Orbital Variety Closures and the Convolution Product in Borel-Moore Homology

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1. Introduction

1.1. The aim of this paper is to prove a twenty year old conjecture [J3, 9.9] on inclusions of orbital variety closures. The description of inclusions of orbital variety closures is a very difficult problem and even in type A an extensive analysis [M1,M3] has not yet provided a complete solution (see 6.7). Our solution is in terms of certain (as yet unknown) geometric data analogous to Verma module multiplicities. It was motivated by the analogy between geometry and representation theory as expressed by the so-called orbit method sometimes referred to as quantization. When one passes to the semisimple case this method encounters a number of difficulties because coadjoint orbits and unitary representations (or primitive ideals) do not quite match up. We have suggested [J5, 1.5] a coherent way out of this difficulty. Namely certain attributes of the geometric and of the representational pictures can be compared and then their detailed description is to be given by exactly the same algorithms in each case but by using slightly different data. In this fashion the characteristic polynomials of orbital varieties were defined [J3, Sect. 2] and conjectured [J3, 9.8] to be expressable in terms of geometric data just as Goldie rank polynomials are given in terms of Verma module multiplicities. Several proofs were later given [R,J5,V]. This result also described the geometric cells, that is the fibres of the Steinberg map, in terms of the above geometric data and gave a way to classify nilpotent orbits so that their relation to primitive ideals becomes transparent. Our present results deepen and refine this relationship describing in particular the geometric cones which determine orbit and orbital variety closures. Its interest may well go beyond semisimple Lie algebras as there is an analogue of the Steinberg variety in the theory of path algebras, see [Re,BN] for example.

1.2. In more detail let $\mathfrak{g}$ be a complex semisimple Lie algebra and fix a Cartan subalgebra $\mathfrak{h}$. In 1977 Duflo [D] showed that every primitive ideal of the enveloping algebra $U(\mathfrak{g})$ is the annihilator of a simple highest weight module, thus giving a surjective map from $\mathfrak{h}^*$ to $Prim\ U(\mathfrak{g})$. A basic question was to determine the fibres of the Duflo map.

Using the centre of $U(\mathfrak{g})$ and the Harish-Chandra isomorphism one is reduced to studying fibres of maps from the Weyl group W to $Prim\ U(\mathfrak{g})$. It was natural to believe this to have a combinatorial solution. Shortly after Duflo's work we noted that Verma module multiplicities should play a key role. This can be expressed by assigning to each $w \in W$ an appropriate Verma module, so that the simple highest weight modules gave a second basis (noted $a(w) : w \in W$ in [J1, 5.7] and see 6.2) of $\mathbb{Q}W$. We noted that this change of basis gives a flag of left ideals of $\mathbb{Q}W$ and suggested that the inclusion relations between them should exactly describe the inclusion relations of primitive ideals (see 6.8). This was partly established [J1, 5.8(ii)] and the proof completed by Vogan [V] almost immediately afterwards. It was a key motivation for the work of Kazhdan and Lusztig [KL1] who conjectured a purely combinatorial algorithm to determine the $a(w)$, whose validity was established shortly afterwards by several authors [BB,BK].

1.3. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form and let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a triangular decomposition. A coadjoint orbit $\mathcal{O}$ is said to be nilpotent if $\mathcal{O} \cap \mathfrak{n}$ is non-empty. After Spaltenstein $\mathcal{O} \cap \mathfrak{n}$ is equidimensional [S1] and one calls its irreducible components orbital varieties. One may show [J3, 7.5] that an orbital variety is Lagrangian with respect to the Kirillov-Kostant symplectic form on $\mathcal{O}$ and hence a natural object to quantize. Despite earlier optimism not all orbital varieties can be quantized giving eventually a primitive ideal [J6, 1.3]; but to determine those which can be is a key open problem on which some recent progress has been reported [B,JM,P]. Here we just view an orbital variety as the natural geometric analogue of a primitive ideal. In this the Steinberg map replaces the Duflo map and the conormals of Bruhat cells in the cotangent bundle on the flag variety are viewed as the geometric analogues of simple highest weight modules. Finally the base change in $\mathbb{Q}W$ necessary to give an analogous description of inclusion of orbital variety closures is provided through the W action on the Borel-Moore homology of the "conormal variety". Here the fundamental classes of the conormals form the required basis $A(w) : w \in W$ of $\mathbb{Q}W$ needed to describe the geometric analogues of cones and cells. That this is the appropriate identification is made particularly evident by the integral formula of [J2, 4.7].

1.4. Once one has understood how to make the correct constructions based on these analogies the proof of our main theorem 5.5 describing orbital variety closures follows relatively easily from known results and techniques. A basic ingredient is that the top Borel-Moore homology on the Steinberg variety has an algebra stucture making it isomorphic to $\mathbb{Q}W$. This was noted in particular by Ginzburg [G]. Here we further note that the top Borel-Moore homology on the conormal variety is a module for this algebra and that both are naturally isomorphic as W modules. The relationship between them is analogous to the Bernstein-Gelfand equivalence [BG] between the Harish-Chandra and highest weight categories. This leads to a geometric analogue of the Enright functor which plays a significant role [J2, 5.7, 5.8] in describing primitive ideal inclusions. Indeed this is how we obtain a right action on the conormal variety rather than the left action which is inherent in Hotta's work [H]. A key observation in our work is that orbital variety closures can be expressed in terms of set theoretic convolution of irreducible components of the Steinberg variety with conormals. It then remains to show that multiplication in homology "sees" all the irreducible components of set theoretic convolution. This is a delicate point as intersections fail to be proper. However enough intersections are proper and furthermore a required equidimensionality results. This result is rather fortuitous in that one cannot hope to obtain explicit formulae. It results from Krull's theorem in a manner analogous to Spaltenstein's proof [S1] of the equidimensionality of $\mathcal{O} \cap \mathfrak{n}$. From these things combined our proof results. We remark that in our work we shall only rely on Ginzburg's formalism [G] as presented in [CG, Sect. 3.4]. This means that $A(w)$ is determined by the transition matrix defined in [CG, 3.4.13]. This avoids having to make any comparison with the approaches of Kazhdan-Lusztig [KL2] or Rossmann [R]. However see the remark in 5.3.

