ELSEVIER Journal of Pure and Applied Algebra 159 (2001) 173–185

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Virtual operad algebras and realization of homotopy types

Vladimir Hinich[1]

Department of Mathematics, University of Haifa, Mt. Carmel, 31905 Haifa, Israel

Received 8 September 1999; received in revised form 3 January 2000
Communicated by C. Kassel

Abstract

We prove that the category of algebras over a cofibrant operad admits a closed model category structure. This leads to the notion of "virtual operad algebra" – the algebra over a cofibrant resolution of the given operad. In particular, virtual commutative algebras can serve to an algebraic description of homotopy p-types as in the recent preprint of Mandell (M. Mandell, E_∞-algebras and p-adic homotopy theory, Hopf preprint server, October, 1998). Our main result allows one to simplify the proof of Mandell's theorem. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 18G55; 55P99

1. Introduction

1.1. Let k be a base commutative ring, $C(k)$ be the category of complexes of k-modules. The category of operads $\mathtt{Op}(k)$ in $C(k)$ admits a closed model category (CMC) structure with quasi-isomorphisms as weak equivalences and surjective maps as fibrations (see [1], Section 6 and also Section 2 below).

Let now $\mathscr{O}$ be a cofibrant operad. The main result of this note (see Theorem 3.1) claims that the category of $\mathscr{O}$-algebras admits as well a CMC structure with quasi-isomorphisms as weak equivalences and surjective maps as fibrations. This allows one, following the pattern of [1], 5.4, to construct the homotopy category of *virtual* $\mathscr{O}$-algebras for any operad $\mathscr{O}$ over $C(k)$ as the homotopy category of $\mathscr{P}$-algebras for a cofibrant resolution $\mathscr{P} \to \mathscr{O}$ of the operad $\mathscr{O}$.

E-mail address: hinich@math.haifa.ac.il (V. Hinich)

[1] The author was partially supported by TMR Grant No. FMRX-CT97-0100.

PII: S0022-4049(00)00054-2

The main motivation of the note was to understand the following main result of Mandell's recent paper [3].

1.2. Theorem. *The singular cochain functor with coefficients in $\bar{\mathbb{F}}_p$ induces a contravariant equivalence from the homotopy category of connected p-complete nilpotent spaces of finite p-type to a full subcategory of the homotopy category of E_∞ $\bar{\mathbb{F}}_p$-algebras.*

In his approach, Mandell realizes the homotopy category of E_∞-algebras as a localization of the category of algebras over a "particular but unspecified" operad $\mathscr{E}$. One of major technical problems was that the category of $\mathscr{E}$-algebras did not seem to admit a CMC structure.

We suggest to choose $\mathscr{E}$ to be a cofibrant resolution of the Eilenberg–Zilber operad. Then according to Theorem 3.1, the category of $\mathscr{E}$-algebras admits a CMC structure. This simplifies the proof of Theorem 1.2.

1.3. Content of Sections. The main body of the note (Sections 2–4) can be considered as a complement to [1] where some general homology theory of operad algebras is presented.

In Section 2 we recall some results of [1] we need in the sequel. In Section 3 we prove the Main Theorem 3.1. In Section 4 we present, using Theorem 3.1, a construction of the homotopy category $\mathtt{Viral}(\mathscr{O})$ of virtual $\mathscr{O}$-algebras.

In Section 5 we review the proof of Mandell's theorem [3], stressing the simplifications due to our Theorem 3.1.

2. Homotopical algebra of operads: a digest of [1]

In this Section we recall some results from [1] and give some definitions we will be using in the sequel.

2.1. Category of operads. Let k be a commutative ring and let $C(k)$ denote the category of complexes of k-modules.

Recall ([1], 6.1.1) that the category $\mathtt{Op}(k)$ of operads in $C(k)$ admits a closed model category (CMC) structure in which weak equivalences are componentwise quasi-isomorphisms and fibrations are componentwise surjective maps.

Cofibrations in $\mathtt{Op}(k)$ are retractions of *standard cofibrations*; a map $\mathscr{O} \to \mathscr{O}'$ is a standard cofibration if $\mathscr{O}' = \varinjlim_{s \in \mathbb{N}} \mathscr{O}_s$ with $\mathscr{O}_0 = \mathscr{O}$ and each $\mathscr{O}_{s+1}$ is obtained from $\mathscr{O}_s$ by adding a set of free generators g_i with prescribed values of $d(g_i) \in \mathscr{O}_s$.

2.2. Algebras over an operad. Let $\mathscr{O} \in \mathtt{Op}(k)$.

The category of $\mathscr{O}$-algebras is denoted by $\mathtt{Alg}(\mathscr{O})$. For $X \in C(k)$ we denote by $F(\mathscr{O}, X)$ the free $\mathscr{O}$-algebra generated by X.

