ reconstructs $a : x \rightarrow y$.

9.5.5. **Lemma.** Let $f : X \rightarrow B$ be a map. The collection of f -cocartesian arrows form a subcategory in X .

Proof. We already know that if $\alpha \sim \alpha'$ and α is cocartesian, then so is α' . Thus, it remains to prove that composition of cocartesian arrows is cocartesian. We are back to the commutative diagram (57).

We now have a map $c : [2] \rightarrow X$ and its composition with f . We make base change of (57) with respect to c . As a result we have a commutative diagram

$$(60) \quad \begin{array}{ccccc} X_{z/} & \longrightarrow & X_{y/} & \longrightarrow & X_{x/} \\ \downarrow & & \downarrow & & \downarrow \\ B_{f(z)/} & \longrightarrow & B_{f(y)/} & \longrightarrow & B_{f(x)/} \end{array}$$

with two cartesian squares. Composition of two cartesian squares is cartesian, so we are done. \square

9.5.6. Definition. A map $f : X \rightarrow B$ is called a cocartesian fibration if for any $x \in X$ and any $a : f(x) \rightarrow b'$ it admits a cocartesian lifting of a .

For a fixed ∞ -category B we define $\mathbf{Coc}(B)$ as the subcategory of $\mathbf{Cat}_{\infty/B}$ spanned by the cocartesian fibrations $X \rightarrow B$, with the arrows being the arrows $X \rightarrow X'$ over B preserving cocartesian liftings.

9.5.7. Lemma. *Let $f : X \rightarrow B$ be a left fibration. Then all arrows in X are cocartesian and f is a cocartesian fibration.*

Proof. By definition, the map $X^{[1]} \rightarrow X \times_B B^{[1]}$ is an equivalence. This implies that for any $x \in X$ the map $X_{x/} \rightarrow B_{f(x)/}$ induced by f , is an equivalence. This implies that any commutative square defined by an arrow in X , is cartesian. Equivalence of $X_{x/}$ with $B_{f(x)/}$ also implies essential surjectivity, which means that any arrow $f(x) \rightarrow b$ is equivalent to the image of an arrow $x \rightarrow x'$ in X . \square

The converse is also true, see Corollary 9.8.10 below.

9.6. Properties of cocartesian fibrations.

9.6.1. Lemma.

1. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{v} & X \\ \downarrow g & & \downarrow f \\ C & \xrightarrow{u} & B \end{array}$$

be a cartesian diagram. If the image of $\alpha : [1] \rightarrow Y$ in X is f -cocartesian, then α is g -cocartesian.

2. *In particular, a base change of a cocartesian fibration is a cocartesian fibration.*

Proof. The second claim immediately follows from the first one. Let $\alpha : y \rightarrow y'$ have images $g(\alpha) : c \rightarrow c'$, $v(\alpha) : x \rightarrow x'$ and $fv(\alpha) : b \rightarrow b'$.

Since $v(\alpha)$ is f -cocartesian, it gives an equivalence

$$(61) \quad X_{x'/j} \rightarrow B_{b'/j} \times_{B_{b/j}} X_{x/j}.$$

To deduce from this the equivalence

$$Y_{y'/j} \rightarrow C_{c'/j} \times_{C_{c/j}} Y_{y/j},$$

it is enough to make base change of (61) with respect to $C_{c'/j} \rightarrow B_{b'/j}$. \square

9.6.2. Lemma. *A composition of cocartesian fibrations is a cocartesian fibration.*

Proof. Given a pair of maps $X \xrightarrow{f} Y \xrightarrow{g} B$ such that f and g are cocartesian fibrations, we will verify that $g \circ f$ -cocartesian lifting of $a : b \rightarrow b'$ can be found in two steps, as f -cocartesian lifting of a g -cocartesian lifting. It is an exercise to provide the necessary details. \square

9.6.3. Lemma. *Let $K \in \mathbf{Cat}_\infty$. Let $f : X \rightarrow B$ be a cocartesian fibration. Then $f^K : X^K \rightarrow B^K$ is also a cocartesian fibration.*

Proof. An arrow in X^K is given by a functor $K \times [1] \rightarrow X$. First of all, we will describe f^K -cocartesian arrows in X^K . These are functors $A : K \times [1] \rightarrow X$ such that for any $k \in K$ the functor $A_k : [1] \rightarrow X$ is f -cocartesian. We have to verify this, as well as to prove that any arrow in B^K admits a cocartesian lifting. Let us explain this second property. Given a commutative diagram

$$(62) \quad \begin{array}{ccc} K & \longrightarrow & X \\ \downarrow & & \downarrow \\ K \times [1] & \xrightarrow{a} & B \end{array}$$

one has to find a lifting $A : K \times [1] \rightarrow X$ such that for any $k \in K$ $A_k : [1] \rightarrow X$ is f -cocartesian. First of all, we make a base change of f with respect to a , so that from now on $B = K \times [1]$. The composition $X \rightarrow K \times [1] \rightarrow [1]$ is a cocartesian fibration, so we reduce the problem to the following special case.

Claim. *Given a cocartesian fibration $f : X \rightarrow [1]$, any map $a : K \rightarrow X_0 = f^{-1}(0)$ extends to a map $A : K \times [1] \rightarrow X$ of cocartesian fibrations over $[1]$.*

We will now verify this claim for the special case $K = [n]$. The general case will be deduced in Proposition 9.8.3.

So, we have a cocartesian fibration $p : X \rightarrow [1]$. Given a map $f : [n] \rightarrow X_0$, we have to prove the existence of $F : [1] \times [n] \rightarrow X$ satisfying the conditions

- $F_{0,*} : [n] \rightarrow X$ is equivalent to a composition $[n] \xrightarrow{f} X_0 \rightarrow X$.
- For any $k \in [n]$ the arrow $F_{*,k} : [1] \rightarrow X$ is cocartesian.

The product $[1] \times [n]$ is a colimit

$$[1] \times [n] = s_0 \sqcup^{d_1} s_1 \sqcup^{d_2} \dots \sqcup^{d_n} s_n,$$

with s_i isomorphic to $[n+1]$ and d_i isomorphic to $[n]$, see below a presentation of the category $[1] \times [n]$

$$(63) \quad \begin{array}{ccccccc} (1, 0) & \longrightarrow & (1, 1) & \longrightarrow & (1, 2) & \cdots & (1, n-1) & \longrightarrow & (1, n) \\ \uparrow & \nearrow & \uparrow & \nearrow & & & \uparrow & \nearrow & \uparrow \\ (0, 0) & \longrightarrow & (0, 1) & \longrightarrow & (0, 2) & \cdots & (0, n-1) & \longrightarrow & (0, n), \end{array}$$

with s_k corresponding to the sequence

$$(0, 0) \rightarrow \dots \rightarrow (0, k) \rightarrow (1, k) \rightarrow \dots \rightarrow (1, n),$$

and d_k being the n -simplex containing the diagonal arrow $(0, k-1) \rightarrow (1, k)$. We will construct a map $F : [1] \times [n] \rightarrow X$ gluing $(n+1)$ -simplices one by one, starting with s_n . To construct a map $s_n \rightarrow X$, we need to choose a cocartesian lifting for $(0, n) \rightarrow (1, n)$ and use that $[n+1] = [n] \sqcup^{[0]} [1]$. Assuming the map F is constructed on

$$s_{k+1} \sqcup^{d_{k+2}} \dots \sqcup^{d_n} s_n,$$

we have to verify that we can find a map $s_k \rightarrow X$ compatible with the given map on d_{k+1} and such that it carries $(0, k) \rightarrow (1, k)$ to a cocartesian arrow of X . To do so, we have to choose a cocartesian image for $(0, k) \rightarrow (1, k)$ and then to decompose the given image of $(0, k) \rightarrow (1, k+1)$ into $(0, k) \rightarrow (1, k) \rightarrow (1, k+1)$. \square

9.6.4. Corollary. *In the notation of Lemma 9.6.3, the map $X_B^K \rightarrow B$, where $X_B^K = B \times_{B^K} X^K$, the base change of X^K with respect to the diagonal embedding, is a cocartesian fibration.*

\square

Given a map $f : X \rightarrow Y$ of cocartesian fibrations over B preserving cocartesian arrows, we can verify whether it is an equivalence, looking at the fibers. One has

9.6.5. Proposition. *A map $f : X \rightarrow Y$ in $\mathbf{Coc}(B)$ is an equivalence iff for any $b \in B$ the respective map of fibers $f_b : X_b \rightarrow Y_b$ is an equivalence in \mathbf{Cat}_∞ .*

Proof. We will have in mind that \mathbf{Cat}_∞ is a full subcategory of $\mathbf{Fun}(\Delta^{\text{op}}, \mathcal{S})$. To prove that f is an equivalence, we have to verify that for any n the map

$$f_n : (X^{[n]})^{eq} \rightarrow (Y^{[n]})^{eq}$$

is an equivalence. One has a map of cocartesian fibrations $X^{[n]} \rightarrow Y^{[n]}$ over $B^{[n]}$ and, therefore, a map of spaces

$$f_n : (X^{[n]})^{eq} \rightarrow (Y^{[n]})^{eq}$$

over $(B^{[n]})^{eq}$. To prove that this map is an equivalence, it is sufficient to verify that all its fibers are equivalences. Some of the fibers of f_n are known to be equivalences — these are the fibers at functors $[n] \rightarrow [0] \xrightarrow{b} B$, as they coincide with the n -th component of f_b . In general, the fiber of $f_n : X^{[n]} \rightarrow B^{[n]}$ at $u : [n] \rightarrow B$ is the category of the sections $\text{Fun}_{[n]}([n], X_u)$ of $X_u \rightarrow [n]$ obtained from f by base change along u . In Corollary 9.8.5 below we give a formula (70) calculating this category. In particular, the formula implies that, if the fibers of f at all $b \in B$ are equivalences, the fibers of f_n are also equivalences. \square

9.6.6. *Cocartesian sections.* If K is an ∞ -category, the projection $K \times B \rightarrow B$ is a cocartesian fibration. This yields a functor $\mathbf{Cat}_\infty \rightarrow \mathbf{Coc}(B)$. This functor has a right adjoint which we will now describe.