2. A Review of the Steinberg Variety

2.1. Let G be a connected, simply-connected complex semisimple algebraic group. Fix a

Borel subgroup B of G. Recall that $N_G(B) = B$ and that the set $\mathcal{B}$ of all Borel subgroups of G is a single G orbit under conjugation. Thus $\mathcal{B}$ identifies with G/B. In this it is convenient to let gB denote either the right B coset defined by $g \in G$ or the B conjugate gBg^{-1}, which we may sometimes also write as ${}^gB$.

2.2. Set $\mathfrak{g} = Lie\ G$ with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h} \oplus \mathfrak{n} = Lie\ B$. Let N^-, H, N be the corresponding connected subgroups of G. Let Δ (resp. $\Delta^{\pm}$) denote the set of root (resp. positive, negative roots) defined by this decomposition and $\pi \subset \Delta^+$ the corresponding set of simple roots. For each $\alpha \in \pi$, let s_α be the corresponding reflection and W the subgroup of $Aut\ \mathfrak{h}^*$ they define, and x_α a root vector of weight α. Of course W identifies with $N_G(H)/H$ and we simply use $w \in W$ to also denote a representative in $N_G(H)$.

2.3. Recall [C, Chap. 8] the Bruhat decomposition

$$G = \coprod_{w \in W} BwB.$$

Using bar to denote Zariski closure one has

$$\overline{BwB} = \coprod_{y \leq w} ByB,$$

where $\leq$ denotes the Bruhat order. The Bruhat cell $X_w := BwB/B$ in G/B has a similar closure property. Again dim $X_w = \ell(w)$, where $\ell(\cdot)$ denotes reduced length. In particular dim $G/B = |\Delta^+| =: n$.

Take $\alpha \in \pi$. Recall [C, Prop. 8.2.4] that

$$Bs_\alpha BwB = Bs_\alpha wB, \quad if \quad s_\alpha w > w,$$

$$\overline{Bs_\alpha BwB} = \overline{BwB}, \quad if \quad s_\alpha w < w.$$

For each $\alpha \in \pi$, let P_α denote the parabolic subgroup it defines. One has $P_\alpha = \overline{Bs_\alpha B} = Bs_\alpha B \amalg B$. Set $\mathfrak{p}_\alpha = Lie\ P_\alpha$. Its nilradical $\mathfrak{m}_\alpha$ has codimension 1 in $\mathfrak{n}$. Let

$\mathbb{P}_\alpha$ denote the set of all Borel subgroups contained in P_α. It identifies with P_α/B and is isomorphic to the projective line $\mathbb{P}^1$.

For each $w \in W$, let ${}^wN$ (resp. ${}^w\mathfrak{n}$) denote the w conjugate of N (resp. ${}^w\mathfrak{n}$) and set $N_w = {}^wN \cap N$. One has $Lie\ N_w = \mathfrak{n} \cap^w \mathfrak{n}$. In particular N_{s_α} is the unipotent radical of P_α and $\mathfrak{n} \cap^{s_\alpha} \mathfrak{n} = \mathfrak{m}_\alpha$.

2.4. Set $\mathcal{N} = G\mathfrak{n}$ which is the cone of nilpotent elements of $\mathfrak{g}$. After Dynkin, $\mathcal{N}/G$ is finite. Each $\mathcal{O} \in \mathcal{N}/G$ is identified with a G orbit in $\mathcal{N}$, called a nilpotent orbit.

Fix $\mathcal{O} \in \mathcal{N}/G$. After Spaltenstein, $\mathcal{O} \cap \mathfrak{n}$ is equidimensional [S1] of dimension $\frac{1}{2}\dim \mathcal{O}$. Each component $\mathcal{V}$ of $\mathcal{O} \cap \mathfrak{n}$ is called an orbital variety (associated to $\mathcal{O}$). They have the following remarkable description due to Steinberg, see [S2] for example.

Fix $w \in W$. Then $G(\mathfrak{n} \cap^w \mathfrak{n})$ is irreducible and a finite union of nilpotent orbits, hence admits a unique dense orbit $\mathcal{O}(w)$. Set $\mathcal{V}(w) = \mathcal{O}(w) \cap \overline{B(\mathfrak{n} \cap^w \mathfrak{n})}$. It is an orbital variety and every orbital variety so obtains. The map $St\ : w \mapsto \mathcal{O}(w)$ (resp. $Str\ : w \mapsto \mathcal{V}(w)$) is called the Steinberg (resp. right Steinberg) map. Observe that $w \in St^{-1}(\mathcal{O})$ if and only if $\mathcal{O} \cap \mathfrak{n} \cap^w \mathfrak{n}$ is dense in $\mathfrak{n} \cap^w \mathfrak{n}$. In particular $w \in St^{-1}(\mathcal{O}) \iff w^{-1} \in St^{-1}(\mathcal{O})$.

2.5. View $\mathcal{B} \times \mathcal{B}$ as a G set under diagonal action and set $Z_w = G(B, {}^wB)$ which is a G orbit in $\mathcal{B} \times \mathcal{B}$. Through Bruhat decomposition one has

$$\mathcal{B} \times \mathcal{B} = \coprod_{w \in W} Z_w$$

and

$$(*) \qquad \bar{Z}_w = \coprod_{y \leq w} Z_y.$$

In particular the Z_w are locally closed in the Zariski topology.

2.6. Set $S = \{(x, B_1, B_2) \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in Lie\ B_1 \cap Lie\ B_2\}$. It is called the Steinberg variety. Let $\pi_i : i = 1, 2, 3$ be the projection of the i-th component of S and set

$\pi = \pi_1, \pi' = \pi_2 \times \pi_3$. Similarly, the cotangent bundle $T = T^*(\mathcal{B}) = G \times^B \mathfrak{n}$ of $\mathcal{B}$ identifies with

$$\{(x, B_1) \in \mathcal{N} \times \mathcal{B} | x \in Lie\ B_1\}.$$

In this presentation the projection of T to the first factor identifies with the moment map. In this fashion S identifies with the fibre product $T \times_{\mathcal{N}} T$. For each $w \in W$, set $S_w = \pi'^{-1}(Z_w)$. It is an irreducible locally closed subset of S of dimension $2n$ and identifies with the conormal to Z_w in $T^*(\mathcal{B} \times \mathcal{B})$. Thus the $\bar{S}_w : w \in W$ are just the irreducible components of S which is hence equidimensional.