For any $d \in \mathbb{Z}$ denote by $W_d \in C(k)$ the contractible complex

$$0 \to k = k \to 0$$

concentrated in degrees $d, d+1$.

2.2.1. Definition. An operad $\mathcal{O} \in \mathtt{Op}(k)$ is called *homotopically amenable* if for any $A \in \mathtt{Alg}(\mathcal{O})$ the natural map

$$A \to A \sqcup F(\mathcal{O}, W_d)$$

is a quasi-isomorphism.

2.2.2. Proposition. (see [1], Theorem. 2.2.1) *Let $\mathcal{O}$ be homotopically amenable. Then the category of $\mathcal{O}$-algebras admits a CMC structure with quasi-isomorphisms as weak equivalences and surjective maps as fibrations.*

2.3. Examples.

2.3.1. First of all, not all operads are homotopically amenable. In fact, let $k = \mathbb{F}_p$, $\mathcal{O} = \mathtt{COM}$ (the operad of commutative algebras). Then the symmetric algebra of W_d fails to be contractible in degree p.

2.3.2. Proposition. (see [1], Theorem. 4.1.1) *Any Σ-split operad* (*see* [1], 4.2) *is homotopically amenable.*

In particular, all operads over $k \supseteq \mathbb{Q}$ are homotopically amenable. Also, all operads of form $\mathcal{T}^{\Sigma}$ where $\mathcal{T}$ is an asymmetric operad, in particular, $\mathtt{ASS}$ (see [1], 4.2.5), are homotopically amenable.

2.3.3. The main result of this note claims that any cofibrant operad is homotopically amenable.

2.4. Base change and equivalence. Let $f : \mathcal{O} \to \mathcal{O}'$ be a map of operads. Then a pair of adjoint functors

$$f^* : \mathtt{Alg}(\mathcal{O}) \to \mathtt{Alg}(\mathcal{O}') : f_* \quad (1)$$

is defined in a standard way.

2.4.1. Proposition. (*see* [1], 4.6.4.) *Let $f : \mathcal{O} \to \mathcal{O}'$ be a map of homotopically amenable operads. The inverse and direct image functors* (1) *induce the adjoint functors*

$$\mathbf{L}f^* : \mathtt{Hoalg}(\mathcal{O}) \to \mathtt{Hoalg}(\mathcal{O}') : \mathbf{R}f_* = f_* \quad (2)$$

between the corresponding homotopy categories.

2.4.2. Definition. A map $f:\mathcal{O}\to\mathcal{O}'$ of operads is called *strong equivalence* if for each $d=(d_1,\ldots,d_n)\in\mathbb{N}^n$, the induced map

$$\mathcal{O}(|d|)\otimes_{\Sigma_d}k\to\mathcal{O}'(|d|)\otimes_{\Sigma_d}k$$

is a quasi-isomorphism.

Here $|d|=\sum d_i$ and $\Sigma_d=\Sigma_{d_1}\times\cdots\times\Sigma_{d_n}\subseteq\Sigma_{|d|}$.

2.4.3. Proposition. *Let $f:\mathcal{O}\to\mathcal{O}'$ be a strong equivalence of homotopically amenable operads. Then the functors $\mathbf{L}f^*$, f_* are equivalences.*

In Section 5 we will be using the following version of Proposition 2.4.3:

2.4.4. Proposition. *Let $f:\mathcal{O}\to\mathcal{O}'$ be a strong equivalence of operads. Suppose $\mathcal{O}$ is homotopically amenable operad. Then for each cofibrant $\mathcal{O}$-algebra A the natural map*

$$A\to f_*(f^*(A))$$

is an equivalence.

2.4.5. Remark. A quasi-isomorphism of Σ-split operads compatible with the Σ-splittings is necessarily a strong equivalence.

Theorem 4.7.4 of [1] actually proves Propositions 2.4.4 and 2.4.3 together with the last Remark.

3. Main theorem

3.1. Theorem. *Any cofibrant operad $\mathcal{O}\in\mathtt{Op}(k)$ is homotopically amenable.*

In particular, the category of algebras $\mathtt{Alg}(\mathcal{O})$ over a cofibrant operad $\mathcal{O}$ admits a CMC structure with quasi-isomorphisms as weak equivalences and epimorphisms as fibrations.

3.2. Proof of the theorem.

3.2.1. First of all, we can easily reduce the claim to the case $\mathcal{O}$ is standard cofibrant. In fact, since $\mathcal{O}$ is cofibrant, it is a retraction of a standard cofibrant operad $\mathcal{O}'$. Let

$$\mathcal{O}\xrightarrow{\alpha}\mathcal{O}'\xrightarrow{\pi}\mathcal{O}$$

be a retraction. Let A be a $\mathcal{O}$-algebra. We can consider A as a $\mathcal{O}'$-algebra via π. Then the map $A\to A\sqcup F(\mathcal{O},M)$ is a retraction of the map $A\to A\sqcup F(\mathcal{O}',M)$. This reduces the theorem to the case $\mathcal{O}$ is standard cofibrant.

