ideal

Problem 2. Let $(X, \mathcal{O}_X)$ be a space with functions, $A = \mathcal{O}_X(X)$. For $f \in A$, let X_f be the open set $\{x : f(x) \neq 0\}$. Suppose $f_1, ..., f_n \in A$ generate A (~~as a k-algebra~~), and each X_{f_i} is affine. Show that X is affine.

Solution:

Let $g : X \to \text{spec}_k(A)$ be the canonical map (explicitly, $g(x)(a) = a(x)$ or $g(x) = \text{ev}_x$). Let $Y = \text{spec}_k(A)$

For every f_i, let $Y_{f_i} = \{y \in Y : f_i(y) \neq 0\} \subseteq Y$. Since $(f_i)_i$ generate A, we know that $\bigcup_i Y_{f_i} = Y$. Explicitly, if $y \in Y$ is not in any Y_{f_i}, then it vanishes on all f_i, thus vanishes on all of A and is hence 0. Note that:

$$g^{-1}(Y_{f_i}) = \{x \in X : f_i(g(x)) \neq 0\} = \{x : f(x) \neq 0\} = X_{f_i}$$

if a/(f_i)^k vanishes on X_{f_i}, then f_i a vanishes on the whole X so f_ia=0 in A so a/(f_i)^k =0 in A_{f_i}

By affinity, each $X_{f_i} = \text{spec}_k(B_i)$ for some ring B_i. Note that if f is a global chart, and s is some power of f_i, then by properties of spaces with functions $1/s \in \mathcal{O}_X(X_{f_i}) = B_i$, hence also $f/s \in B_i$. This means that $A_{f_i} \subseteq B_i$. see on the left

Now let $b \in B_i$. $X_{f_i} \cap X_{f_j}$ is a principal subset of (the affine) X_{f_j} given by $(X_{f_j})_{f_i|_{X_j}}$. This principal subset has the ring of functions $(B_j)_{f_i|_{X_j}}$, and hence

$$b|_{X_{f_i} \cap X_{f_j}} = \frac{b_j}{f_i|_{X_j}{}^{r_i}}$$

(and on $X_i \cap X_i$ we can just let $b_i = f, r_i = 0$). Without loss of generality, we can assume $r = r_1 = \cdots = r_n$ (otherwise let $R = \max r_i$ and multiply each b_i appropriately).

We wish to define a global map f by letting $f|_{X_{f_j}} = f_i \cdot b_j$. This function is clearly regular on each X_{f_j}. If $x \in X_{f_j} \cap X_{f_k}$, then either $f_i(x) = 0$ and then clearly the expression for $f|_{X_{f_j}}$ agrees with that of $f|_{X_k}$. Otherwise $x \in X_{f_i}$ and then both expressions are equal to $b \cdot f_i^{r+1}$. Hence f is well defined and regular locally, thus it is regular. Note that on X_{f_i} it is the case that $f = b \cdot f_i^{r+1}$, i.e. $b = f/f_i^{r+1}$, meaning that $b \in A_{f_i}$.

and =0 if X_{f_j} has no intersection with X_{f_i}

In total we've shown that $B_i = A_{f_i}$.

If we apply g to $x \in X_{f_i}$, we get that $g(x)(f_i) = f_i(x) \neq 0$, i.e. $g(X_{f_i}) = D(f_i) \subseteq Y$. This map has a clear inverse given by sending $y \in D(f_i)$ to the element $g^{-1}(y)(a/(f_i^n)) = a(y)/(f_i(y)^n)$. This means that $g|_{X_{f_i}}$ is an isomorphism. By problem 1, it follows that g itself is an isomorphism, i.e. $X = \text{spec}_k(A)$ is affine.