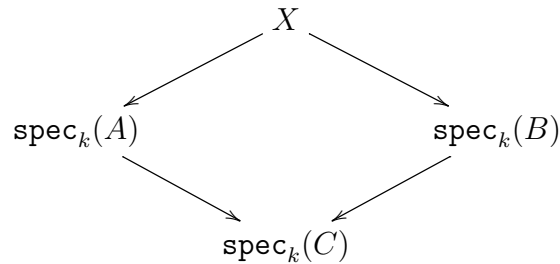


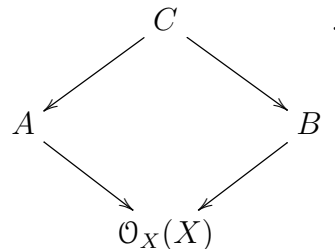
## SOLUTION - EXERCISE 7, QUESTIONS 1 AND 5

Here are some solutions.

1. Let  $(X, \mathcal{O}_X)$  be a space with functions and let  $Z$  be a subset of  $X$ . We define a structure of space with functions on  $Z$  as follows. The topology on  $Z$  is induced from that on  $X$ , that is open subsets in  $Z$  are those  $V$  such that there exists  $U$  open in  $X$  such that  $V = Z \cap U$ . A function  $f : V \rightarrow k$  on such  $V$  is regular if for any  $x \in V$  there exists an open neighborhood  $U$  of  $x$  in  $X$  and a function  $g \in \mathcal{O}_X(U)$  such that  $f|_{Z \cap U} = g|_{Z \cap U}$ . Verification of the axioms of a space with functions, as well as the universal property are very easy.
3. Using universality of affine varieties, a commutative diagram



can be equivalently described by the commutative diagram in  $\text{Alg}_k^{fr}$



Using universality of tensor products of commutative algebras, we can rewrite the latter as a morphism  $X \rightarrow \text{spec}_k(D)$  where  $D$  is the quotient of  $A \otimes_C B$  by the nilradical. Note that the nilradical can be nonzero.

5. To construct a map to a product, one has to construct a pair of maps to each factor. The map  $\text{proj}(A \otimes B) \rightarrow \text{spec}(A \otimes B)_0 = \text{spec}(B)$  is canonical, whereas the map  $\text{proj}(A \otimes B) \rightarrow \text{proj}(A)$  is induced by the graded ring homomorphism  $A \rightarrow A \otimes B$  carrying  $a \in A$  to  $a \otimes 1$  (one has to verify that the latter homomorphism does induce a map of  $\text{proj}$ 's). Choose homogeneous  $f_i \in A_+$  generating  $A_+$  as an ideal of  $A$ . Then  $D_+(f_i)$  cover  $\text{proj}(A)$ ,  $f_i \otimes 1$  generate  $B_+$  as an ideal in  $B$  and

$D_+(f_i \otimes 1)$  cover  $\mathbf{proj}(A \otimes B)$ . Finally,  $D_+(f_i \otimes 1) = \mathbf{spec}(A \otimes B)_{(f_i \otimes 1)} = \mathbf{spec}(A_{(f_i)} \otimes B) = D_+(f_i) \times \mathbf{spec}(B)$ . This proves that the map we constructed is locally an isomorphism of spaces with functions, so it is an isomorphism.

Finally, about the blowing-up. We have  $A = k[x_1, \dots, x_n]$ ,  $B_n = (x_1, \dots, x_n)^n t^n = (x_1 t, \dots, x_n t)^n$ , so that  $B \subset A[t]$ ,  $\deg(t) = 1$ . We denote  $C = A[Y_1, \dots, Y_n]$  with  $\deg(Y_i) = 1$  so that there is a graded surjective homomorphism  $p : C \rightarrow B$  carrying  $Y_i$  to  $x_i t$ . It induces a closed embedding  $\mathbf{proj}(B) \rightarrow \mathbf{proj}(C)$  and we have  $\mathbf{proj}(C) = \mathbf{spec}(A) \times \mathbf{proj}(k[Y_1, \dots, Y_n]) = \mathbf{spec}(A) \times \mathbb{P}^{n-1}$ .