3.2.2. Let $\mathcal{O}=\varinjlim_{s\in\mathbb{N}}\mathcal{O}_s$ (see notation of 2.1, $\mathcal{O}_0=0$) be a standard cofibrant operad. Let $\{g_i\}$, $i\in I$ be a set of free (homogeneous) generators of $\mathcal{O}$.

Let a function $s: I \to \mathbb{N}$ be given so that $\mathcal{O}_s$ is freely generated as a graded operad by g_i with $s(i) \le s$ and, of course, $dg_i \in \mathcal{O}_{s(i)-1}$.

Let, finally, $\text{val}: I \to \mathbb{N}$ and $d: I \to \mathbb{Z}$ be the valence and the degree functions defined by the condition $g_i \in \mathcal{O}(\text{val}(i))^{d(i)}$.

The collection $\mathcal{I} = (I, s, \text{val}, d)$ will be called *a type* of $\mathcal{O}$.

Since we deal with free operads and free algebras, it is worthwhile to have an appropriate notion of tree. Fix a type $\mathcal{I} = (I, s, \text{val}, d)$.

Put $I^+ = I \cup \{a, m\}$ (a and m will be special marks on some terminal vertices of our trees) and extend the functions $\text{val}: I \to \mathbb{N}$ and $d: I \to \mathbb{Z}$ to I^+ by setting $\text{val}(a) = \text{val}(m) = d(a) = d(m) = 0$.

3.2.3. Definition. *A $\mathcal{I}$-tree* is a finite connected directed graph such that any vertex has ≤ 1 ingoing arrows; each vertex is marked by an element $i \in I^+$ so that $\text{val}(i)$ equals the number of outgoing arrows which are numbered by $1, \ldots, \text{val}(i)$.

The set of vertices of a tree T will be denoted by $V(T)$. Terminal vertices of a $\mathcal{I}$-tree are the ones having no outgoing arrows. In particular, all vertices marked by a or by m are terminal.

3.2.4. Definition. A $\mathcal{I}$-tree T is called *proper* if the following property (P) is satisfied:
(P) For any vertex v of T one of the possibilities (a)–(c) below occurs:
(a) v is terminal;
(b) v admits an outgoing arrow to a non-terminal vertex;
(c) v admits an outgoing arrow to a vertex marked by m.

We denote by $\mathcal{P}(\mathcal{I})$ the set of isomorphism classes of proper $\mathcal{I}$-trees. The following easy result justifies the notion of proper tree:

3.2.5. Proposition. *Let $\mathcal{O}$ be a standard cofibrant operad of type $\mathcal{I} = (I, s, \text{val}, d)$, A be a $\mathcal{O}$-algebra and $M \in C(k)$. Then the coproduct $B := A \sqcup F(M)$ is given, as a graded k-module, by the formula*

$$B = \bigoplus_{T \in \mathcal{P}(\mathcal{I})} A^{\otimes a(T)} \otimes M^{\otimes m(T)}[d(T)] \tag{3}$$

where $a(T)$ (respectively, $m(T)$) is the number of vertices of type a (respectively, of type m) in T and $d(T) = \sum_{v \in V(T)} d(v)$.

Proof. Since the operad $\mathcal{O}$ is freely generated by $\{g_i\}$ as an operad of graded k-modules, its action on a graded k-module B is given by a collection of operations γ_i, $i \in I$,

$$\gamma_i : B^{\otimes \text{val}(i)} \to B[d(i)],$$

defined by g_i.

Denote for $T \in \mathcal{P}(\mathcal{I})$

$$B_T = A^{\otimes a(T)} \otimes M^{\otimes m(T)}[d(T)], \tag{4}$$

so that the graded k-module B defined in (3) is the direct sum of B_T. Let $T_0 \in \mathcal{P}(\mathcal{I})$ be the single vertex marked by a. Then $B_{T_0} = A$.

The map $\gamma_i : B^{\otimes \mathrm{val}(i)} \to B[d(i)]$ is defined as follows. Its restriction on $B_{T_0}^{\otimes \mathrm{val}(i)}$ coincides with the action of g_i on $\mathcal{O}$-algebra A. The restriction of γ_i on $B_{T_1} \otimes \cdots \otimes B_{T_{\mathrm{val}(i)}}$ in the case when at least one of T_j is different from T_0, is given by glueing the trees T_j to the corollary (tree having no internal vertices) marked by $i \in I$. The resulting tree will be necessarily proper.