2.7. For each $\mathcal{O} \in \mathcal{N}/G$, set $S(\mathcal{O}) = \pi^{-1}(\mathcal{O})$. It is a locally closed subset of dimension $2n$. Set $S_w(\mathcal{O}) = S(\mathcal{O}) \cap S_w$. Their disjoint union is $S(\mathcal{O})$. One has $w \in St^{-1}(\mathcal{O})$, if and only if $\overline{S_w(\mathcal{O})} = \bar{S}_w$. Since $\overline{S(\mathcal{O})}$ is equidimensional [S2] the latter are just the irreducible components of $\overline{S(\mathcal{O})}$.

2.8. Set $Y = \{(x, B_1) \in \mathfrak{n} \times \mathcal{B} \,\big|\, x \in Lie\ B_1\}$ and let $\pi_i : i = 1, 2$ be the projection of the i-th component of Y. Of course Y identifies with $\pi_3^{-1}(B) \subset S$. In this manner Y further identifies with the fibre product $T \times_{\mathcal{N}} \mathfrak{n}$.

For each $w \in W$, set $Y_w = \pi_2^{-1}(X_w)$. It is an irreducible locally closed subset of Y. It identifies with the conormal $\{(b(\mathfrak{n} \cap^w \mathfrak{n}), bwB) : b \in B\}$ of BwB in $T^*(\mathcal{B})$ and so has dimension n. Thus the $\bar{Y}_w : w \in W$ are just the irreducible components of Y which is hence equidimensional. We call Y the conormal variety.

3. The Conormal Variety.

3.1. We investigate some properties of the conormal variety Y defined in 2.8.

3.2. It is convenient to identify Y with $(Y, B) \subset S$ and Y_w with $(Y_w, B) \subset S_{w^{-1}}$. Clearly Y (resp. S) is B (resp. G) stable under diagonal action and $S = GY = G \times^B Y$, $S_{w^{-1}} = GY_w = G \times^B Y_w$. Moreover $\bar{S}_{w^{-1}} = G\bar{Y}_w = G \times^B \bar{Y}_w$ since G/B is complete.

3.3. For each locally closed subvariety $\mathcal{W}$ of $\mathfrak{n}$ set $Y(\mathcal{W}) = \pi^{-1}(\mathcal{W})$. By definition

$Y = Y(\mathfrak{n})$. For each $\mathcal{O} \in \mathcal{N}/G$, $w \in W$, set $Y_w(\mathfrak{n} \cap \mathcal{O}) = Y_w \cap Y(\mathfrak{n} \cap \mathcal{O})$. Identify $Y_w(\mathfrak{n} \cap \mathcal{O})$ with $(Y_w(\mathfrak{n} \cap \mathcal{O}), B)$ in $S_{w^{-1}}(\mathcal{O})$. Then $Stab_G(Y_w(\mathfrak{n} \cap \mathcal{O}), B) = N_G(B) = B$ and $G(Y_w(\mathfrak{n} \cap \mathcal{O}), B) = S_{w^{-1}}(\mathcal{O})$. Consequently

$$(*) \qquad \dim\ Y_w(\mathfrak{n} \cap \mathcal{O}) = \dim\ S_{w^{-1}}(\mathcal{O}) - n.$$

Lemma. *For all $\mathcal{O} \in \mathcal{N}/G$ one has*

(i) $Y(\mathfrak{n} \cap \mathcal{O}) = \coprod_{w \in W} Y_w(\mathfrak{n} \cap \mathcal{O}).$

(ii) $\overline{Y(\mathfrak{n} \cap \mathcal{O})} = \coprod_{w \in St^{-1}(\mathcal{O})} \overline{Y_w(\mathfrak{n} \cap \mathcal{O})}.$

(iii) $\overline{Y(\mathfrak{n} \cap \mathcal{O})}$ *is equidimensional with irreducible components* $\overline{Y_w(\mathfrak{n} \cap \mathcal{O})}$: $w \in St^{-1}(\mathcal{O})$ *of dimension n.*

Proof. (i) is trivial. For (ii) observe that $Y(\mathfrak{n} \cap \mathcal{O})$ and $Y_w(\mathfrak{n} \cap \mathcal{O})$ are B stable and recall that G/B is complete. Thus

$$\begin{aligned} G(\overline{Y(\mathfrak{n} \cap \mathcal{O})}) &= \overline{GY(\mathfrak{n} \cap \mathcal{O})}, \\ &= \overline{S(\mathcal{O})}, \\ &= \bigcup_{w \in St^{-1}(\mathcal{O})} \overline{S_w(\mathcal{O})}, \quad \text{by 2.7}, \\ &= G\left(\bigcup_{w \in St^{-1}(\mathcal{O})} \overline{Y_w(\mathfrak{n} \cap \mathcal{O})} \right). \end{aligned}$$

Since all these sets are B stable and $N_G(B) = B$ one obtains (ii) by taking intersection with $\pi_3^{-1}(B)$.

Recall that $w \in St^{-1}(\mathcal{O})$ if and only if $\mathfrak{n} \cap^w \mathfrak{n} \cap \mathcal{O}$ is dense in $\mathfrak{n} \cap^w \mathfrak{n}$. Thus if $w \in St^{-1}(\mathcal{O})$ it follows that $Y_w(\mathfrak{n} \cap \mathcal{O}) = B(\mathfrak{n} \cap^w \mathfrak{n} \cap \mathcal{O}, wB)$ is irreducible. Moreover in this case we obtain from $(*)$ that

$$\dim\ \overline{Y_w(\mathfrak{n} \cap \mathcal{O})} = \dim\ \overline{S_{w^{-1}}(\mathcal{O})} - n = n.$$

Hence (iii). $\square$

3.4. Since $\bar{Y}_w \supset \overline{Y_w(\mathcal{O} \cap \mathfrak{n})}$ is irreducible of dimension n, we obtain the

Corollary. *The $\bar{Y}_w : w \in St^{-1}(\mathcal{O})$ are the irreducible components of $\overline{Y(\mathfrak{n} \cap \mathcal{O})}$.*

3.5. Set

$$DC(w) = St^{-1}(St(w)),\ \overline{DC(w)} = St^{-1}(\overline{St(w)}).$$

These are subsets of W called the geometric double cells (resp. cones). As we shall see $\overline{DC(w)}$ naturally defines a two-sided ideal in the group algebra $\mathbb{Q}W$. Moreover we show that the resulting $W \times W$ module structure determines the form of $\overline{St(w)}$. Observe that $DC(w) = DC(w^{-1})$.

