# SOLUTION - EXERCISE 7, QUESTIONS 1 AND 5

Here are some solutions.

1. Let $(X, \mathcal{O}_X)$ be a space with functions and let $Z$ be a subset of $X$. We define a structure of space with functions on $Z$ as follows. The topology on $Z$ is induced from that on $X$, that is open subsets in $Z$ are those $V$ such that there exists $U$ open in $X$ such that $V = Z \cap U$. A function $f : V \to k$ on such $V$ is regular if for any $x \in V$ there exists an open neighborhood $U$ of $x$ in $X$ and a function $g \in \mathcal{O}_X(U)$ such that $f|_{Z\cap U} = g|_{Z\cap U}$. Verification of the axioms of a space with functions, as well as the universal property are very easy.
3. Using universality of affine varieties, a commutative diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathtt{spec}_k(A) \arrow[dr] & & \mathtt{spec}_k(B) \arrow[dl] \\ & \mathtt{spec}_k(C) & \end{tikzcd}$$

can be equivalently described by the commutative diagram in $\mathtt{Alg}_k^{fr}$

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A \arrow[dr] & & B \arrow[dl] \\ & \mathcal{O}_X(X) & \end{tikzcd} .$$

Using universality of tensor products of commutative algebras, we can rewrite the latter as a morphism $X \to \mathtt{spec}_k(D)$ where $D$ is the quotient of $A \otimes_C B$ by the nilradical. Note that the nilradical can be nonzero.
5. To construct a map to a product, one has to construct a pair of maps to each factor. The map $\mathtt{proj}(A \otimes B) \to \mathtt{spec}(A \otimes B)_0 = \mathtt{spec}(B)$ is canonical, whereas the map $\mathtt{proj}(A \otimes B) \to \mathtt{proj}(A)$ is induced by the graded ring homomorphism $A \to A \otimes B$ carrying $a \in A$ to $a \otimes 1$ (one has to verify that the latter homomorphism does induce a map of $\mathtt{proj}$'s). Choose homogeneous $f_i \in A_+$ generating $A_+$ as an ideal of $A$. Then $D_+(f_i)$ cover $\mathtt{proj}(A)$, $f_i \otimes 1$ generate $B_+$ as an ideal in $B$ and

$D_+(f_i \otimes 1)$ cover $\texttt{proj}(A \otimes B)$. Finally, $D_+(f_i \otimes 1) = \texttt{spec}(A \otimes B)_{(f_i \otimes 1)} = \texttt{spec}(A_{(f_i)} \otimes B) = D_+(f_i) \times \texttt{spec}(B)$. This proves that the map we constructed is locally an isomorphism of spaces with functions, so it is an isomorphism.

Finally, about the blowing-up. We have $A = k[x_1, \ldots, x_n]$, $B_n = (x_1, \ldots, x_n)^n t^n = (x_1 t, \ldots, x_n t)^n$, so that $B \subset A[t]$, $\deg(t) = 1$. We denote $C = A[Y_1, \ldots, Y_n]$ with $\deg(Y_i) = 1$ so that there is a graded surjective homomorphism $p : C \to B$ carrying $Y_i$ to $x_i t$. It induces a closed embedding $\texttt{proj}(B) \to \texttt{proj}(C)$ and we have $\texttt{proj}(C) = \texttt{spec}(A) \times \texttt{proj}(k[Y_1, \ldots, Y_n]) = \texttt{spec}(A) \times \mathbb{P}^{n-1}$.