The tree T_0 defines a map $A \to B$; the trees $T \in \mathcal{P}(\mathcal{I})$ satisfying the condition $a(T) = 0$ define a map $F(M) \to B$. Any $\mathcal{O}$-algebra C endowed with maps $A \to C$ and $F(M) \to C$ gives rise to a unique map $B \to C$. This proves the assertion. □

3.2.6. Let $\mathcal{W}$ be the set of maps $\mathbb{N} \to \mathbb{N}$ having finite support. Endow $\mathcal{W}$ with the following lexicographic order. For $f, g \in \mathcal{W}$ we will say that $f > g$ if there exists a $s \in \mathbb{N}$ such that $f(s) > g(s)$ and $f(t) = g(t)$ for all $t > s$.

The set $\mathcal{W}$ well-ordered.

Our next step is to define a filtration of $B = A \sqcup F(M)$ indexed by $\mathcal{W}$.

3.2.7. Definition. Let $T \in \mathcal{P}(\mathcal{I})$. The weight of T, $w(T) \in \mathcal{W}$ is the function $\mathbb{N} \to \mathbb{N}$ which assigns to any $s \in \mathbb{N}$ the number of vertices v of T whose mark $i \in I$ satisfies $s(i) = s$.

Now we are able to define a filtration on B.

3.2.8. Let $A, M, B = A \sqcup F(M)$ be as above. For each $f \in \mathcal{W}$ define

$$\mathcal{F}_f(B) = \bigoplus_{T : w(T) \le f} B_T,$$

the graded k-modules B_T being defined by formula (4).

The homogeneous components of the associated graded complex are defined as

$$\mathrm{gr}_f^{\mathcal{F}}(B) = \mathcal{F}_f(B) \Big/ \sum_{g < f} \mathcal{F}_g(B) .$$

The following properties of the filtration $\mathcal{F}$ are obvious:

3.2.9. Proposition. (1) *For each $f \in \mathcal{W}$ the graded submodule $\mathcal{F}_f$ is a subcomplex of B.*
(2) *One has $\mathcal{F}_0 = A \oplus M$.*
(3) *Suppose M is a contractible complex. Then for each $f > 0$ the homogeneous components $\mathrm{gr}_f^{\mathcal{F}}$ are contractible.*

3.2.10. Proposition. 3.2.9 *immediately implies that a standard cofibrant operad is homotopically amenable. This, together with* 3.2.1, *gives Theorem* 3.1.

4. Virtual algebras

4.1. Theorem. 3.1 suggests the following definition:

Let $\mathscr{O} \in \mathtt{Op}(k)$. The homotopy category of virtual $\mathscr{O}$-algebras $\mathtt{Viral}(\mathscr{O})$ is defined as $\mathtt{Hoalg}(\mathscr{P})$ where $\mathscr{P} \to \mathscr{O}$ is a cofibrant resolution of $\mathscr{O}$ in the category of operads.

One should, however, do some work, to ensure the definition above make sense.

4.2. Base change. Any morphism $f:\mathscr{P} \to \mathscr{Q}$ of operads induces a pair of adjoint functors

$$f^*:\mathtt{Alg}(\mathscr{P}) \rightleftarrows \mathtt{Alg}(\mathscr{Q}):f_*. \tag{5}$$

Theorem 3.1 together with Proposition 2.4.1 give immediately the following:

4.2.1. Proposition. *For any morphism $f:\mathscr{P} \to \mathscr{Q}$ of cofibrant operads the adjoint functors* (5) *induce a pair of adjoint functors*

$$\mathbf{L}f^*:\mathtt{Hoalg}(\mathscr{P}) \rightleftarrows \mathtt{Hoalg}(\mathscr{Q}):\mathbf{R}f_* = f_* \tag{6}$$

between the homotopy categories.

4.2.2. Proposition. (1) *Let $f:\mathscr{P} \to \mathscr{Q}$ be a weak equivalence of cofibrant operads. Then f is a strong equivalence. In paticular, the derived functors of inverse and direct image* (6) *establish an equivalence of the homotopy categories.*

(2) *Let $f,g:\mathscr{P} \to \mathscr{Q}$ be homotopic maps between cofibrant operads. Then there is an isomorphism of functors*

$$f_*, g_*:\mathtt{Hoalg}(\mathscr{Q}) \to \mathtt{Hoalg}(\mathscr{P}).$$

This isomorphism depends only on the homotopy class of the homotopy connecting f with g.

Proof. (1) Let $d=(d_1,\ldots,d_n)$, $|d| = \sum d_i$ and let $\Sigma_d = \prod \Sigma_{d_i} \subseteq \Sigma_{|d|}$.

We have to check that the map

$$\mathscr{P}(|d|) \otimes_{\Sigma_d} k \to \mathscr{Q}(|d|) \otimes_{\Sigma_d} k,$$

induced by f, is a quasi-isomorphism.

Since $\mathscr{P}$ and $\mathscr{Q}$ are cofibrant operads, $\mathscr{P}(|d|)$ and $\mathscr{Q}(|d|)$ are cofibrant as complexes of $k(\Sigma_{|d|})$-modules. Therefore, their quasi-isomorphism is a homotopy equivalence of $k(\Sigma_{|d|})$-modules and therefore is preserved after tensoring by k.