Set

$$C(w) = Str^{-1}(Str(w)),\ \overline{C(w)} = Str^{-1}(\overline{Str(w)}).$$

These are subsets of W called the right geometric cells (resp. cones). As we shall see $\overline{C(w)}$ naturally defines a right ideal of $\mathbb{Q}W$. Moreover we show that the resulting W module structure determines the form of $\overline{Str(w)}$. Through the dimension estimate in 2.4 one has $C(w) = DC(w) \cap \overline{C(w)}$. Through the Steinberg variety one checks (see [J3, 9.5] for example) that

$$(*) \qquad DC(w) = \coprod_{y \in C(w^{-1})} C(y).$$

Lemma. *For each $y \in C(w^{-1})$, there exists a subset $C_y \subset DC(w)$ such that the $\bar{Y}_z : z \in C_y$ are the irreducible components of $\overline{Y(\mathcal{V}(y))}$.*

Proof. Let $\overline{Y(\mathcal{V}(y))}_i : i = 1, 2, \cdots, s_y$, be the irreducible components of $\overline{Y(\mathcal{V}(y))}$. Let $\mathcal{O} = \mathcal{O}(w)$. By $(*)$ one has

$$\mathfrak{n} \cap \mathcal{O} = \bigcup_{y \in C(w^{-1})} \mathcal{V}(y).$$

It follows that

$$\overline{Y(\mathfrak{n}\cap\mathcal{O})} = \bigcup_{y\in C(w^{-1})} \bigcup_{i=1}^{s_y} \overline{Y(\mathcal{V}(y))_i}$$
$$= \bigcup_{z\in DC(w)} \bar{Y}_z, \quad \text{by 3.4.}$$

Then the assertion follows by uniqueness of decomposition into irreducible components. □

3.6. It follows in particular from 3.5 that $\overline{Y(\mathcal{V}(y))}$ is equidimensional. More precisely we obtain the

Proposition. *For all $w \in W$,*

(i) The irreducible components of $Y(\overline{\mathcal{V}(w)})$ are the $\bar{Y}_y : y \in \overline{C(w)}$.

(ii) The irreducible components of $\overline{Y(\mathcal{V}(w))}$ are the $\bar{Y}_y : y \in C(w)$.

Proof. Consider (ii). Since $\mathcal{B}$ is projective, π is a closed map. Hence $\pi(\bar{Y}_y)$ is closed, contains $B(\mathfrak{n}\cap^y\mathfrak{n})$ and is contained in $\overline{B(\mathfrak{n}\cap^y\mathfrak{n})}$. Thus we obtain

$$(*) \qquad \pi(\bar{Y}_y) = \overline{\mathcal{V}(y)}.$$

Now by 3.5

$$\bigcup_{y\in C_w} \bar{Y}_y = \overline{Y(\mathcal{V}(w))} \subset Y(\overline{\mathcal{V}(w)}),$$

and so

$$\overline{\mathcal{V}(y)} \subset \pi(Y(\overline{\mathcal{V}(w)})) = \overline{\mathcal{V}(w)}.$$

By definition this gives $y \in \overline{C(w)}$ and so $C_w \subset \overline{C(w)} \cap DC(w) = C(w)$. On the other hand $\mathcal{V}(y) = \mathcal{V}(w)$, for all $y \in C(w)$ and since $C_w \neq \emptyset$, this forces $C_w \supset C(w)$. Hence (ii).

The proof of (i) is similar. First of all

$$\overline{Y(\mathfrak{n}\cap\overline{\mathcal{O}(w)})} = \bigcup_{y\in\overline{DC(w)}} \overline{Y(\mathfrak{n}\cap\mathcal{O}(y))},$$
$$= \bigcup_{y\in\overline{DC(w)}} \bar{Y}_y, \quad \textit{by } 3.4.$$

Yet $\mathfrak{n} \cap \mathcal{O}(y) = \bigcup_{z \in C(y^{-1})} \mathcal{V}(z)$ and so there exists a subset $D_w \subset \overline{DC(w)}$ such that the $\overline{\mathcal{V}(z)} : z \in D_w$ are the irreducible components of $\mathfrak{n} \cap \overline{\mathcal{O}(w)}$. Since

$$\dim\ \overline{\mathcal{V}(y)} = \frac{1}{2} \dim\ \mathcal{O}(w) \quad \text{if} \quad y \in DC(w)$$

and

$$\dim\ \overline{\mathcal{V}(y)} < \frac{1}{2} \dim\ \mathcal{O}(w) \quad \text{if} \quad y \in \overline{DC(w)} \setminus DC(w)$$

we may deduce that $D_w \supset DC(w) \ni w$. Then as in 3.4 uniqueness of decomposition into irreducible components gives a subset $\bar{C}_w \subset \overline{DC(w)}$ such that the $\bar{Y}_y : y \in \bar{C}_w$ are the irreducible components of $Y(\overline{\mathcal{V}(w)})$. Through $(*)$ we conclude as in (i) and the definition of $\overline{C(w)}$, that $\overline{C_w} = \overline{C(w)}$. □

3.7. We see from 3.6 that $Y(\overline{\mathcal{V}(w)})$ and $\overline{Y(\mathcal{V}(w))}$ are both equidimensional of dimension n. One may remark that $\overline{\mathcal{V}(w)}$ is badly behaved in that it is not in general a union of orbital varieties, see [T] and [M, 4.2.5]. (On the other hand by [J3, 7.4] any closed irreducible B stable involutive subvariety $\mathcal{V}$ of $\mathfrak{n}$ is the closure of an orbital variety corresponding to the unique dense orbit in $G\mathcal{V}$). In this sense $Y(\overline{\mathcal{V}(w)})$ is better.