Given a cocartesian fibration $f : X \rightarrow B$, we define the category of its cocartesian sections, $\text{Fun}_B^{\text{coc}}(B, X)$, as follows. The category of all sections is defined as the fiber,

$$\text{Fun}_B(B, X) = \{\text{id}\} \times_{\text{Fun}(B, B)} \text{Fun}(B, X).$$

Cocartesian sections form a full subcategory in the above, spanned by the functors $s : B \rightarrow X$ carrying all arrows of B to cocartesian arrows.

One has a canonical equivalence of spaces

$$(64) \quad \text{Map}(K, \text{Fun}_B^{\text{coc}}(B, X)) \xrightarrow{\sim} \text{Map}_{\mathbf{Coc}(B)}(K \times B, X).$$

9.7. **Locally cocartesian fibrations.** Let $f : X \rightarrow B$ be a functor in \mathbf{Cat}_∞ . Fix an object $x \in X$ and an arrow $a : f(x) = b \rightarrow b'$ in B . Put $X_0 = f^{-1}(b)$ and $X_1 = f^{-1}(b')$.

We will define a locally cocartesian lifting of a as an object of X_1 (co)representing the functor $\psi_{x,a} : X_1 \rightarrow \mathcal{S}$ defined by the formula

$$(65) \quad \psi_{x,a}(y) = \text{Map}_X(x, y) \times_{\text{Map}_B(b, b')} \{a\}$$

(compare to the definition of cocartesian lifting in 9.5).

9.7.1. **Definition.** A functor $f : X \rightarrow B$ is called a locally cocartesian fibration if any pair (x, a) as above admits a locally cocartesian lifting.

It is clear that any cocartesian lifting is a locally cocartesian lifting, so, in particular, any cocartesian fibration is a locally cocartesian fibration. The converse is wrong already for conventional categories, see 9.1.2. However, one has the following.

9.7.2. **Proposition.** *Let $f : X \rightarrow B$ be a locally cocartesian fibration. The following is equivalent.*

- f is a cocartesian fibration.
- Locally f -cocartesian arrows in X are closed under composition.

Proof. If f is a cocartesian fibration, cocartesian arrows are locally cocartesian arrows and they are closed under the composition. We will now prove the converse. Thus, f is a locally cocartesian fibration such that composition of locally cocartesian arrows is locally cocartesian. We will prove that any locally cocartesian arrow is in fact cocartesian. Let $\alpha : x \rightarrow x'$ be locally cocartesian and let $z \in X$ over $b'' \in B$. We have to verify that the map of spaces

$$\mathrm{Map}_X(x', z) \rightarrow \mathrm{Map}_B(b', b'') \times_{\mathrm{Map}_B(b, b'')} \mathrm{Map}_X(x, z)$$

is an equivalence. This is enough to verify fiberwise, over any $a' \in \mathrm{Map}_B(b', b'')$. Fix such a' and let $\alpha' : x' \rightarrow x''$ be a locally cocartesian lifting of a' . The arrow α' yields an equivalence

$$\mathrm{Map}_{f^{-1}(b'')}(x'', z) \rightarrow \mathrm{Map}_X(x', z) \times_{\mathrm{Map}_B(b', b'')} \{a'\}.$$

On the other side, the composition $\alpha' \circ \alpha$, which is also locally cocartesian lifting by the assumption, yields an equivalence

$$\mathrm{Map}_{f^{-1}(b'')}(x'', z) \rightarrow \mathrm{Map}_X(x, z) \times_{\mathrm{Map}_B(b, b'')} \{a' \circ a\}.$$

This proves the assertion. \square

Remark. Originally Grothendieck defined cocartesian fibration as a locally cocartesian fibration for which composition of locally cocartesian arrows (he called them cocartesian) compose.

9.7.3. Corollary. *Let $f : X \rightarrow B$ be a functor in \mathbf{Cat}_∞ .*

1. f is a locally cocartesian fibration iff for any $a : [1] \rightarrow B$ the base change $X \times_B [1] \rightarrow [1]$ is a locally cocartesian fibration.
2. f is a cocartesian fibration iff for any $a : [2] \rightarrow B$ the base change $X \times_B [2] \rightarrow [2]$ is a locally cocartesian fibration.
3. For $B = [1]$ the notions of cocartesian and locally cocartesian fibration coincide.

\square

Left fibrations in \mathbf{Cat}_∞ can be similarly characterized. The following easily follows from the characterization of left fibrations in Section 7.

9.7.4. Lemma. *Let $f : X \rightarrow B$ be a map in \mathbf{Cat}_∞ . It is a left fibration if and only if for any $a : [1] \rightarrow B$ the base change $X_a = [1] \times_B X \rightarrow [1]$ is a left fibration.*

Proof. The claim is about ∞ -category \mathbf{Cat}_∞ which can be realized as the subcategory of the ∞ -category underlying \mathbf{ssSet} with the Reedy model structure. The map f can be presented by a Reedy fibration. It is a left fibration in \mathbf{Cat}_∞ if and only if it is a left fibration in \mathbf{ssSet} . We know that base change of a left fibration is a left fibration. It remains to prove the opposite direction. We will verify the condition 3 of Lemma 7.2.4.

First of all, since X and B are CSS, the vertical arrows in the commutative diagram

$$(66) \quad \begin{array}{ccc} X_n & \longrightarrow & X_0 \times_{B_0} B_n \\ \downarrow & & \downarrow \\ X_1 \times_{X_0} \cdots \times_{X_0} X_1 & \longrightarrow & X_0 \times_{B_0} B_1 \times_{B_0} \cdots \times_{B_0} B_1 \end{array}$$

are equivalences. Thus, it is enough to verify that the natural map $X_1 \rightarrow X_0 \times_{B_0} B_1$ is an equivalence. This is a fibration of spaces, so it is sufficient to verify that the fibers are contractible. The latter property is satisfied if $X_a \rightarrow [1]$ is a left fibration for all $a : [1] \rightarrow B$. \square

9.8. Cocartesian fibrations over $[1]$. In this subsection we do the following. In the case $B = [1]$, we establish an equivalence between cocartesian fibrations over B and functors $B \rightarrow \mathbf{Cat}$. Later on (see 9.9) we will establish this equivalence for all B . Finally, we prove the equivalence between the functors $X \rightarrow [1]$ and the correspondences between the fibers X_0 and X_1 .

9.8.1. Categories over $[1]$ and correspondences. In 8.3.2 we defined a correspondence from \mathcal{C} to \mathcal{D} as a left fibration over $\mathcal{C} \times \mathcal{D}^{\text{op}}$. In this subsection we will show that correspondences can be equivalently defined as functors $f : X \rightarrow [1]$, so that \mathcal{C} and \mathcal{D} are the fibers of f at 0 and 1 respectively.

Let us recall how a functor $f : X \rightarrow [1]$ defines a correspondence.

In what follows we will denote by X_0 and X_1 the fibers of f at 0 and 1 respectively.

Given a functor $f : X \rightarrow [1]$, one defines a functor $\tilde{f} : X_1 \rightarrow P(X_0)$ as the composition

$$X_1 \rightarrow X \xrightarrow{Y} P(X) \rightarrow P(X_0),$$

the last map being the restriction of a presheaf to a subcategory.

The map \tilde{f} can be interpreted as a left fibration $\bar{f} : E \rightarrow X_0^{\text{op}} \times X_1$. It can also be interpreted as a functor

$$(67) \quad X_0^{\text{op}} \rightarrow P(X_1^{\text{op}}).$$

9.8.2. Proposition. *The map $f : X \rightarrow [1]$ is a cocartesian fibration iff the left fibration $\bar{f} : E \rightarrow X_0^{\text{op}} \times X_1$ is left-representable, that is, if for any $x \in X_0$ the base change $\{x\} \times_{X_0^{\text{op}}} E$ is a representable presheaf on X_1^{op} .*

Proof. This is just a reformulation of the definition of locally cocartesian lifting. \square

Thus, a map $f : X \rightarrow [1]$ is a cocartesian fibration if and only if the map (67) factors through the Yoneda embedding $X_1^{\text{op}} \rightarrow P(X_1^{\text{op}})$.

We continue studying cocartesian fibrations over $[1]$.

9.8.3. Proposition. *Let $f : X \rightarrow [1]$ be a cocartesian fibration, $X_0 = f^{-1}(0)$. The “evaluation at 0” functor*

$$(68) \quad \text{Fun}_{[1]}^{\text{coc}}([1], X) \rightarrow X_0$$

is an equivalence.

Proof. The fiber of $\text{Fun}_{[1]}([1], X) \rightarrow X_0$ at x is $(X_1)_{x/}$, the category having an initial object. The fiber of (68) at x is, therefore, a contractible space. By the verified special case of lemma 9.6.3, the same claim will remain true if we replace X and X_0 with $X_{[1]}^{[n]}$ (notation of 9.6.4) and $X_0^{[n]}$.

The claim now follows from the following result which we leave as an exercise. \square

9.8.4. Exercise. A map $f : X \rightarrow Y$ in \mathbf{Cat}_∞ is an equivalence iff all fibers of the maps $f^{[n]} : X^{[n]} \rightarrow Y^{[n]}$ are contractible spaces.

9.8.5. Corollary. *In the notation of Proposition 9.8.3 one has equivalences*

$$(69) \quad \text{Fun}_{[1]}([1], X) \leftarrow \text{Fun}_{[1]}^{\text{coc}}([1], X) \times_{X_1} \text{Fun}([1], X_1) \rightarrow X_0 \times_{X_1} \text{Fun}([1], X_1),$$

where the map $f : X_0 \rightarrow X_1$ is defined by Proposition 9.8.2.

\square

9.8.6. Corollary. *Let $f : X \rightarrow [n]$ be a cocartesian fibration, $X_i = f^{-1}(i)$, and let $s_i : X_i \rightarrow X_{i+1}$ be defined by the restriction of f to the embedding $[1] \rightarrow [n]$ carrying 0 to i and 1 to $i + 1$. Then there exists an equivalence*

$$(70) \quad \text{Fun}_{[n]}([n], X) \xrightarrow{\sim} X_0 \times_{X_1} X_1^{[1]} \times_{X_2} X_2^{[1]} \dots \times_{X_n} X_n^{[1]},$$

where the maps $X_i^{[1]} \rightarrow X_i$ are induced by the embedding $\{0\} \rightarrow [1]$, and $X_i^{[1]} \rightarrow X_{i+1}$ are compositions $X_i^{[1]} \rightarrow X_i \xrightarrow{s_i} X_{i+1}$.