(2) We present here a proof which is identical to the proof of Lemma 5.4.3(2) of [1].

Let $\mathscr{Q} \xrightarrow{\alpha} \mathscr{Q}^I \overset{p_0,p_1}{\rightrightarrows} \mathscr{Q}$ be a path diagram for $\mathscr{Q}$ (see [4, Chapter 1]) so that α is an acyclic cofibration. Since the functors p_{0*} and p_{1*} are both quasi-inverse to an equivalence $\alpha_*:\mathtt{Hoalg}(\mathscr{Q}^I) \to \mathtt{Hoalg}(\mathscr{Q})$, there is a natural isomorphism $\theta: p_{0*} \to p_{1*}$.

Therefore, any homotopy $F:\mathscr{P} \to \mathscr{Q}^I$ between f and g defines an isomorphism $\theta_F = F_* \circ \theta$ between f_* and g_*. Let now $F_0, F_1 : \mathscr{P} \to \mathscr{Q}^I$ be homotopic. The homotopy can be realized by a map $h:\mathscr{P} \to \mathscr{R}$ where $\mathscr{R}$ is taken from a path diagram

$$\mathscr{Q}^I \xrightarrow{\beta} \mathscr{R} \xrightarrow{q_0 \times q_1} \mathscr{Q}^I \times_{\mathscr{Q}\times\mathscr{Q}} \mathscr{Q}^I \tag{7}$$

where β is an acyclic cofibration, $q_0 \times q_1$ is a fibration, $q_i \circ h = F_i$, $i = 0, 1$. Passing to the corresponding homotopy categories we get the functors $q_{i*} \circ p_{j*} : \mathtt{Hoalg}(\mathscr{Q}) \to \mathtt{Hoalg}(\mathscr{R})$ which are quasi-inverse to $\alpha_* \circ \beta_* : \mathtt{Hoalg}(\mathscr{R}) \to \mathtt{Hoalg}(\mathscr{Q})$. One has $\theta_{q_0} = \theta_{q_1}$ since the equivalence β_* applied to both sides gives the same result. This implies that $\theta_{F_0} = \theta_{F_1}$. □

4.3. Virtual operad algebras. Our construction of the category of virtual $\mathscr{O}$-algebras follows the construction of virtual modules in [1], 5.4.

Let $\mathtt{Op}^c(k)$ denote the category of cofibrant operads in $C(k)$. For each $\mathscr{P} \in \mathtt{Op}^c(k)$ let $\mathtt{Hoalg}(\mathscr{P})$ be the homotopy category of $\mathscr{P}$-algebras. These categories form a fibred category $\mathtt{Hoalg}$ over $\mathtt{Op}^c(k)$, with the functors $\mathbf{R}f_* = f_*$ playing the role of "inverse image functors".

Let $\mathscr{O} \in \mathtt{Op}(k)$. Let $\mathtt{Op}^c(k)/\mathscr{O}$ be the category of maps $\mathscr{P} \to \mathscr{O}$ of operads with cofibrant $\mathscr{P}$. The obvious functor

$$c_{\mathscr{O}} : \mathtt{Op}^c(k)/\mathscr{O} \to \mathtt{Op}^c(k)$$

assigns the cofibrant operad $\mathscr{P}$ to an arrow $\mathscr{P} \to \mathscr{O}$.

4.3.1. Definition. The (homotopy) category $\mathtt{Viral}(\mathscr{O})$ of virtual $\mathscr{O}$-algebras is the fibre of $\mathtt{Hoalg}$ at $c_{\mathscr{O}}$. In other words, an object of $\mathtt{Viral}(\mathscr{O})$ consists of a collection $A_a \in \mathtt{Hoalg}(\mathscr{P}_a)$ for each $a:\mathscr{P}_a \to \mathscr{O}$ in $\mathtt{Op}^c(k)/\mathscr{P}$ and of compatible collection of isomorphisms $\phi_f : A_a \to f_*(A_b)$ given for every $f:\mathscr{P}_a \to \mathscr{P}_b$ in $\mathtt{Op}^c(k)/\mathscr{O}$.

4.3.2. Corollary. *Let $\alpha:\mathscr{P} \to \mathscr{O}$ be a weak equivalence of operads with cofibrant $\mathscr{P}$. Then the obvious functor*

$$q_\alpha : \mathtt{Viral}(\mathscr{O}) \to \mathtt{Hoalg}(\mathscr{P})$$

is an equivalence of categories.