Since $Y(\overline{\mathcal{V}(w)}) = \pi^{-1}(\overline{\mathcal{V}(w)}) = \pi^{-1}\pi(\bar{Y}_w)$, we can obtain from 3.6 a homological approach to the description of $\overline{C(w)}$. Similarly $\pi^{-1}\pi(\bar{S}_w) = \pi^{-1}(\overline{\mathcal{O}(w)}) = S(\overline{\mathcal{O}(w)})$. Then 2.7 gives a homological approach to the description of $\overline{DC(w)}$.

4. Set Theoretic Convolution.

4.1. Let X be a set. As in [CG, 2.7.5] we may define a convolution product on subsets of $X \times X$ as follows. Let $p_{i,j} : X \times X \times X \to X \times X$ be the projection onto the i,j-th factor. Given $Y, Z \subset X \times X$, set

$$Y \circ Z := p_{13}(p_{12}^{-1}(Y) \cap p_{23}^{-1}(Z)).$$

We call it the convolution product of Y with Z. It is associative; but not commutative. Clearly the diagonal $\Delta \subset X \times X$ is the identity for this product.

4.2. Our interest in the convolution product comes from the following easy but crucial result. Take $X = T := T^*(\mathcal{B})$ and view S, Y as closed subsets of $T \times T$. Recall the moment map $\pi : T^*(\mathcal{B}) \to \mathcal{N}$.

Lemma. *For any $A \subset S$ (resp. $C \subset Y$) one has*

(i) $S \circ A \circ S = \pi^{-1}\pi(A)$

(ii) $S \circ C = \pi^{-1}\pi(C)$

(iii) *The formula $C \mapsto GC = G \times^B C$ defines a one-to one correspondence between B invariant subsets $C \subset Y$ and G invariant subsets of S. If $A \subset S$ is G invariant and $C \subset Y$ is B-invariant, one has*

$$A \circ GC = G(A \circ C).$$

In particular,

$$S \circ (GC) = G(\pi^{-1}\pi(C)).$$

Moreover all the above sets are closed if A is closed in S, and C is closed in Y and is B stable.

Proof. Recall that $S = T \times_{\mathcal{N}} T = \{(t_1, t_2) \in T \times T \mid \pi(t_1) = \pi(t_2)\}$. Thus $\pi^{-1}\pi(A) = \{(s,t) \in S \mid \exists (a_1, a_2) \in A \mid \pi(s) = \pi(a_1) = \pi(a_2) = \pi(t)\}$. On the other hand $(s,t) \in S \circ A \circ S$ means that we can find $(a_1, a_2) \in A$ such that $s \in \pi^{-1}\pi(a_1), t \in \pi^{-1}\pi(a_2)$. Comparison with the previous formula gives (i).

Recall that $Y = \{(t,x) \in T \times \mathfrak{n} \mid \pi(t) = x\}$. Thus $\pi^{-1}\pi(C) = \{(s,x) \in Y \mid \exists (t,x) \in C \text{ with } \pi(s) = x\}$. On the other hand $(s,x) \in S \circ C$ means that we can find $(t,x) \in C$ such that $s \in \pi^{-1}\pi(t)$. Hence (ii).

Since $S = G \times^B Y$, the assignment $A \mapsto A \cap Y$ from the set of G invariant subsets of S to the set of B-invariant subsets of Y is inverse to the map $C \mapsto GC$.

Suppose $A = \{(a_1, a_2)\} \subset S$ and $C = \{(t,x)\} \subset Y$. Then

$$A \circ C = \begin{cases} (a_1, x) & \text{if} \quad a_2 = t, \\ \emptyset & \text{otherwise.} \end{cases}$$

Recall that the group G acts diagonally on $T \times T$. The above formula implies that for any $g \in G$ $g(A \circ C) = g(A) \circ g(C)$. Therefore,

$$G(A \circ C) = A \circ GC$$

if A is G invariant.

In particular $S \circ (GC) = G(S \circ C)$, so the last formula of the lemma follows from (ii).

The last claim of the lemma follows from the fact that π is a closed continuous map and G/B is complete. $\square$

Remark. Note that the correspondence described in (iii) assigns S_w to $Y_{w^{-1}}$.

4.3. It is instructive to first make some computations with T replaced by $\mathcal{B}$.

Lemma. *Take $x, y \in W$ such that $\ell(x) + \ell(y) = \ell(xy)$. Then $Z_x \circ Z_y = Z_{xy}$. Moreover the intersection $p_{12}^{-1}(Z_x) \cap p_{23}^{-1}(Z_y)$ is transversal and the restriction*

$$p_{13} \; : p_{12}^{-1}(Z_x) \cap p_{23}^{-1}(Z_y) \longrightarrow Z_{xy}$$

is a birational isomorphism.

Proof. One has $p_{12}^{-1}(Z_x) = \{(gB, gxB, B') \mid g \in G, B' \in \mathcal{B}\}$ and $p_{23}^{-1}(Z_y) = \{(B'', g'B, g'yB\} \mid g' \in G, B'' \in \mathcal{B}\}$. Now $(gB, gxB, B') = (B'', g'B, g'yB)$ implies that $g^{-1}g' = xb$, for some $b \in B$. Then this common element equals $g(B, xB, xbyB)$ which has image $g(B, xbyB)$ under p_{13}. Since $\ell(x) + \ell(y) = \ell(xy)$ implies that $BxByB = BxyB$ we conclude that $Z_x o Z_y = Z_{xy}$.

Given $w \in W$, set $S(w) = \{\alpha \in \Delta^+ \mid w\alpha \in \Delta^-\}$. Under the hypothesis that lengths add it follows that

$$S((xy)^{-1}) = xS(y^{-1}) \amalg S(x^{-1}).$$

Thus $Lie(xyN^-(xy)^{-1}\cap N)$ is the direct sum of $Lie\ x(yN^-y^{-1}\cap N)x^{-1}$ and $Lie(xN^-x^{-1}\cap N)$. On the other hand for any $w \in W$ the multiplication map induces an isomorphism $(wN^-w^{-1} \cap N) \times wB/B$ onto BwB/B and the tangent space at wB/B is just $Lie(wN^-w^{-1} \cap N)$. It follows that the multiplication map induces an isomorphism of $(xN^-x^{-1} \cap N) \times x(yN^-y^{-1} \cap N)x^{-1}$ onto $xyN^-(xy)^{-1} \cap N$ and hence an isomorphism of $BxB \times ByB$ onto $BxyB$ (as is well-known).