The above calculation is used in the proof of Proposition 9.6.5 saying that equivalence of cocartesian fibrations can be verified fiberwise.

9.8.7. Grothendieck construction for $B = [1]$. We have just seen that any cocartesian fibration $X \rightarrow [1]$ gives rise to an arrow $X_0 \rightarrow X_1$ in \mathbf{Cat}_∞ .

We claim that this construction is functorial, that is, that it defines a functor $\Gamma : \mathbf{Coc}([1]) \rightarrow \text{Fun}([1], \mathbf{Cat}_\infty)$ compatible with the equivalence $\mathbf{Coc}(\partial[1]) = \text{Fun}(\partial[1], \mathbf{Cat}_\infty)$. This requires some work.

The main step is to make Yoneda embedding $Y_X : X \rightarrow P(X)$ functorial in X , that is, to present it as a functor $Y : \mathbf{Cat}_\infty \rightarrow \text{Fun}([1], \mathbf{CAT}_\infty)$ carrying X to $Y_X : X \rightarrow P(X)$ (we write \mathbf{CAT}_∞ for a category of categories, “big” enough to contain $P(X)$). This is done as follows. We use the identification $\mathbf{Cat}_\infty = \mathbf{CS}(\mathcal{S})$. The assignment of the left fibration $\text{Tw}(X) \rightarrow X \times X^{\text{op}}$, as presented in 8.1.3, is obviously functorial as it is defined purely in terms of the pair of natural

transformations $\text{id} \rightarrow \tau$, $\text{op} \rightarrow \tau$ from Δ to itself. Then one should verify that the Grothendieck construction for spaces $\text{Left}(B) \rightarrow \text{Fun}(B, \mathcal{S})$ discussed in Section 7 is functorial in base in the sense that it yields a morphism of functors $\text{Cat}_\infty^{\text{op}} \rightarrow \text{Cat}_\infty$.

We will not do it here.

Let us define the functor in the opposite direction. Given a map $f : X_0 \rightarrow X_1$, we will construct a cocartesian fibration over $[1]$ with the fibers X_0 and X_1 .

We define the category X by the formula

$$(71) \quad X = (X_0 \times [1]) \coprod^{X_0} X_1,$$

where the map $X_0 \rightarrow X_0 \times [1]$ is given by the embedding $\{1\} \rightarrow [1]$. The map $p : X \rightarrow [1]$ is given by the projection on $X_0 \times [1]$ and carrying X_1 to $\{1\} \in [1]$.

9.8.8. Lemma. *The cylinder construction (71) provides a cocartesian fibration $X \rightarrow [1]$.*

Proof. We have to describe the map spaces $\text{Map}_X(x, y)$ for $x \in X_0$ and $y \in X_1$. Together they form a category $\text{Fun}_{[1]}([1], X)$, so we would like to have an analog of formula (69) for X . This is not obvious as X is defined as colimit, so maps to X are difficult to describe. Fortunately, we can present X as a fiber product of categories (they are all special cases of the cylinder construction). The cylinder construction is functorial in the morphism $X_0 \rightarrow X_1$. This allows us to construct a commutative diagram

$$(72) \quad \begin{array}{ccc} \text{Cyl}(X_0 \rightarrow X_1) & \longrightarrow & \text{Cyl}(X_1 \rightarrow X_1) \\ \downarrow & & \downarrow \\ \text{Cyl}(X_0 \rightarrow [0]) & \longrightarrow & \text{Cyl}(X_1 \rightarrow [0]) \end{array}$$

which can be easily verified to be cartesian (see Exercise 9.12.1).

Now we apply the functor $\text{Fun}_{[1]}([1], -)$ to the cartesian diagram. We get the formula

$$(73) \quad \text{Fun}_{[1]}([1], X) = X_0 \times_{X_1} X_1^{[1]}.$$

This formula immediately implies that for any $x \in X_0$ the image of the horizontal arrow $\{x\} \times [1] \rightarrow X_0 \times [1]$ in X is cocartesian. This proves the claim. \square

9.8.9. Proposition. *The cylinder construction above defines a functor inverse to $\Gamma : \text{Coc}([1]) \rightarrow \text{Fun}([1], \text{Cat}_\infty)$.*

Proof. We have to construct an equivalence of two compositions, $\Gamma \circ \text{Cyl}$ and $\text{Cyl} \circ \Gamma$, with identity. This is clear for the first composition; for the second we have to present an equivalence $\text{Cyl}(X_0 \xrightarrow{s} X_1) \rightarrow X$ where X is a cocartesian fibration over $[1]$ and $s : X_0 \rightarrow X_1$ is $\Gamma(X)$. By Proposition 9.8.3 one has an

equivalence $X_0 \rightarrow \text{Fun}_{[1]}^{\text{coc}}([1], X)$ which can be rewritten as a map $i : X_0 \times [1] \rightarrow X$ of cocartesian fibrations over $[1]$. The map of fibers at 1, $X_0 \rightarrow X_1$, is precisely s . Thus, the map i canonically extends to a map $\text{Cyl}(s) \rightarrow X$ that induces an equivalence of the fibers at 0 and at 1. According to 9.6.5, this implies that the map is an equivalence. \square

The adjoint pair of equivalences (Γ, Cyl) is what we call Grothendieck construction for $B = [1]$.

9.8.10. Corollary. *Let $f : X \rightarrow B$ be a cocartesian fibration. The following properties are equivalent.*

1. *All arrows of X are cocartesian.*
2. *All fibers of f are spaces.*
3. *f is a left fibration.*

Proof. The implication (3) \Rightarrow (1) is already known and (1) \Rightarrow (2) is obvious as an arrow a in X whose image in B is an equivalence, is cocartesian if and only if it is an equivalence. The implication (2) \Rightarrow (1) is also easy: as f is a cocartesian fibration, $f(a)$ has a cocartesian lifting for any arrow a in X , so there is a commutative triangle in X presenting a as a composition of a cocartesian lifting with a (vertical) equivalence. It remains to prove (2) \Rightarrow (3). By Lemma 9.7.4, the claim reduces to the case $B = [1]$. In this case $f : X \rightarrow [1]$ corresponds to a map $s : X_0 \rightarrow X_1$ where X_0 and X_1 are spaces. Then X is equivalent to the cylinder of s which is a left fibration. \square

9.9. Grothendieck construction: general case. We construct a functor $G : \text{Coc}(B) \rightarrow \text{CS}(\text{Left}(B))$ as follows. To any cocartesian fibration $f : X \rightarrow B$ we assign a simplicial object $G(X)$ in $\text{Left}(B)$ defined by the formula

$$G(X)_n = \text{Fun}_B([n], X)^{\text{coc}}.$$

Here, for a cocartesian fibration X over B , we denote X^{coc} the left fibration over B defined by the cocartesian arrows in X .

Note that even when working with conventional categories, one should be careful to define a functor both on objects and on arrows. This is even more so with infinity categories. So let us be more careful. The functor G is the composition of two functors²³. The first one carries X to the simplicial object $n \mapsto \text{Fun}([n], X)$. The second one is the functor $\text{Coc}(B) \rightarrow \text{Left}(B)$, carrying X to X^{coc} ; it is right adjoint to the embedding.

The functor $X \mapsto X^{\text{coc}}$ preserves the limits. The simplicial object $n \mapsto \text{Fun}_B([n], X)$ is a complete Segal object in $\text{Coc}(B)$, therefore, $G(X)$ is a complete Segal object in $\text{Left}(B)$.

²³Here it makes sense to talk about "the composition" — following Drinfeld's suggestion for a "homotopy definite article".

We will prove that G is an equivalence. This is done as follows. First of all, we construct a left adjoint functor $F : \mathbf{CS}(\text{Left}(B)) \rightarrow \mathbf{Coc}(B)$. We will verify that the adjoint pair of functors (F, G) is functorial in B . We prove F and G are mutually inverse equivalences verifying this for $B = *$ and using Proposition 9.6.5.

9.9.1. *The functor $F : \mathbf{CS}(\text{Left}(B)) \rightarrow \mathbf{Coc}(B)$.* We construct the functor F left adjoint to G as (a sort of) geometric realization functor.

Given a simplicial object X_\bullet in $\mathbf{Cat}_{\infty/B}$, we define $F(X_\bullet)$ as the colimit of the functor $\mathfrak{F} : \text{Tw}(\Delta) \rightarrow \mathbf{Cat}_{\infty/B}$ given by the formula

$$(74) \quad \mathfrak{F}(a : [m] \rightarrow [n]) = X_m \times [n].$$

We will now verify that F carries complete Segal objects in $\text{Left}(B)$ to cocartesian fibrations over B . Let X_\bullet be a complete Segal object in $\text{Left}(B)$. One has a canonical map $X_0 \rightarrow F(X_\bullet)$ over B . We claim that the image of this map consists of locally cocartesian arrows. Since the image is closed under compositions, this will immediately imply the claim.

Our claim is quite easy in case $B = [0]$. The functor F in this case is the composition of the rightwards arrows in the diagram below.

$$(75) \quad \mathbf{CS}(\mathcal{S}) \xrightarrow{i} \mathcal{S}^{\Delta^{\text{op}}} \xrightarrow{j} \mathbf{Cat}^{\Delta^{\text{op}}} \xrightarrow{\mathbb{F}} \mathbf{Cat},$$

\longleftarrow
 K

\longleftarrow
 G

where i and j are the obvious embeddings and \mathbb{F} is the colimit functor described above. The functors G and K are right adjoint to \mathbb{F} and j respectively, K being the maximal subspace functor and G carrying X to $\{n \mapsto \text{Fun}([n], X)\}$. The composition $K \circ G$ is known to identify \mathbf{Cat}_∞ with $\mathbf{CS}(\mathcal{S})$: therefore the functor F is an equivalence.

For B varying, the functor $F : \text{Fun}(\Delta^{\text{op}}, \text{Left}(B)) \rightarrow \mathbf{Cat}_{\infty/B}$ commutes with the base change $B' \rightarrow B$ ²⁴. Thus, in order to prove that the image of the map $X_0 \rightarrow F(X_\bullet)$ consists of locally cocartesian arrows, it is sufficient to assume $B = [1]$.