Proof. We will construct a quasi-inverse functor $q^\alpha : \mathtt{Hoalg}(\mathscr{P}) \to \mathtt{Viral}(\mathscr{O})$. For this choose for any map $\beta:\mathscr{Q} \to \mathscr{O}$ a map $f_\beta : \mathscr{Q} \to \mathscr{P}$ making the corresponding triangle homotopy commutative. Then, for any $A \in \mathtt{Hoalg}(\mathscr{P})$ we define $q^\alpha(A)$ to be the collection of $f_{\beta*}(A) \in \mathtt{Hoalg}(\mathscr{Q})$. According to Proposition 4.2.2, the definition does not depend on the choice of $f_\beta's$. □

The corollary means that the homotopy category of virtual $\mathscr{O}$-algebras is really the category of algebras over a cofibrant resolution of $\mathscr{O}$.

4.3.3. Any map $f:\mathcal{O}\to\mathcal{O}'$ defines an obvious functor $\mathtt{Op}^c(k)/\mathcal{O}\to\mathtt{Op}^c(k)/\mathcal{O}'$. This induces a direct image functor

$$f_*:\mathtt{Viral}(\mathcal{O}')\to\mathtt{Viral}(\mathcal{O}).$$

According to Corollary 4.3.2, this functor admits a left adjoint inverse image functor f^* which can be calculated using cofibrant resolutions for $\mathcal{O}$ and $\mathcal{O}'$.

4.4. The Σ -split case.

Proposition 4.4.1 below was communicated to the author by the referee. Let $\mathcal{O}$ be a Σ-split operad. This means that a collection of Σ_n-equivariant maps

$$t(n):\mathcal{O}(n)\to\mathcal{O}(n)\otimes k\Sigma_n$$

splitting the Σ_n-action on $\mathcal{O}(n)$ and satisfying some compatibility properties is given — see [1], 4.2.4.

Choose a cofibrant resolution $f:\mathcal{P}\to\mathcal{O}$. For each n the map

$$\mathcal{P}(n)\otimes k\Sigma_n\to\mathcal{O}(n)\otimes k\Sigma_n$$

is a cofibrant resolution of complexes of $k\Sigma_n$-modules. Therefore, there exists a Σ_n-equivariant map

$$s(n):\mathcal{P}(n)\to\mathcal{P}(n)\otimes k\Sigma_n$$

commuting with $t(n)$. The composition of $s(n)$ with the obvious map $\mathcal{P}(n)\otimes k\Sigma_n\to\mathcal{P}(n)$ is homotopic to identity since it commutes with $\mathrm{id}:\mathcal{O}(n)\to\mathcal{O}(n)$.

Now for any $d=(d_1,\dots,d_n)\in\mathbb{N}^n$ the map

$$\tilde{f}:\mathcal{P}(|d|)\otimes k\Sigma_{|d|}\otimes_{\Sigma_d}k\to\mathcal{O}(|d|)\otimes k\Sigma_{|d|}\otimes_{\Sigma_d}k \tag{8}$$

induced by f is obviously a quasi-isomorphism. The maps $s(|d|)$ and $t(|d|)$ give a presentation of

$$H(\mathcal{P}(|d|)\otimes_{\Sigma_d}k)\to H(\mathcal{O}(|d|)\otimes_{\Sigma_d}k)$$

as a retract of $H(\tilde{f})$ in (8). Thus, the map $f:\mathcal{P}\to\mathcal{O}$ is a strong equivalence.

This proves the following:

4.4.1. Proposition. *For a Σ-split operad $\mathcal{O}$ there is an equivalence of categories*

$$\mathtt{Viral}(\mathcal{O})\to\mathtt{Hoalg}(\mathcal{O}).$$

4.4.2. Remark. We do not know whether the above equivalence takes place for any homotopically amenable operad $\mathcal{O}$.

5. Application: realization of homotopy *p*-types

Theorem 3.1 allows one to simplify Mandell's proof of Theorem 1.2.

In his approach, Mandell realizes the homotopy category of E_∞-algebras as a localization of the category of algebras over a “particular but unspecified” operad $\mathscr{E}$. Then the category of $\mathscr{E}$-algebras does not seem to admit a CMC structure. Nevertheless, Mandell proves that the category of $\mathscr{E}$-algebras admits a structure which “looks like” a CMC structure and allows one to use the standard model category arguments in constructing derived functors.

Theorem 3.1 suggests to choose $\mathscr{E}$ to be a cofibrant resolution of the Eilenberg–Zilber operad. Then the category of $\mathscr{E}$-algebras admits a CMC structure.

In this section we give a summary of the proof Mandell’s Theorem 1.2. We also present Proposition 5.2.3 which, together with the usage of a cofibrant operad $\mathscr{E}$, makes Mandell’s agrument much shorter.

5.1. Adjoint functors C^* and U

5.1.1. Recall [2] that the cochain complex $C^*(X)$ of an arbitrary simplicial set $X \in \Delta^{\mathrm{op}}\mathtt{Ens}$ admits a canonical structure of algebra over the Eilenberg–Zilber operad $\mathscr{Z}$ which is weakly equivalent to the operad $\mathtt{COM}$ of commutative algebras. Choose any cofibrant resolution $\mathscr{E}$ of $\mathscr{Z}$. The category of virtual commutative algebras $\mathtt{Viral(COM)}$ is canonically equivalent to $\mathtt{Hoalg}(\mathscr{E})$.

