SOLUTION - EXERCISE 5, QUESTION 3

Q3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be an affine morphism. Prove that preimage of any open affine subspace of Y is affine.

Solution.

1. First of all let us verify that, if $f : X \to Y$ is affine and $U \subset Y$ is open, then the restriction $f^{-1}(U) \to U$ is also affine. In fact, if U_i form an open affine covering of Y with $f^{-1}(U_i)$ affine, the open subsets $(U_i)_g$ for $g \in \mathcal{O}_Y(U_i)$ are also affine with an affine preimage and they form a basis of open sets in Y.

2. Now, let $f : X \to Y$ be affine and $U \subset Y$ be an affine subset. By the first step we can assume that $U = Y$ is affine. Given an affine open covering U_i of Y with affine preimages, each U_i is a union of certain $D(g)$; if $U_i \supset D(g)$ then the preimage of $D(g)$ is affine as $D(g) = (U_i)_{\bar{g}}$ where $\bar{g}$ is the restriction of $g \in \mathcal{O}_Y(Y)$ to U_i. Then we can use Question 2 to deduce that X is affine.