One can easily see that, after identification of $\text{Fun}([1], \mathbf{Cat}_\infty)$ with $\mathbf{CS}(\text{Left}([1]))$, the functor $F : \mathbf{CS}(\text{Left}([1])) \rightarrow \mathbf{Cat}_{\infty/[1]}$ identifies with $\text{Cyl} : \text{Fun}([1], \mathbf{Cat}_\infty) \rightarrow \mathbf{Cat}_{\infty/[1]}$. For, if $s_\bullet : X_\bullet \rightarrow Y_\bullet$ is a map of categories presented as complete Segal objects in \mathcal{S} , we convert s_\bullet into a complete Segal object in $\text{Left}([1])$, and get $\{n \mapsto \text{Cyl}(s_n)\}$. Then F carries this to

$$(76) \quad \text{colim}_{a:m \rightarrow n \in \text{Tw}(\Delta)} ((X_m \times [1]) \sqcup^{X_m} Y_m) \times [n].$$

Alternatively, one has $X = \text{colim}(X_m \times [n])$, $Y = \text{colim}(Y_m \times [n])$, so that $\text{Cyl}(s) = X \times [1] \sqcup^X Y$ can be also expressed in terms of colimits and products with $[n]$. Two expressions are canonically equivalent as colimits in \mathbf{Cat}_∞ commute with products (with $[n]$).

²⁴Since the base change $\mathbf{Cat}_{\infty/B} \rightarrow \mathbf{Cat}_{\infty/B'}$ has right adjoint functor $Y \mapsto \text{Fun}_{B'}(B, Y)$.

Identification of $F(X_\bullet)$ with the cylinder yields, in particular, our description of cocartesian arrows.

Finally, we have an adjoint pair

$$F : \mathbf{CS}(\mathrm{Left}(B)) \rightleftarrows \mathbf{Coc}(B) : G.$$

This pair commutes with the base change. Since we know that for $B = [0]$ this is an equivalence of categories, the unit and the counit of the adjunction are pointwise equivalences, and so, by 9.6.5, they are equivalences.

9.10. Categories over $[1]$. In the beginning of the previous subsection we presented, for a category X over $[1]$, a correspondence between the fibers X_0 and X_1 , that is, a functor $X_1 \rightarrow P(X_0)$. We will now present a construction in the opposite direction. Let $f : X_1 \rightarrow P(X_0)$ be given. According to 9.8, this yields a cartesian fibration $f : \mathcal{P} \rightarrow [1]$ such that $\mathcal{P}_0 = P(X_0)$ and $\mathcal{P}_1 = X_1$. We define X as the full subcategory of \mathcal{P} spanned by the objects:

- representable presheaves on X_0 over $\{0\} \in [1]$.
- all objects of \mathcal{P} over $\{1\} \in [1]$.

We claim that the category X over $[1]$ so defined has the required fibers and gives rise to the required functor $X_1 \rightarrow P(X_0)$.

9.10.1. Proposition. *The construction above establishes an equivalence between the categories $X \rightarrow [1]$ over $[1]$ and the correspondences $X_1 \rightarrow P(X_0)$.*

9.11. Limits and colimits in terms of cocartesian fibrations.

9.11.1. Limits. In 9.6.6 we presented an adjoint pair

$$F : \mathbf{Cat}_\infty \rightleftarrows \mathbf{Coc}(B) : G,$$

with $F(X)$ defined as the constant family $X \times B \rightarrow B$, and $G(Y) = \mathrm{Fun}_B^{\mathrm{coc}}(B, Y)$ being the category of cocartesian sections of Y . Identifying $\mathbf{Coc}(B)$ with $\mathrm{Fun}(B, \mathbf{Cat}_\infty)$, we can describe F as assigning to $X \in \mathbf{Cat}_\infty$ the constant functor from B to \mathbf{Cat}_∞ with value X . Thus, the functor of cocartesian sections identifies with the limit of a functor to \mathbf{Cat}_∞ .

This fact was first described by Grothendieck (SGA1) where he defines **LIMIT** of a (pseudo) functor to categories, given by a cocartesian fibration $p : X \rightarrow B$ as the category of cocartesian sections of p . In our version this definition has become a theorem.

9.11.2. Colimits. Let $p : X \rightarrow B$ be a cocartesian fibration. We claim that the colimit functor has also a nice description in terms of cocartesian fibrations.

The result is left as an exercise.

Exercise. Colimit of a functor from B to \mathbf{Cat}_∞ given by a cocartesian fibration $p : X \rightarrow B$ is the localization of X with respect to the collection of p -cocartesian arrows.

Needless to say that this also appeared as a definition in SGA1.

9.12. Exercises.

9.12.1. **Exercise.** Prove that the diagram (72) is cartesian. *Hint.* Use CSS model structure; verify that the vertical arrows are quasifibrations.

9.12.2. **Exercise.** 1. Let $f : C \rightarrow D$ be fully faithful. Then the diagram

$$\begin{array}{ccc} \mathrm{Tw}(C) & \longrightarrow & \mathrm{Tw}(D) \\ \downarrow & & \downarrow \\ C \times C^{\mathrm{op}} & \xrightarrow{f \times f^{\mathrm{op}}} & D \times D^{\mathrm{op}} \end{array}$$

is cartesian.

2. Let $f : C \rightarrow P(D)$ be a functor. Then the corresponding left fibration over $C \times D^{\mathrm{op}}$ is described by the cartesian diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Tw}(P(D)) \\ \downarrow & & \downarrow \\ C \times D^{\mathrm{op}} & \xrightarrow{f \times Y^{\mathrm{op}}} & P(D) \times P(D)^{\mathrm{op}} \end{array}$$

3. Let D be a full subcategory of $P(C)$ containing C . Then the functors $C \rightarrow D$ and $D \rightarrow P(C)$ define equivalent functors $D \times C^{\mathrm{op}} \rightarrow \mathcal{S}$.

9.12.3. **Exercise.** 1. Given an adjoint pair of functors $\mathcal{C} \xrightarrow{\leftarrow} \mathcal{D}$ and a category B , construct an adjoint pair $\mathcal{C}^B \xrightarrow{\leftarrow} \mathcal{D}^B$.

2. Deduce from the above that if a category \mathcal{C} has limits / colimits, the category \mathcal{C}^B has also limits / colimits which are calculated pointwise.

10. STABLE CATEGORIES

10.1. Introduction: stable homotopy theory, spectra.

10.1.1. *Pointed spaces.* We will say a few words about pointed topological spaces. The category \mathbf{Top}_* has coproducts called *wedge sum*. One has a natural embedding $X \vee Y \rightarrow X \times Y$.

Smash product of pointed spaces is defined as $(X, x) \wedge (Y, y) = (X \times Y) / (X \vee Y)$.

In a good category of pointed spaces smash product is left adjoint to the internal Hom functor.

Smash product with (pointed) circle is called (reduced) suspension functor. For good topological spaces the suspension functor has right adjoint, the loop space functor.

10.1.2. A functor $h : (\mathbf{Top}_*)^{\text{op}} \rightarrow \mathbf{Ab}^{\mathbb{Z}}$ endowed with a natural isomorphism $h^{n+1}(\Sigma(X)) = h^n(X)$, is a (reduced) cohomology theory, if

- It carries weak homotopy equivalence of pointed spaces into isomorphism.
- Let $i : X \rightarrow Y$ be a cofibration (e.g., embedding of a subcomplex into a complex) and let $j : Y \rightarrow Z$ be the embedding of Y into the cone $Z = * \sqcup^X (X \times [0, 1]) \sqcup^X Y$. Then the sequence of graded abelian groups

$$h^*(Z) \rightarrow h^*(Y) \rightarrow h^*(X)$$

is exact.

- For any set of pointed spaces X_i the natural map

$$h^*(\vee X_i) \rightarrow \prod h^*(X_i)$$

is an isomorphism.

As a result, for each n the functor $h^n : \mathbf{Top}_* \rightarrow \mathbf{Ab}$ is represented by some E_n in the sense that $h^n(X) = [X, E_n]$, so that the abelian group structure on $h^n(X)$ comes from H-space structure on E_n . Now the isomorphism $h^{n+1}(\Sigma X) = h^n(X)$ can be rewritten as

$$[\Sigma X, E_{n+1}] = [X, E_n],$$

that is, as a weak equivalence $E_n = \Omega(E_{n+1})$.

A collection of pointed spaces E_n and equivalences $E_n \rightarrow \Omega(E_{n+1})$ is called Ω -spectrum.

We do not intend to define here the category of spectra (see the book of Adams [A]). In any case, the theory of infinity categories says that the classical definition is imprecise and should be replaced with an infinity notion which we present below.

The passage from \mathbf{Top}_* to spectra will be formalized as a stabilization procedure assigning to any infinity category with finite limits a stable infinity category.

Homotopy category of any stable category will have a canonical structure of triangulated category. Stable categories are higher analogs of abelian categories

— and the stabilization is a higher analog of the procedure described by Quillen in 60-ies, illustrated below by an exercise.

10.1.3. **Exercise.** Let \mathcal{A} be the category of associative (or commutative, or Lie) algebras. Let $A \in \mathcal{A}$. Prove that the category of bimodules (or modules) over A can be equivalently described as the category of abelian group objects in \mathcal{A}/A .

10.2. Definitions.

10.2.1. *Pointed category.* A (infinity) category \mathcal{C} is pointed if its initial object is terminal. Examples of pointed categories include complexes, pointed spaces. If \mathcal{C} has a terminal object $*$, the category $\mathcal{C}_{*/}$ is pointed.

The object that is initial and terminal, is denoted 0 .

Exercise. If \mathcal{C} has initial object 0 and terminal object 1 , it is pointed iff $\text{Map}(1, 0) \neq \emptyset$.

10.2.2. *Stable category.* A triangle in a pointed category \mathcal{C} is a commutative diagram

$$(77) \quad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

Recall that we live in infinity world and commutative diagram actually means a functor from $[1] \times [1]$ to \mathcal{C} . A triangle is fiber sequence if the diagram is cartesian; it is cofiber sequence if the diagram is cocartesian.