Now recall that the multiplication map induces an isomorphism of $N^- \times B$ onto its image N^-B which is open dense in G. Then $N^-(B, xB, BxByB)$ is open dense in $p_{12}^{-1}(Z_x) \cap p_{23}^{-1}(Z_y)$ and isomorphic to $N^- \times (B, BxB, BxByB)$ which maps under p_{13} to $N^- \times (B, BxByB)$. By the result of the paragraph above this restriction of p_{13} is an isomorphism.

Finally the tangent space to the point $(B, xB, xbyB)$ in $p_{12}^{-1}(Z_x) \cong Z_x \times \mathcal{B}$ (resp. in $p_{23}^{-1}(Z_y) \cong \mathcal{B} \times Z_y$) is by the above isomorphic to $\mathfrak{n}^- \oplus {}^x\mathfrak{n}^- \cap \mathfrak{n} \oplus {}^{xby}\mathfrak{n}^-$ (resp. $\mathfrak{n}^- \oplus {}^x\mathfrak{n}^- \oplus {}^{xb}({}^y\mathfrak{n}^- \cap \mathfrak{n})$). Its intersection is $\mathfrak{n}^- \oplus ({}^x\mathfrak{n}^- \cap \mathfrak{n}) \oplus^{xb} ({}^y\mathfrak{n}^- \cap \mathfrak{n})$ which has dimension $n + \ell(x) + \ell(y)$. Since this equals dim $p_{12}^{-1}(Z_x)$ + dim $p_{23}^{-1}(Z_y)$ − dim $\mathcal{B} \times \mathcal{B} \times \mathcal{B}$, the intersection $p_{12}^{-1}(Z_x) \cap p_{23}^{-1}(Z_y)$ is transversal. $\square$

4.4. Recall that S_x is just the conormal to Z_x in $T^*(\mathcal{B} \times \mathcal{B})$. Then by [CG, 2.7.26] applied to 4.3 or directly we obtain

Corollary. *Take $x, y \in W$ such that $\ell(x) + \ell(y) = \ell(xy)$. Then $S_x \circ S_y = S_{xy}$. Moreover the intersection $p_{12}^{-1}(S_x) \cap p_{23}^{-1}(S_y)$ is transversal and the restriction*

$$p_{13} \ : p_{12}^{-1}(S_x) \cap p_{23}^{-1}(S_y) \longrightarrow S_{xy}$$

is a birational isomorphism.

4.5. It is easy to see how the conclusion of 4.3 is modified if lengths do not add. Define a new associative product on W through $x * y = xy$ if lengths add and $s_\alpha * s_\alpha = s_\alpha$, for all $\alpha \in \pi$. (This is just the Hecke algebra multiplication defined as in say [KL1] at $q = 0$).

Then $\overline{Z_x \circ Z_y} = \bar{Z}_{x*y}$, the corresponding intersection is still proper and the restriction of p_{13} has generic fibre of dimension $\ell(x) + \ell(y) - \ell(x * y)$. Moreover one may directly check that $\overline{S_x \circ S_y} = \bar{S}_{x*y}$. However the corresponding intersection is not proper having a dimension which is too great by $\ell(x) + \ell(y) - \ell(x * y)$ which is also the dimension of the generic fibre.

4.6. By 4.2 the set $\bar{Z}_x \circ \bar{Z}_y$ is closed and its precise form can be read off from 2.5(∗), 4.4 and 4.5. However the determination of $\bar{S}_x \circ \bar{S}_y$ is a much more subtle question and indeed it takes a much more complex form. For our purposes it is enough to compute this convolution product when $x = s_\alpha$ for some $\alpha \in \pi$. The simplification that results is due to the fact that we can precisely determine $\bar{S}_{s_\alpha}$. Indeed $Y_{s_\alpha} = B({}^{s_\alpha}\mathfrak{n} \cap \mathfrak{n}, s_\alpha B) = (\mathfrak{m}_\alpha, B s_\alpha B)$, since $\mathfrak{m}_\alpha$ is B stable. Then $\bar{Y}_{s_\alpha} = \mathfrak{m}_\alpha \times \mathbb{P}_\alpha$. Finally $\bar{S}_{s_\alpha} = G \times^B \bar{Y}_{s_\alpha}$, since G/B is complete. It follows in particular that $\bar{S}_{s_\alpha}$ is smooth.

4.7. Suppose $A, A' \subset S$. The following consideration is convenient in computing the intersection $p_{12}^{-1}(A) \cap p_{23}^{-1}(A')$. Since S identifies with $T \times_{\mathcal{N}} T \subset T \times T$, the intersection $p_{12}^{-1}(S) \cap p_{23}^{-1}(S)$ identifies with $T \times_{\mathcal{N}} T \times_{\mathcal{N}} T$ which is the collection of quadruples

$$\{(x, B_1, B_2, B_3) \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \times \mathcal{B} | x \in Lie(B_1) \cap Lie(B_2) \cap Lie(B_3)\}.$$

Fix $\alpha \in \pi$ and write s_α simply as s.

Lemma. *Suppose $w \in W$ satisfies $sw < w$. Then $\bar{S}_s \circ \bar{S}_w = \bar{S}_w$.*

Proof. One has $p_{12}^{-1}(\bar{S}_s) = G(\mathfrak{m}_\alpha \times \{B\} \times \mathbb{P}_\alpha \times \mathcal{B})$ whilst $p_{23}^{-1}(\bar{S}_w) = G(\overline{B(\mathfrak{n} \cap^w \mathfrak{n}, \mathcal{B}, \{B\}, \{wB\})})$. In the second term $\mathcal{B}$ may be taken out of the closure operation. Then as in 4.3 matching the second and third entries we conclude that the resulting intersection is the G saturation set of $(\mathfrak{m}_\alpha \times \{B\} \times \mathbb{P}_\alpha \times \mathcal{B}) \cap P_\alpha(\overline{B(\mathfrak{n} \cap^w \mathfrak{n}, \mathbb{P}_\alpha, \{B\}, \{wB\})})$. Under the hypothesis of the lemma $\mathfrak{n} \cap^w \mathfrak{n} \subset \mathfrak{m}_\alpha$. Thus this intersection is just the right hand factor in which the second entry is reduced to $\{B\}$. Its G saturation set maps under p_{13} to $\bar{S}_w$, as required. □

4.8. Let $A, A' \subset S$ be closed subvarieties. We shall say that $A \circ A'$ is proper if the intersection $p_{12}^{-1}(A) \cap p_{23}^{-1}(A')$ is proper. If $sw < w$, then $\bar{S}_\alpha \circ \bar{S}_w$ is *not* proper. Indeed the corresponding intersection has dimension which exceeds the value to be proper by 1.