5.1.2. For each commutative ring k define

$$C^*(\_,k):(\Delta^{\mathrm{op}}\mathtt{Ens})^{\mathrm{op}} \to \mathtt{Alg}(k\otimes\mathscr{E}) \tag{9}$$

(here and below $\otimes$ means tensoring over $\mathbb{Z}$) to be the functor of normalized k-valued cochains. This functor admits an obvious left adjoint functor

$$U_k:\mathtt{Alg}(k\otimes\mathscr{E}) \to (\Delta^{\mathrm{op}}\mathtt{Ens})^{\mathrm{op}} \tag{10}$$

given by the formula

$$U_k(A)_n = \mathrm{Hom}(A, C^*(\Delta^n,k)) \tag{11}$$

The pair of functors $C^*(\_,k)$ and U_k satisfies the requirements of Quillen’s theorem [4, Section 4, Theorem 3].

Since the functor $C^*(\_,k)$ preserves weak equivalences, one therefore obtains a pair of derived adjoint functors

$$\mathbb{U}_k:\mathtt{Viral(COM)} = \mathtt{Hoalg}(k\otimes\mathscr{E}) \rightleftarrows \mathscr{H}o : C^*(\_,k), \tag{12}$$

$\mathscr{H}o$ being the homotopy category of simplicial sets.

5.2. A simplicial set X is called *k-resolvable* if the natural map

$$u_X:X \to \mathbb{U}_k C^*(X,k)$$

is a weak equivalence.

The following two lemmas allow one to construct resolvable spaces:

5.2.1. Lemma. (Mandell [3], Theorem 1.1) *Let X be the limit of a tower of Kan fibrations*

$$\cdots \to X_n \to \cdots \to X_0.$$

Assume that the canonical map from H^*X to colim *H^*X_n is an isomorphism. If each X_n is k-resolvable, then X is k-resolvable.*

5.2.2. Lemma. (Mandell [3], Theorem 1.2) *Let X, Y and Z be connected simplicial sets of finite type, and assume that Z is simply connected. Let $X \to Z$ and $Y \to Z$ be given, so that $Y \to Z$ is a Kan fibration. Then, if X, Y and Z are k-resolvable then so is the fibre product $X \times_Z Y$.*

Lemma 5.2.1 follows from the fact that the functor $\mathbb{U}$ carries homotopy colimits in $\mathtt{Alg}(\mathscr{E})$ into homotopy limits in $\Delta^{\mathrm{op}}\mathtt{Ens}$. The proof of Lemma 5.2.2 is similar, but needs an extra argument which can be deduced from Proposition 5.2.3 below.

Using the CMC structure on $\mathtt{Op}(k)$, one can embed the obvious map of operads $\mathtt{ASS} \to \mathtt{COM}$ into the following commutative diagram

$$\begin{array}{ccccc} \mathtt{ASS}_\infty & \xrightarrow{\alpha} & \mathscr{E} & & \\ \downarrow & & \downarrow{\scriptstyle\tau} & & \\ \mathtt{ASS} & \xrightarrow{\bar\alpha} & \bar{\mathscr{E}} & \xrightarrow{\pi} & \mathtt{COM} \end{array},$$

where $\mathtt{ASS}_\infty$ is the operad of A_∞-algebras, α is a cofibration, π is a weak equivalence and the square is cocartesian.

5.2.3. Proposition. (compare to [3], Lemma 5.2). *Let $A \to B$ and $A \to C$ be cofibrations of cofibrant $\mathscr{E}$-algebras. Let $\bar A = \tau^*(A)$, and similarly for $\bar B, \bar C$. Then the natural maps*

$$B \sqcup^A C \xrightarrow{t} \bar B \sqcup^{\bar A} \bar C \xleftarrow{r} \bar B \otimes_{\bar A} \bar C$$

are quasi-isomorphisms in $C(k)$. Here t is induced by τ and r is induced by the composition

$$\bar B \otimes \bar C \to (\bar B \sqcup^{\bar A} \bar C) \otimes (\bar B \sqcup^{\bar A} \bar C) \xrightarrow{\text{mult.}} \bar B \sqcup^{\bar A} \bar C.$$

Proof. (1) *t is a quasi-isomorphism.* The functor τ^* commutes with colimits. Therefore, it is enough to prove that the natural map $A \to \tau_*\tau^*(A)$ is a weak equivalence for a cofibrant algebra A. According to 2.4.4, it is enough to check that $\tau : \mathscr{E} \to \bar{\mathscr{E}}$ is a strong equivalence of operads.