We now define stable categories.

Definition. A category \mathcal{C} is stable if it is pointed, there exist fiber and cofiber for each arrow, and a triangle is fiber sequence iff it is a cofiber sequence.

Examples (details later): spectra, derived category.

10.3. **Example: derived category of an abelian category.** Let A be an abelian category. Since Grothendieck and Verdier homological algebra has been developed in derived category of A , denoted $D(A)$, defined as follows.

First, one constructs $C(A)$, the category of complexes of objects in A . A map $f : X \rightarrow Y$ in $C(A)$ is called quasiisomorphism (quism) if it induces an isomorphism of homology. The derived category $D(A)$ is the localization (in a “naive” sense) of $C(A)$ with respect to quasiisomorphisms.

$D(A)$ is a triangulated category. This means it is additive, admits a translation autoequivalence $T : D(A) \rightarrow D(A)$ and a collection of “exact triangles” $X \rightarrow Y \rightarrow Z \rightarrow T(A)$ satisfying a list of properties which we will list later.

We now have a better replacement for $D(A)$ — this is the infinity version $D_\infty(A)$ defined as a DK localization of $C(A)$ with respect to quasiisomorphisms. We will see that $D_\infty(A)$ is an example of stable category.

A triangle in $D_\infty(A)$ is a commutative diagram (77); to construct a fiber sequence for $f : Y \rightarrow Z$, we can present f with a surjective map of complexes and choose X to be the kernel of f . To construct a cofiber sequence, we can take $g : X \rightarrow Y$ to be injective, and choose Z to be the cokernel of g .

This proves the existence of fiber and cofiber sequences. Looking at short exact sequences $X \rightarrow Y \rightarrow Z$, we see that fiber and cofiber sequences actually coincide — and they correspond to the distinguished triangles in $D(A)$.

10.4. Homotopy category of a stable infinity-category. We will now prove that $\mathrm{Ho}(\mathcal{C})$ has a structure of triangulated category if \mathcal{C} is stable.

10.4.1. Suspension functor. Let us first assume that \mathcal{C} is pointed and admits cofibers. Under these assumptions we can define suspension functor $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ as follows. Look at the category $\mathcal{Q} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$, the full subcategory consisting of cocartesian diagrams having zeros outside of the diagonal. One has two projections $p_X, p_Y : \mathcal{Q} \rightarrow \mathcal{C}$, and p_X is an equivalence. This yields a functor $\Sigma = p_Y \circ p_X^{-1} : \mathcal{C} \rightarrow \mathcal{C}$. In case \mathcal{C} is stable, Σ is obviously an equivalence since the composition $\Omega = p_X \circ p_Y^{-1}$ is its inverse. We write $X[n]$ instead of $\Sigma^n(X)$. Let us add a few words. By definition, $X[1]$ can be calculated as a colimit of the diagram $0 \leftarrow X \rightarrow 0$. Colimits are defined up to contractible space of choices. The choice of the functor $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ gives a specific choice of the colimit for each X . This means that any cocartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y \end{array}$$

yields an equivalence $X[1] \rightarrow Y$, unique in the usual sense. This diagram is symmetric. What happens to the equivalence when we transpose the diagram? The equivalence belongs to $\mathrm{Map}(X[1], Y)$, which is, by definition, the fiber product $\mathrm{Map}(0, Y) \times_{\mathrm{Map}(X, Y)} \mathrm{Map}(0, Y)$, that is, the loop space of $\mathrm{Map}(X, Y)$. In particular, $\pi_0(\mathrm{Map}(X[1], Y))$ is a group.

Lemma. *The equivalences corresponding to transposed diagrams are invert to each other.*

Proof. If we carefully check the identification of the loop space with the (homotopy) fiber product, we will see that transposition of the factors inverts the loop. \square

From now on we assume \mathcal{C} is pointed, admits cofibers and such that $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence. These conditions are clearly fulfilled if \mathcal{C} is stable. ²⁵

²⁵Actually, stability of \mathcal{C} follows from these conditions, see 10.5.4.

10.4.2. *Additivity: coproducts.* We will prove now that \mathcal{C} admits finite products and finite coproducts, and the natural map from coproducts to products is an equivalence.

Let us prove the existence of coproducts of two objects X and Y . A priori we only know about the existence of cofibers — this is why the claim is not completely obvious. But we also know that Σ is an equivalence. Thus, X is cofiber of $f : X[-1] \rightarrow 0$ and Y is cofiber of $g : 0 \rightarrow Y$. Coproduct of these two diagrams exists — this is just the zero map $h : X[-1] \rightarrow Y$. Let us prove that cofiber preserves (existing) colimits. In fact, it is left adjoint to the functor $\mathcal{C} \rightarrow \text{Fun}([1], \mathcal{C})$ carrying X to $0 \rightarrow X$.

Remark. Since the opposite of a stable category is also stable (the definition is self-dual), stable categories admit finite products as well.

We, however, prefer not to use this as we intend to deduce additivity for any \mathcal{C} which is pointed, has cofibers and such that Σ is equivalence.

10.4.3. *\mathbb{Z} -enrichment.* One has a canonical equivalence $\text{Map}(\Sigma X, Y) = \Omega \text{Map}(X, Y)$. This endows $\pi_0(\text{Map}(X, Y)) = \pi_2(\text{Map}(\Sigma^{-2}X, Y))$ with the structure of abelian group.

Exercise. Prove the composition is bilinear, that is that $\text{Ho}(\mathcal{C})$ is \mathbb{Z} -enriched.

Now, if a conventional category has \mathbb{Z} -enrichment and if it admits finite coproducts, then it is additive.

This proves $\text{Ho}(\mathcal{C})$ is additive.

10.4.4. *$\text{Ho}(\mathcal{C})$ is triangulated.* Let us identify exact triangles in $\text{Ho}(\mathcal{C})$. These are sequences of arrows

$$(78) \quad X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\text{Ho}(\mathcal{C})$ which have representatives $\tilde{f}, \tilde{g}, \tilde{h}$ coming from a diagram

$$(79) \quad \begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \tilde{g} & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{\tilde{h}} & W \end{array}$$

whose both squares are cocartesian, and equivalence $X[1] \rightarrow W$ is given by the rectangle diagram.

Remark. Note that one could have put two squares one upon the other, instead of putting them aside. This would give a difference in sign.

10.4.5. *Axiom (Tr1)*.

- Any arrow embeds into a distinguished triangle.
- Any triangle isomorphic to a distinguished triangle is distinguished.
- $X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow X[1]$ is a distinguished triangle.

This is quite obvious as the category of diagrams (79) consisting of two cofiber diagrams, is equivalent to the category of arrows in \mathcal{C} .

10.4.6. *Axiom (Tr2)*. A diagram (78) is a distinguished triangle iff

$$(80) \quad Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is distinguished.

Let (79) represent the distinguished triangle (78). Let us construct a cofiber diagram for \tilde{h} as follows.

$$(81) \quad \begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \tilde{g} & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{\tilde{h}} & W \\ & & \downarrow & & \downarrow \tilde{u} \\ & & 0 & \longrightarrow & V \end{array}$$

There is a map from the upper rectangle of the diagram to the right rectangle (these are the rectangles consisting of two cells). One of them induces an equivalence $X[1] \rightarrow W$, whereas the other one induces $Y[1] \rightarrow V$. Thus, we have a commutative diagram

$$\begin{array}{ccc} X[1] & \longrightarrow & Y[1] \\ \downarrow & & \downarrow \\ W & \longrightarrow & V \end{array}$$

This proves that the vertical rectangle yields what is needed (remember the sign!)

It remains to prove the “if” part of the axiom. We leave this as an exercise.

Hint: use that Σ is an equivalence.

10.4.7. *Axiom (Tr3)*. Any commutative square (thought of as a morphism of arrows) extends to a morphism of respective distinguished triangles. *In particular, distinguished triangle corresponding to an arrow, is unique up to (noncanonical) isomorphism.*

Any commutative square in $\text{Ho}(\mathcal{C})$ can be lifted to a commutative square in \mathcal{C} . Since the category of diagrams (79) is equivalent to the category of arrows, this yields a morphism of distinguished triangle.

Noncanonicity in this axiom of triangulated categories is now explained by the fact that a commutative diagram in $\text{Ho}(\mathcal{C})$ can be presented by essentially different commutative diagrams in \mathcal{C} .

10.4.8. *Axiom (Tr₄) (Octahedron Axiom)*. Given three distinguished triangles constructed on the sides of a commutative triangle as shown below,

$$(82) \quad \begin{array}{c} X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1], \\ \\ Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1] \\ \\ X \xrightarrow{g \circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1] \end{array}$$

there exists one more distinguished triangle

$$Y/X \xrightarrow{\phi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} Y/X[1],$$

such that the following diagram is commutative.

$$(83) \quad \begin{array}{ccccccc} X & \xrightarrow{g \circ f} & Z & \xrightarrow{v} & Z/Y & \xrightarrow{\theta} & Y/X[1] \\ & \searrow f & & \searrow w & & \searrow d' & \\ & & Y & & Z/X & & Y[1] \\ & & & \nearrow u & & \nearrow \phi & \\ & & & & Y/X & \xrightarrow{d} & X[1] \end{array}$$

Here is a more symmetric presentation of the axiom. The distinguished triangles (82) can be equivalently presented by the solid arrows in the pair of diagrams presented below.²⁶

$$(84) \quad \begin{array}{ccc} \begin{array}{ccccc} Z/Y & \xleftarrow{v} & Z & & \\ & \searrow +1 & \nearrow g & & \\ \theta = uod' & \searrow d' & Y & & \\ & \nearrow u & \nearrow f & & \\ Y/X & \xrightarrow{+1} & X & & \end{array} & \begin{array}{ccccc} Z/Y & \xleftarrow{v} & Z & & \\ & \searrow \psi & \nearrow w & & \\ \theta = uod' & \searrow +1 & Z/X & & \\ & \nearrow \phi & \nearrow d'' & & \\ Y/X & \xrightarrow{+1} & X & & \end{array} \end{array}$$

The axiom claims existence of ϕ and ψ completing the picture.

²⁶The diagrams consists of commutative triangles and of exact triangles.