Proposition. *Suppose $w \in W$ satisfies $sw > w$. Then $\bar{S}_s \circ \bar{S}_w$ is proper. Moreover $\bar{S}_s \circ \bar{S}_w$ is equidimensional of dimension $2n$, up to components of codimension 1 in $\bar{S}_w$.*

Proof. As in 4.7 matching the second and third entries the corresponding intersection is just the G saturation set of $(\mathfrak{m}_\alpha \times \{B\} \times \mathbb{P}_\alpha \times \mathcal{B}) \cap P_\alpha(\overline{B(\mathfrak{n} \cap^w \mathfrak{n}, \mathbb{P}_\alpha, \{B\}, \{wB\})})$. In calculating this intersection we may ignore the second and third entries. Then this intersection is just the P_α saturation set of $(\mathfrak{m}_\alpha \times \mathcal{B}) \cap \overline{B(\mathfrak{n} \cap^w \mathfrak{n}, \{wB\})}$. Now $\overline{B(\mathfrak{n} \cap^w \mathfrak{n}, \{wB\})} = \bar{Y}_w$ lies in $\mathfrak{n} \times \mathcal{B}$ in which $\mathfrak{m}_\alpha \times \mathcal{B}$ has codimension 1. Under the hypothesis of the proposition $\mathfrak{n} \cap^w \mathfrak{n} \not\subset \mathfrak{m}_\alpha$ and so $\bar{Y}_w \not\subset \mathfrak{m}_\alpha \times \mathcal{B}$. It follows by Krull's theorem (associated primes of a principal ideal have height at most one) that this intersection is equidimensional of codimension one. Let $C_1, C_2, \ldots, C_s$ be its irreducible components. These are obviously B stable and have dimension dim $\bar{Y}_w - 1 = n - 1$.

Suppose $P_\alpha C_i \supsetneq C_i$. Then $P_\alpha C_i$ which is closed, irreducible, lies in Y and has dimension n, must be some $\bar{Y}_z : z \in W$, by 2.8. Now consider the reinsertion of the second and third ignored entries. The third ignored entry will be modified by the action of P_α; but this is of no consequence since it is eliminated on applying p_{13}. Reinsertion of the second ignored entry which is $\{B\}$ and the G saturation of the result give $\bar{S}_z$ again up to the irrelevant third entry. Thus this component, namely C_i gives a contribution $\bar{S}_z$ to $\bar{S}_s \circ \bar{S}_w$.

Finally suppose $P_\alpha C_i = C_i$. Then $C_i \subset \bar{Y}_w$ and has codimension 1. Reinsertion of the third ignored entry and applying P_α will replace it by $\mathbb{P}_\alpha$. This increases the dimension to n. Finally reinserting the second ignored entry which is $\{B\}$ and acting by G will further increase the dimension to $2n$. Applying p_{13} yields a closed irreducible subvariety of $\bar{S}_w$. Since p_{13} is G equivariant it follows as in 4.3 that p_{13} has generic fibre isomorphic to $\mathbb{P}_\alpha$ (coming from the third entry). Thus C_i gives a contribution to an irreducible subvariety of $\bar{S}_w$ of codimension 1. $\square$

4.9. We can make 4.8 more precise as follows.

Lemma. *Suppose $w \in W$ satisfies $sw > w$. There exists a subset $F_{s,w} \subset \{z \in W \mid z < w, sz < z\}$ such that*

$$\bar{S}_s \circ \bar{S}_w = \bar{S}_{sw} \bigcup \bigcup_{z \in F_{s,w}} \bar{S}_z \tag{$*$}$$

up to components of codimension 1 in $\bar{S}_w$.

Proof. Already $\bar{S}_s \circ \bar{S}_w \supset \overline{S_s \circ S_w} = \bar{S}_{sw}$, by 4.4. The remaining terms in (*) come from $\bar{S}_w \setminus S_w$ which has codimension ≥ 1 in $\bar{S}_w$. Now $\bar{S}_w \subset \pi'^{-1}(\bar{Z}_w)$ and so this complement lies in $\bigcup_{z<w} S_z$ by 2.5, $(*)$. Consider $X := \bar{S}_w \cap S_z$ and suppose $sz > z$ with $z \neq w$. Then $\bar{S}_s \circ X = S_s \circ X \cup X$ and $S_s \circ X \subset S_s \circ S_z = S_{sz}$, by 4.4. Already dim $X \leq 2n-1$ and we claim that dim $S_s \circ X \leq 2n-1$. As in 4.7 we may match up the second and third entries of $p_{12}^{-1}(\bar{S}_s)$ and $p_{23}^{-1}(X)$ to deduce that their intersection is just the G saturation set of $(\mathfrak{m}_\alpha \times \{B\} \times \mathbb{P}_\alpha \times \mathcal{B}) \cap P_\alpha(\overline{B(\mathfrak{n} \cap^w \mathfrak{n}, \mathbb{P}_\alpha, \{B\}, \{wB\})} \cap B(\mathfrak{n} \cap^z \mathfrak{n}, \mathbb{P}_\alpha, \{B\}, \{zB\}))$. As before we may ignore the second and third entries in calculating this intersection which then becomes $P_\alpha((\mathfrak{m}_\alpha \times \mathcal{B}) \cap (\bar{Y}_w \cap Y_z))$, which is B stable. Since the action of G increases dimension by at most dim $G/B = n$ (actually exactly by n in view of the second ignored entry) it is enough to show that $(\mathfrak{m}_\alpha \times B) \cap \bar{Y}_w \cap Y_z$, which is B stable, has dimension $\leq n-2$. Yet $sw > w$ and $sz > z$ so both $\mathfrak{n} \cap^w \mathfrak{n}$ and $\mathfrak{n} \cap^z \mathfrak{n}$ contain $\mathbb{C}x_\alpha$. Consequently $\bar{Y}_w \cap Y_z$ which has dimension $\leq n-1$ contains $\mathbb{C}x_\alpha$ and so intersection with $\mathfrak{m}_\alpha \times \mathcal{B}$ further drops the dimension by 1, as required. It follows that such terms give no contribution to right hand side of $(*)$.