Since α is a cofibration, $\bar\alpha$ is a cofibration as well. Therefore, both $\mathscr{E}(n)$ and $\bar{\mathscr{E}}(n)$ are cofibrant over $k\Sigma_n$. Then the strong equivalence of $\mathscr{E}$ and $\bar{\mathscr{E}}$ follows from their weak equivalence.

(2) *r is a quasi-isomorphism.*

Suppose A is standard cofibrant and the maps $A \to B$, $A \to C$ are standard cofibrations. Let $\{e_i, i \in I\}$, $\{e_j, j \in I \cup J\}$, $\{e_k, k \in I \cup K\}$, be graded free bases of A, B and C respectively (the index sets I, J, K are disjoint).

The sets I, J and K are well-ordered and the differential of e_i is expressed through $e_{i'}$ with $i' < i$.

Put $S = I \cup J \cup K$ with the order given by $i < j < k$ for $i \in I$, $j \in J$, $k \in K$. Let $\tilde{S}$ be the set of maps $S \to \mathbb{N}$ with finite support and with the lexicographic order as in 3.2.6.

For $f \in \tilde{S}$ denote $|f| = \sum_{s \in S} f(s)$.

The algebra $\bar{B} \sqcup^{\bar{A}} \bar{C}$ has an obvious increasing filtration by subcomplexes $\{F_f\}$ indexed by $f \in \tilde{S}$.

The homogeneous component of the associated graded complex for $f \in \tilde{S}$ takes form

$$\mathrm{gr}_f(F) = \bar{\mathscr{E}}(|f|) \otimes_{\Sigma_f} e^f$$

where $e^f = \prod_{s \in S} e_s^{f(s)}$ and $\Sigma_f = \prod_{s \in S} \Sigma_f(s)$.

Define a filtration $\{F'_f\}$ of $\bar{B} \otimes_{\bar{A}} \bar{C}$ indexed by the same set $\tilde{S}$. It is given by the formula

$$F'_f = \bigoplus_{g < f} \bar{\mathscr{E}}(|g|_1) \otimes \bar{\mathscr{E}}(|g|_2) \otimes_{\Sigma_g} e^g$$

where $|g|_1 = \sum_{s \in I \cup J} g(s)$ and $|g|_2 = \sum_{s \in K} g(s)$. The homogeneous component for $f \in \tilde{S}$ is given by

$$\mathrm{gr}_f(F') = \bar{\mathscr{E}}(|f|_1) \otimes \bar{\mathscr{E}}(|f|_2) \otimes_{\Sigma_f} e^f.$$

The map $r : \bar{B} \otimes_{\bar{A}} \bar{C} \to \bar{B} \sqcup^{\bar{A}} \bar{C}$ is compatible with the filtrations. The corresponding map of the homogeneous components

$$\mathrm{gr}_f(r) : \bar{\mathscr{E}}(|f|) \otimes_{\Sigma_f} e^f \to \bar{\mathscr{E}}(|f|_1) \otimes \bar{\mathscr{E}}(|f|_2) \otimes_{\Sigma_f} e^f$$

is induced by the map

$$\bar{\mathscr{E}}(|f|_1) \otimes \bar{\mathscr{E}}(|f|_2) \to \bar{\mathscr{E}}(|f|), \tag{13}$$

which is obviously quasi-isomorphism. The assertion then follows from the observation that both the left and the right hand side of (13) are cofibrant over $k(\Sigma_f)$. □

5.3. The following theorem implies that the Eilenberg–MacLane space $K(\mathbb{Z}/p, n)$ is $\bar{\mathbb{F}}_p$-resolvable:

5.3.1. Theorem. (cf. [3], Proposition A.7). *The space $K(\mathbb{Z}/p, n)$ is k-resolvable iff $k \supseteq \mathbb{F}_p$ and the frobenius $F : k \to k$ gives rise to a short exact sequence of abelian groups*

$$0 \to \mathbb{F}_p \to k \xrightarrow{1-F} k \to 0. \tag{14}$$

This, together with 5.2.1 and 5.2.2, yields Theorem 1.2.

Acknowledgements

This work was made during my stay at the Max-Planck Institut für Mathematik at Bonn. I express my gratitude to the Institute for hospitality and excellent working conditions. I am grateful to P. Salvatore for a useful discussion. I am also grateful to the referee for numerous suggestions. In particular, he suggested Proposition 4.4.1 (and its proof) instead of some partial claims of the first version of the paper.

References

[1] V. Hinich, Homological Algebra of homotopy algebras, Comm. Algebra 25 (10) (1997) 3291–3323.
[2] V. Hinich, V. Schechtman, Homotopy limit of homotopy algebras, Lecture Notes in Math. 1289 (1987) 240–264.
[3] M. Mandell, E_∞-algebras and p-adic homotopy theory, Hopf preprint server, October, 1998.
[4] D. Quillen, Homotopical algebra, Lecture Notes in Math. 43 (1967).