In order to verify the octahedron axiom, we construct the following diagram, cell by cell, making sure that each cell is cocartesian.

$$(85) \quad \begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y/X & \longrightarrow & Z/X & \longrightarrow & X' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & Z/Y & \longrightarrow & Y' \longrightarrow Y'/X' \end{array} .$$

Looking at the respective rectangles, we deduce canonical isomorphisms $X[1] \rightarrow X'$, $Y[1] \rightarrow Y'$, $(Y/X)[1] \rightarrow Y'/X'$. One can also find in this picture four rectangles determining four distinguished triangles in the homotopy category (find them!)

10.4.9. The stable categories are enriched over spectra in the same sense as the general infinity categories are enriched over spaces. In fact, given $X, Y \in \mathcal{C}$ one can define $\text{Map}^{st}(X, Y)$ as the collection of spaces $n \mapsto \text{Map}(\Sigma^n X, Y)$. One can also define $\text{Ext}^n(X, Y)$ as $\text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y[n])$.

10.5. Elementary properties.

10.5.1. **Lemma.** *Let \mathcal{C} be stable. Then $\text{Fun}(K, \mathcal{C})$ is stable for any K .*

□

A full subcategory \mathcal{C}_0 of a stable category \mathcal{C} is called stable subcategory if \mathcal{C} contains 0 and is closed under formation of fibers and cofibers.

10.5.2. **Lemma.** *Let $\mathcal{C}_0 \subset \mathcal{C}$ be a subcategory of a stable category containing 0 and closed under cofibers and translations. Then \mathcal{C}_0 is a stable subcategory.*

Proof. Since \mathcal{C} is stable, $\text{Ho}(\mathcal{C})$ is triangulated, which implies that fibers are cofibers shifted by -1 . □

10.5.3. **Definition.** A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ between two stable categories is exact if it carries 0 to 0 and preserves (co)fiber sequences.

10.5.4. **Lemma.** *The following properties of a functor between stable categories are equivalent.*

- 1 f is exact.
- 2 f is right exact (that is, it preserves finite colimits).
- 3 f is left exact.

Proof. Obviously, (2) implies (1). Conversely, an exact functor preserves coproducts (see description of coproducts in stable categories) and coequalizers (can be expressed via cofibered sequences and direct sums). □

10.5.5. **Proposition.** *Let \mathcal{C} be a pointed category. It is stable iff*

1. \mathcal{C} has finite limits and colimits.
2. A commutative square is cartesian iff it is cocartesian.

Proof. “If” direction is clear. Let us prove that a stable category has finite colimits. We already know it has coproducts. A standard fact then reduces everything to existence of coequalizers.²⁷ But the coequalizer of a pair of maps f, g coincides with the cofiber of the difference $f - g$. The notion of stable category is self-dual, so stable categories have also finite limits.

It remains to prove that any pushout square is also a pullback. We already know that \mathcal{C} has finite colimits. Thus, one can present pushout as a functor

$$(86) \quad \sqcup : \text{Fun}(\Lambda, \mathcal{C}) \rightarrow \text{Fun}([1] \times [1], \mathcal{C}),$$

where Λ is the category $\bullet \leftarrow \bullet \rightarrow \bullet$. The functor \sqcup preserves colimits, so is exact.

Look at the category of pullback squares $\mathcal{D} \subset \text{Fun}([1] \times [1], \mathcal{C})$. \mathcal{D} is stable under limits and shifts, so it is stable subcategory. Therefore, $\mathcal{D}' = \sqcup^{-1}(\mathcal{D})$ is an exact subcategory of $\text{Fun}(\Lambda, \mathcal{C})$. We will show \mathcal{D}' is the whole thing. One can easily see that the diagram of a special type

$$Z \leftarrow Z \rightarrow Z, \quad 0 \leftarrow 0 \rightarrow Z, \quad Z \leftarrow 0 \rightarrow 0$$

are all in \mathcal{D}' , and that any diagram is a colimit of such. □

10.6. **Exact functors.** We define a subcategory Cat^{ex} of Cat consisting of stable categories and exact functors between them.

10.6.1. **Theorem.** *The category Cat^{ex} has small limits and the forgetful functor $\text{Cat}^{ex} \rightarrow \text{Cat}$ preserves them.*

Proof. It is enough to prove that products and fiber products of stable categories are stable. □

Later on we will construct a functor $\text{Cat} \rightarrow \text{Cat}^{ex}$ left adjoint to the embedding. This implies that the embedding preserves small limits (however, we know this from the construction).

10.7. **Stabilization.** Toward the end we become more and more sketchy.

Stable categories admit finite limits; the forgetful functor

$$\text{Cat}^{st} \rightarrow \text{Cat}^{fl}$$

to the category of categories having finite limits (and functors preserving them), has a right adjoint; it is called stabilization.

We will now present the construction of stabilization functor denoted St in the sequel.

²⁷More precisely, this is an infinty categorical version of the standard fact.

10.7.1. *Finite pointed spaces.* We denote $\mathcal{S}_*^{\text{fin}}$ the category of finite pointed spaces. This is a full subcategory of the category of pointed spaces $\mathcal{S}_* := \mathcal{S}_{*/}$ spanned by the *finite spaces* — these are spaces presentable by a finite simplicial set.

An equivalent definition: a finite space is a finite colimit of a constant functor with the value $*$.

10.7.2. **Definition.** Let \mathcal{C} have finite colimits and a terminal object. A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ is called *excisive* if it carries cocartesian diagrams in \mathcal{C} to cartesian diagrams in \mathcal{D} . It is called *reduced* if it carries a terminal object of \mathcal{C} to a terminal object of \mathcal{D} . The full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ of excisive reduced functors is denoted $\text{Exc}_*(\mathcal{C}, \mathcal{D})$.

Note that if \mathcal{C} is stable, reduced excisive functors are those preserving fiber products and terminal objects, that is, left exact functors.

Vice versa, if \mathcal{D} is stable, f is reduced excisive iff it is right exact.

If both \mathcal{C} and \mathcal{D} are stable, f is reduced excisive iff it is exact.

10.7.3. *The functor St.* For \mathcal{C} having finite limits we define $\text{St}(\mathcal{C}) = \text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$.

The main properties of the functor St are presented in the sequence of claims below. The proofs (see [L.HA], 1.4) require more effort than we are ready to make now.

- Let \mathcal{C} be pointed with finite colimits and let \mathcal{D} have finite limits. Then $\text{Exc}_*(\mathcal{C}, \mathcal{D})$ is stable. This implies that $\text{St}(\mathcal{C}) = \text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$ is stable.
- Let \mathcal{C} have finite limits and let $\mathcal{C}_* = \mathcal{C}_{*/}$ be the respective pointed category. Then the forgetful functor $\mathcal{C}_* \rightarrow \mathcal{C}$ induces an equivalence $\text{St}(\mathcal{C}_*) \rightarrow \text{St}(\mathcal{C})$.
- Let \mathcal{C} be a pointed category having finite limits. Then $\text{St}(\mathcal{C})$ can be indentified with the limit

$$\dots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}.$$

- A category \mathcal{C} is stable iff the functor $\Omega^\infty : \text{St}(\mathcal{C}) \rightarrow \mathcal{C}$ defined as evaluation at $S^0 \in \mathcal{S}_*^{\text{fin}}$, is an equivalence.
- Let \mathcal{C} be pointed with finite colimits, \mathcal{D} have finite limits. Then composition with Ω^∞ defines an equivalence

$$\text{Exc}_*(\mathcal{C}, \text{St}(\mathcal{D})) = \text{Exc}_*(\mathcal{C}, \mathcal{D}).$$

In particular, applying this to \mathcal{C} stable, we deduce that St is right adjoint to the embedding $\mathbf{Cat}^{st} \rightarrow \mathbf{Cat}^{fl}$.

- In good cases²⁸ (for instance, $\mathcal{C} = \mathcal{S}$), the functor Ω^∞ has a left adjoint denoted Σ_+^∞ .

²⁸If \mathcal{C} is presentable

10.8. Presentable categories. We have to say a few words about presentable categories. Without giving a precise definition, we will say only that presentable categories have colimits and are generated by filtered colimits from a small subcategory.

A typical example of a presentable category is $P(\mathcal{C})$ where \mathcal{C} is small.

In \mathcal{C} has finite colimits, the category $\text{Ind}(\mathcal{C})$ defined, as for conventional categories, as the full subcategory of $P(\mathcal{C})$ spanned by the filtered colimits of representable presheaves, is presentable.

The most important property of presentable categories is the following.

10.8.1. Proposition. *Let \mathcal{C} be presentable. A functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}$ is representable iff it preserves small limits.*

This implies the following.

10.8.2. Proposition. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between presentable categories. F admits a right adjoint iff it preserves colimits.*

For a pair of presentable categories \mathcal{C}, \mathcal{D} one defines $\text{Fun}^L(\mathcal{C}, \mathcal{D})$ as the full subcategory of colimit-preserving functors, and $\text{Fun}^R(\mathcal{C}, \mathcal{D})$ the full subcategory of functors admitting left adjoint (these are precisely limit-preserving functors which also preserve κ -filtered colimits for some cardinal κ).

One obviously has $\text{Fun}^L(\mathcal{C}, \mathcal{D}) = \text{Fun}^R(\mathcal{D}, \mathcal{C})$.

We are now back to stable categories and the stabilization functor.

10.8.3. Proposition. • *Let \mathcal{C} be a presentable category. Then $\text{St}(\mathcal{C})$ is also presentable. In particular, $\mathbf{Sp} = \text{St}(\mathcal{S})$ is presentable.*

• *The functor $\Omega^\infty : \text{St}(\mathcal{C}) \rightarrow \mathcal{C}$ has left adjoint $\Sigma_+^\infty : \mathcal{C} \rightarrow \text{St}(\mathcal{C})$.*

In particular, $S = \Sigma_+^\infty(*) \in \mathbf{Sp}$ is the sphere spectrum.

Let \mathcal{C} be a small category. The stabilization of $P(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$ identifies with $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Sp})$. This is a presentable stable category and one has a functor $\Sigma_+^\infty : P(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Sp})$ left adjoint to Ω^∞ .

