SOLUTION - EXERCISE 6, QUESTIONS 3–6

Here are some solutions.

3. If $\texttt{spec}(A) = Z_1 \sqcup Z_2$ then the functions $f_i = 1$ at Z_i and 0 outside Z_i are regular so belong to A. These are nontrivial idempotents. In the opposite direction, assume $e \in A$ is a nontrivial idempotent. The vector subspace $E = \mathrm{Span}_k\{e, 1-e\}$ is an algebra isomorphic to $k \times k$. We have an injective algebra homomorphism $E \to A$ that induces a dominant map $\texttt{spec}(A) \to \texttt{spec}(E) = \texttt{pt} \sqcup \texttt{pt}$. Here dominant implies surjective so $\texttt{spec}(A)$ is the disjoint union of two nonempty components, the preimages of the points.

4a If $A_+ \neq 0$, it is not nilpotent so there is $\phi : A \to k$ such that ϕ does not vanish at A_+. Then $\texttt{proj}(A) \neq \emptyset$. The converse is obvious.

4b $A_+ = (x)$ so $\texttt{proj}(A) = D_+(x)$. We have already proven that $D_+(x) = \texttt{spec}(A_{(x)})$. Since $A_x = A_0[x, x^{-1}]$, $A_{(x)} = A_0$.

5a If $b \in B_+$ is homogeneous, $b^n \in A_+$ for some n. So, for any $\phi : B \to k$ with $B_+ \not\subset \mathrm{Ker}(\phi)$ one has $\mathrm{Ker}(\phi \circ f) \not\supset A_+$. Thus the map $f^* : \texttt{proj}(B) \to \texttt{proj}(A)$ is defined. Each homogeneous $a \in A_+$ defines two basic open subsets $D_+(a)$ in $\texttt{proj}(A)$ and in $\texttt{proj}(B)$ that are identified by f^* as the are deduced from the isomorphism of the graded localizations $A_{(a)} \to B_{(a)}$ (that is easy as for $b \in B$ one has $\frac{b}{a^k} = \frac{a^{N-k}b}{a^N}$).

5b Very similar to the above. It is convenient to replace $A^{(d)}$ with another graded algebra B, $B_k = 0$ for $d \not| k$ and $B_{dk} = A_{dk}$. Then $\texttt{proj}(A^{(d)})$ and $\texttt{proj}(B)$ are obviously isomorphic as they only differ by grading, and we can argue as in 5a about the morphism of $\texttt{proj}$'s induced by the graded ring homomorphism $B \to A$.

6 To prove that the restriction of π to $\pi^{-1}(D(I))$ is an isomorphism, it is sufficient to prove that its restriction to any $pi^{-1}(D(f))$ is isomorphism for any $f \in I$. This easily follows from 4b applied to the polynomial ring over A_f as $B_f = A_f[x]$.