10.8.4. Proposition. *Let \mathcal{C} be small and \mathcal{D} be a presentable stable category.*

The composition with Σ_+^∞ induces an equivalence

$$(87) \quad \text{Fun}^L(\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Sp}), \mathcal{D}) \rightarrow \text{Fun}^L(P(\mathcal{C}), \mathcal{D}) = \text{Fun}(\mathcal{C}, \mathcal{D}).$$

10.9. Representations, finite spectra. A representation of \mathcal{C} is, by definition, a functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Sp}$. The category of representations is denoted $\text{Rep}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Sp})$. It can be otherwise defined as the stabilization of $P(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$. The composition $y = \Sigma_+^\infty \circ Y$,

$$\mathcal{C} \xrightarrow{Y} P(\mathcal{C}) \xrightarrow{\Sigma_+^\infty} \text{Rep}(\mathcal{C}),$$

is a stable version of Yoneda embedding.

We define \mathbf{Sp}^{fin} as the smallest full subcategory of \mathbf{Sp} containing the sphere spectrum and closed under cofiber sequences and Σ_+^∞ . It is stable.

The category $\mathbf{Rep}^{\text{fin}}(\mathcal{C})$ is defined as $\text{Fun}(\mathcal{C}, \mathbf{Sp}^{\text{fin}})$. This is the smallest stable subcategory of $\mathbf{Rep}(\mathcal{C})$ containing the image of the stable Yoneda embedding. Proposition 10.8.4 implies the following.

10.9.1. Proposition. *Let \mathcal{C} be a small category and \mathcal{D} a stable category. Composition with $y : \mathcal{C} \rightarrow \mathbf{Rep}^{\text{fin}}(\mathcal{C})$ induces an equivalence*

$$(88) \quad \text{Fun}^{\text{ex}}(\mathbf{Rep}^{\text{fin}}(\mathcal{C}), \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D}).$$

Proof. Here is the idea of the proof: if \mathcal{D} is stable, it has finite colimits, so, assuming it is small, $\text{Ind}(\mathcal{D})$ is stable presentable. Applying 10.8.4 to $\text{Ind}(\mathcal{D})$, we get an equivalence of two categories containing those we need. It remains to see that the subcategories correspond to each other. \square

Thus, $\mathbf{Rep}^{\text{fin}}$ is left adjoint to the embedding $\mathbf{Cat}^{\text{st}} \rightarrow \mathbf{Cat}$.

10.9.2. Remark. One can prove that the objects of $\mathbf{Rep}^{\text{fin}}(\mathcal{C})$ are compact in $\mathbf{Rep}(\mathcal{C})$ and that the latter identifies with $\text{Ind}(\mathbf{Rep}^{\text{fin}}(\mathcal{C}))$.

10.10. Waldhausen construction.

10.10.1. Let \mathcal{C} be a category. The assignment $n \mapsto \text{Map}([n], \mathcal{C})$ yields a simplicial space \mathcal{C}_n which is the complete Segal space corresponding to \mathcal{C} .

10.10.2. In case \mathcal{C} is pointed and has cofiber sequences, one has more symmetries which allow one to have a structure of simplicial space on the collection of spaces

$$S_n(\mathcal{C}) := \mathcal{C}_{n-1} \quad (S_0(\mathcal{C}) = 0).$$

Here is the Waldhausen's construction.

For each n we define $S_n(\mathcal{C})$ as the space of functors $F : \text{Fun}([1], [n]) \rightarrow \mathcal{C}$ satisfying the following properties.

(W1) $F(\text{id}_k) = 0$ for all k .

(W2) For $p \leq q \leq r$ the sequence

$$F(p \rightarrow q) \rightarrow F(p \rightarrow r) \rightarrow F(q \rightarrow r)$$

is a cofiber sequence.

The collection of spaces $S_n(\mathcal{C})$ is obviously a simplicial space. On the other hand, the map $S_\bullet(\mathcal{C}) \rightarrow \mathcal{C}_\bullet$ induced by the embedding $[n-1] \rightarrow \text{Fun}([1], [n])$ carrying k to $0 \rightarrow k+1$, is an equivalence.

10.10.3. In the case \mathcal{C} is stable, the space $\mathcal{C}_{n-1} = S_n(\mathcal{C})$ has even more symmetries. One can see this studying the simplicial object $n \mapsto \mathbf{Rep}^{\text{fin}}([n])$ representing the functor $\mathcal{C} \mapsto \mathcal{C}_\bullet$, see [L.R].

11. MULTIPLICATIVE STRUCTURES

11.1. Algebras in a category with finite products.

11.1.1. Following a Segal's idea, an associative monoid in a category \mathcal{C} with products can be defined as a "special Δ -space", that is a functor

$$A : \Delta^{\text{op}} \rightarrow \mathcal{C}$$

satisfying the following properties

- (special) A_0 is a terminal object.
- (Segal) the map $p_n : A_n \rightarrow (A_1)^n$ defined by n embeddings of $[1]$ to $[n]$ carrying $0, 1$ to $i, i + 1$ ($i = 0, \dots, n - 1$), is an equivalence.

This definition should be understood as follows. The algebra object is A_1 . The multiplication $A_1 \times A_1 \rightarrow A_1$ is given (uniquely up to homotopy) by the composition of $d_1 : A_2 \rightarrow A_1$ with the quasi-inverse of $p_2 : A_2 \rightarrow (A_1)^2$.

The above definition can be generalized to include algebras over any *planar operad* \mathcal{O} in \mathcal{C} . We will only present a concrete example of such notion, when the operad $\mathcal{O} = \mathbf{LMod}$ describes pairs (A, M) where A is an associative algebra and M is a left A -module.²⁹

11.1.2. **Definition.** A left module over a monoid in a category \mathcal{C} with products is a functor

$$F : \Delta^{\text{op}} \times [1] \rightarrow \mathcal{C}$$

satisfying the following properties.

- the simplicial object in \mathcal{C} given by the restriction of F to $\{1\} \in [1]$, is an algebra object.
- The pair of maps $([n], 0) \rightarrow ([n], 1)$ and $(\{n\}, 0) \rightarrow ([n], 0)$ are carried by F to a product diagram.

Equivalently, the functor F can be described as map between two simplicial objects in \mathcal{C} , obtained by restriction of F to $\{0\}$ and to $\{1\}$ respectively. Pretending everything is strict, one should have $F([n], 1) = A^n$ for an certain algebra object A ; and $F([n], 0) = A^n \times M$ for certain M . Multiplication in A is given by the map $F([2], 1) \rightarrow F([1], 1)$ induced by d_1 . Multiplication $A \times M \rightarrow M$ is given by the map $F([1], 0) \rightarrow F([0], 0)$ induced by d_1 .

11.1.3. *Example: spaces.* In case $\mathcal{C} = \mathcal{S}$ we get an almost classical notion of Segal monoid acting on a space. The simplicial space presenting a monoid A is actually the *classifying space* $B(A)$. The action of A on M is described by a map from the simplicial space $F(-, 0)$ to $B(A) = F(-, 1)$, with M appearing as the fiber.

²⁹The category $\Delta^{\text{op}} \times [1]$ appearing in the definition below is not, properly speaking, the whole planar operad \mathbf{LMod} ; but it is enough to define the notion of left module.

11.1.4. *Example: categories.* The category \mathbf{Cat} of infinity categories is also cartesian. So everything said above applies to it.

An algebra in \mathbf{Cat} is called a monoidal (infinity) category. A left module is a category left tensored over a monoidal category.

Basic conventional examples: vector spaces (or finite dimensional vector spaces) form a monoidal category. Any additive k -linear category is left-tensored over the finite dimensional vector spaces.

Note that the ∞ -categorical definition of monoidal category is simpler than the conventional definition. Once we use the ∞ -categorical notion of associative algebra, all associativity constraints disappear.

11.2. **Microcosm principle.** The above definition of algebra is not the most general; it does not even include conventional algebras over a field.

A solution is given by the idea nowadays known as Baez-Dolan microcosm principle. It says:

<p><i>The most general notion of algebraic structure should be defined in a category having a categorified version of the same algebraic structure.</i></p>	For in-
---	---------

stance, associative algebras should be defined in monoidal categories; a pair (algebra, module) should be defined in a pair of categories, one monoidal and the other left-tensored over it.

Here is how the definition goes.

Let \mathcal{C} be a monoidal category. By definition, this is a functor $\mathcal{C} : \Delta^{\text{op}} \rightarrow \mathbf{Cat}$ satisfying some properties. We can reformulate this definition using Grothendieck construction. We get a cocartesian fibration

$$p : \mathcal{C}^{\otimes} \rightarrow \Delta^{\text{op}}.$$

What is the meaning of Segal condition in this language? Once more, we have to look at the arrows $[n] \rightarrow [1]$ in Δ^{op} corresponding to the embeddings $\rho_i : [1] \rightarrow [n]$, $\rho_i(0) = i$, $\rho_i(1) = i + 1$, and we should require that the cocartesian liftings of ρ_i present $\mathcal{C}_n = p^{-1}([n])$ as a product of n copies of \mathcal{C}_1 .

The arrows ϕ_i as above are examples of *inert* arrows. These correspond to maps $\rho : [m] \rightarrow [n]$ in Δ given by the formula $\rho(k) = k + i$ for certain $i \in \{0, \dots, n - m\}$. Cocartesian liftings of inert arrows are called *inert arrows in \mathcal{C}^{\otimes}* . Assuming for a moment that everything is strict, \mathcal{C}_n consists of collections (c_1, \dots, c_n) with $c_i \in \mathcal{C}_1$, inert arrows throw out some of the components, and the monoidal structure is given by the cocartesian lifting of $d_1 : [2] \rightarrow [1]$.

Now we are ready to give a general definition of associative algebra in a monoidal category.

11.2.1. **Definition.** An algebra in a monoidal category \mathcal{C} is a section $A : \Delta^{\text{op}} \rightarrow \mathcal{C}^{\otimes}$ preserving inert arrows.

Similarly, a left-tensored category can be described by a cocartesian fibration

$$p : \mathcal{C}^{\otimes} \rightarrow \Delta^{\text{op}} \times [1],$$

and a left module is defined by a section $(A, M) : \Delta^{\text{op}} \times [1] \rightarrow \mathcal{C}^{\otimes}$ preserving the inert arrows. Here is the description of inerts in $\Delta^{\text{op}} \times [1]$.

- (α, id_1) when α is inert in Δ^{op} .
- $(\alpha, \{0\} \rightarrow \{1\})$, α inert.
- (α, id_0) when $\alpha : [m] \rightarrow [n]$ is inert with $\alpha(m) = n$.

11.3. Where do they come from?

11.3.1. *Cartesian structure.* We have two definitions of an algebra: one for categories with products, and another for monoidal categories. No doubt, the former is a special case of the latter.

More precisely, given a category \mathcal{C} with products, there is an explicit construction of a monoidal category³⁰ $p : \mathcal{C}^{\times} \rightarrow \Delta^{\text{op}}$ leading to equivalence of the two notions of associative algebra. This construction also satisfies a certain universal property, but we won't present it here.

The fiber of \mathcal{C}^{\times} over $[n] \in \Delta^{\text{op}}$ is equivalent to \mathcal{C}^n . Let $a : [m] \rightarrow [n]$ in Δ , $x = (x_1, \dots, x_n) \in \mathcal{C}_n^{\times}$ and $y = (x_1, \dots, y_m) \in \mathcal{C}_m^{\times}$. Then the space $\text{Map}_{\mathcal{C}^{\times}}^a(x, y)$ of maps from x to y over a^{op} is equivalent to the product

$$\prod_{j=1}^m \text{Map}_{\mathcal{C}} \left(\prod_{i=a(j-1)+1}^{a(j)} x_i, y_j \right).$$

11.3.2. *Monoidal subcategory.* Another way to define a monoidal category is to present it as a subcategory of another monoidal category.

Given a monoidal category presented by a cocartesian fibration $p : \mathcal{C}^{\otimes} \rightarrow \Delta^{\text{op}}$, a subcategory $\mathcal{D}^{\otimes} \subset \mathcal{C}^{\otimes}$ will define a monoidal subcategory if

- the composition $\mathcal{D}^{\otimes} \rightarrow \Delta^{\text{op}}$ is a monoidal category.
- the map $\mathcal{D}^{\otimes} \subset \mathcal{C}^{\otimes}$ preserves cocartesian liftings.

A monoidal subcategory gives rise to a monoidal functor (that is, a map of algebras) $\mathcal{D} \rightarrow \mathcal{C}$.

If one weakens the second condition, requiring preservation of cocartesian liftings for inerts only, one gets a lax monoidal functor $\mathcal{D} \rightarrow \mathcal{C}$. Here is an interesting example.

11.3.3. **Example.** We define $\mathbf{Cat}^{L, \otimes} \subset \mathbf{Cat}^{\times}$ as the subcategory whose objects over $[n]$ consist of collections of categories *with colimits*. A morphism $\prod X_i \rightarrow Y$ is in $\mathbf{Cat}^{L, \otimes}$ if it preserves colimits in each argument separately. This defines a monoidal category $\mathbf{Cat}^{L, \otimes}$ of categories with colimits, together with a forgetful functor to \mathbf{Cat}^{\times} which is lax monoidal.

³⁰Actually, a symmetric monoidal category.

11.3.4. *From a monoidal model category.* Model categories are an important source of infinity categories. Given a model category \mathcal{C} , its underlying ∞ -category $L(\mathcal{C})$ is the (infinity-) localization of \mathcal{C} with respect to the weak equivalences. It can be explicitly presented by, say, Dwyer-Kan localization followed by a fibrant replacement.

A model category \mathcal{C} endowed with a structure of monoidal category, is called *monoidal model category* if it satisfies the following conditions.

- Given a pair of cofibrations $i : A \rightarrow B$ and $j : C \rightarrow D$, the induced map

$$(A \otimes D) \sqcup^{A \otimes B} (B \otimes C) \rightarrow B \otimes D$$

is a cofibration, trivial, if i or j is a trivial cofibration.

- For any cofibrant replacement $Q \rightarrow \mathbf{1}$ and any cofibrant $A \in \mathcal{C}$, the maps $A \otimes Q \rightarrow A$ and $Q \otimes A \rightarrow A$ induced by $Q \rightarrow \mathbf{1}$, are equivalences.

31

Given a monoidal model category \mathcal{C} , once can easily define a monoidal structure on $L(\mathcal{C})$, realizing the latter as the localization of the subcategory \mathcal{C}^c of cofibrant objects with respect to weak equivalences.

It was recently proven [NS] that the monoidal category $L(\mathcal{C})$ so defined can be described by a universal property (the universal lax monoidal functor $\mathcal{C} \rightarrow L(\mathcal{C})$ carrying weak equivalences to equivalences).

11.3.5. *Endomorphisms.* As we already hinted, the only reasonable way to get a monoidal category (or any algebra in a monoidal category) is by a certain universal property. Here is a very important construction suggested by Lurie.

Let \mathcal{C} be a monoidal category and let \mathcal{A} be left-tensored over \mathcal{C} . Fixing an object $A \in \mathcal{A}$, one can ask what is the universal algebra object E in \mathcal{C} acting on A . Lurie's construction goes as follows. He constructs a monoidal category $\mathcal{C}[A]$ together with a monoidal functor to \mathcal{C} , equivalent to the category of pairs (E, A) endowed with an arrow $E \otimes A \rightarrow A$. If the category $\mathcal{C}[A]$ has a terminal object, it acquires automatically an algebra structure whose image in \mathcal{C} is what we need. This is the endomorphism object of A in \mathcal{C} .

11.3.6. **Example.** Let $\mathcal{C} = \text{Pr}^L$ be the category of presentable categories and colimit preserving functors. We consider it as left-tensored over itself. Let \mathcal{A} be the category of left R -modules over an associative algebra A (with values in spectra or in complexes of vector spaces). Then $E = \text{End}(A)$ will be the category of R -bimodules.

11.3.7. *Limits.* Let \mathcal{C} be a monoidal category having limits (for instance, $\mathcal{C} = \text{Cat}$). The forgetful functor $\text{Alg}(\mathcal{C}) \rightarrow \mathcal{C}$ preserves limits. Thus, if we have a functor $F : K \rightarrow \text{Alg}(\text{Cat})$ to monoidal categories, the limit will also have a

³¹this is not a standard definition.

monoidal structure. This is the way one defines the monoidal category of quasi-coherent sheaves on a scheme (or derived scheme, or derived stack): this is the limit of the corresponding categories assigned to affine schemes.

REFERENCES

- [A] J.F. Adams, *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics, 1974.
- [B1] J. Bergner, A model category structure on the category of simplicial categories. *Trans. Amer. Math. Soc.* 359 (2007), no. 5, 2043–2058.
- [B2] J. Bergner, Complete Segal spaces arising from simplicial categories, *Trans. Amer. Math. Soc.* 361 (2009), no. 1, 525–546.
- [BK1] C. Barwick, D. Kan, Relative categories: another model for the homotopy theory of homotopy theories. *Indag. Math. (N.S.)* 23 (2012), no. 1–2, 42–68.
- [BK2] C. Barwick, D. Kan, A characterization of simplicial localization functors and a discussion of DK equivalences. *Indag. Math. (N.S.)* 23 (2012), no. 1–2, 69–79.
- [BV] M. Boardmann, R. Vogt, *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer 1973.
- [CP] J.-M. Cordier, T. Porter, Vogt’s theorem on categories of homotopy coherent diagrams. *Math. Proc. Cambridge Philos. Soc.* 100 (1986), no. 1, 65–90.
- [D] D. Dugger, Combinatorial model categories have presentations. *Adv. Math.* 164 (2001), no. 1, 177–201.
- [DK1] W. Dwyer, D. Kan, Simplicial localizations of categories. *J. Pure Appl. Algebra* 17 (1980), no. 3, 267–284.
- [DK2] W. Dwyer, D. Kan, Calculating simplicial localizations. *J. Pure Appl. Algebra* 18 (1980), no. 1, 17–35.
- [DK3] W. Dwyer, D. Kan, Function complexes in homotopical algebra. *Topology* 19 (1980), no. 4, 427–440.
- [GM] S. Gelfand, Y. Manin, *Methods of homological algebra*, Springer.
- [H.L] V. Hinich, Dwyer-Kan localization revisited, *Homology Homotopy Appl.* 18 (2016), no. 1, 27–48.
- [Hir] P. Hirschhorn, *Model categories and their localizations*, AMS Math surveys and monographs 99, 2002.
- [Ho] M. Hovey, *Model categories*, AMS Math surveys and monographs 63, 1999.
- [JT] A. Joyal, M. Tierney, *Quasicategories vs Segal spaces*, arXiv:0607820v2, *Categories in algebra, geometry and mathematical physics*, 277–326, *Contemp. Math.*, 431, Amer. Math. Soc., Providence, RI, 2007.
- [KV] D. Kazhdan, Y. Varshavsky, Yoneda lemma for complete Segal spaces, arXiv:1401.5656, *Funct. Anal. Appl.* 48 (2014), no. 2, 81–106.
- [KV1] D. Kazhdan, Y. Varshavsky, *Infinity category of infinity categories*, in preparation.
- [L.T] J. Lurie, *Higher Topos theory*, *Annals of Mathematics studies*, 120, Cambridge university press.
- [L.G] J. Lurie, $(\infty, 2)$ categories and the Goodwillie calculus, Preprint, available at author’s homepage.
- [L.HA] J. Lurie, *Higher algebra*, preprint, available at author’s homepage.
- [L.R] J. Lurie, *Rotation invariance in algebraic K-theory*, preprint, available at author’s homepage.
- [M] S. MacLane, *Categories for the working mathematician*, Springer, 1971, GTM
- [NS] Th. Nikolaus, P. Scholze, *Cyclotomic spectra*, preprint arXiv 1707.01799.
- [Q] D. Quillen, *Homotopical algebra*, *LNM* 43 (1967).
- [R] C. Rezk, A model for the homotopy theory of homotopy theory, *Trans. Amer. Math. Soc.* 353 (2001), 973–1007.

- [T] B. Toen, Vers une axiomatisation de la théorie des catégories supérieures, *K-Theory* 34 (2005), no. 3, 233—263.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, MOUNT CARMEL, HAIFA 3498838,
ISRAEL

Email address: `hinich@math.haifa.ac.il`