Finally suppose X is a closed irreducible component of $\bar{S}_w \cap \bar{S}_z$ with $sz < z$. Then $\bar{S}_s \circ X \subset \bar{S}_s \circ \bar{S}_z = \bar{S}_z$, by 4.8. Since $z < w$, right hand side of $(*)$ may obtain a term of the form $\bar{S}_z : z \in F_{s,w}$, as required. $\square$

4.10. Now we can give the corresponding result for the $Y_w : w \in W$.

Theorem. *For all $w \in W, \alpha \in \pi$ one has*

(i) *If $ws_\alpha < w$, then $\bar{S}_\alpha \circ \bar{Y}_w = \bar{Y}_w$.*

(ii) *If $ws_\alpha > w$, then $\bar{S}_\alpha \circ \bar{Y}_w$ is proper and*

$$\bar{S}_{s_\alpha} \circ \bar{Y}_w = \bar{Y}_{ws_\alpha} \bigcup \bigcup_{z \in F_{s_\alpha, w^{-1}}} \bar{Y}_{z^{-1}}$$

up to components of codimension 1 in $\bar{Y}_w$.

Proof. Applying Lemma 4.2(iii) to $A = \bar{S}_s$ and $C = \bar{Y}_w$ we immediately deduce the result from 4.9.

□

4.11. The set $F_{s_\alpha, w}$ is far from empty and in general difficult to determine. We shall see that it is determined implicitly by singularity data — combine 5.3, 5.7 and 5.12.

5. Convolution in Borel-Moore homology

5.1. In what follows the coefficient ring is assumed to be $\mathbb{Q}$ and topological spaces are assumed locally compact. For such a space let $H_i^S(X)$ denote its i-th singular homology space and $\hat{X} = \{X, \infty\}$ its one-point compactification. The i-th Borel-Moore homology space $H_i(X)$ of X is defined to be the i-th relative singular homology spaces, that is

$$H_i(X) := H_i^S(\hat{X}, \infty).$$

5.2. Let X be a complex algebraic variety of complex dimension m considered in its complex topology. Let $\{X_\sigma\}_{\sigma \in \Sigma_m}$ denote the set of irreducible components of X having complex dimension m. Their fundamental classes $[X_\sigma] : \sigma \in \Sigma_m$, form a basis for $H_{2m}(X)$, that is

$$H_{2m}(X) = \bigoplus_{\sigma \in \Sigma_m} \mathbb{Q}[X_\sigma].$$

5.3. Recall that the Steinberg variety S has complex dimension $2n$, where $n = |\Delta^+|$. Moreover S is equidimensional and its irreducible components are the $\bar{S}_w : w \in W$. Then by 5.2

$$H_{4n}(S) = \bigoplus_{w \in W} \mathbb{Q}[\bar{S}_w].$$

According to Ginzburg [G, CG, 3.4] the algebra $H_{4n}(S)$ identifies with the group algebra $\mathbb{Q}W$. For $w \in W$ we will denote by the same letter the corresponding element of $H_{4n}(S)$. The unit of $H_{4n}(S)$ is given by the class $1 = [\bar{S}_1]$.

Thus, the vector space $H_{4n}(S)$ admits two bases, the one $\{w | w \in W\}$, and the other $\{[\bar{S}_w] | w \in W\}$, and so one may look at the coefficients of the transition matrix

$$[\bar{S}_w] = \sum_{y \in W} A(w, y) y.$$

They satisfy $A(w, w) = 1$ and $A(w, y) = 0$ unless $y \leq w$. According to [CG], last paragraph in 3.4.13, the coefficients of the inverse matrix are positive integers.

Remark. Kazhdan and Lusztig [KL2] had previously given a topological construction of a $W \times W$ action on $H_{4n}(S)$ making it isomorphic to $\mathbb{Q}W$ with $\bar{S}_e$ corresponding to e. To obtain Ginzburg's result is enough to show that this left W action commutes with the right multiplication in $H_{4n}(S)$. Rossmann [R], whose work relates to both [G] and [KL2], clarified the geometric nature of the coefficients $A(w, y)$. He proved that for $y \leq w$

$$A(w, y) = (-1)^{\ell(w) - \ell(y)} \mathrm{Eu}_y(S_w)$$

where $\mathrm{Eu}_y(S_w)$ is the Euler number of the point $(y, 0) \in \bar{S}_w$ (see also [Dub,Mac,K]). It is a strictly positive integer if $y \leq w$, and equal to 1 if $(y, 0)$ is a smooth point.

Note the following

Lemma. *One has $[\bar{S}_s] = s - 1$, for any simple reflection s.*

Proof. Since $\bar{S}_s \circ \bar{S}_s = \bar{S}_s$, the class $[\bar{S}_s]$ should be idempotent up to a constant. The properties of $A(w,y)$ imply that $[\bar{S}_s] = s + c$ where c is a negative integer and only $c = -1$ gives $(s+c)^2$ proportional to $s+c$. $\square$

The lemma also follows from Rossmann's formula since $\bar{S}_s$ is smooth.

Through the involution $(x,y) \mapsto (y,x)$ on $\mathcal{B} \times \mathcal{B}$ one obtains

$$(*) \qquad A(w,y) = A(y^{-1}, w^{-1}).$$

5.4. As noted in the introduction, following our theory of $Prim\ U(\mathfrak{g})$ concerning in particular order relations and Goldie rank polynomials, it was rather natural to conjecture that the presence of two such bases for $\mathbb{Q}W$ should lead to similar results for orbital varieties. Thus in [J2, Sect. 2] we described "characteristic polynomials" for orbital varieties and conjectured [J2, 9.8] that their precise form should be given in terms of the basis $A(w) := \sum_{y \in W} A(w,y)y$ of $\mathbb{Q}W$. This conjecture was first established by Rossmann [R, Sect. 11]. Different proofs were given in [J5] and in [V].

5.5. The conjecture [J2, 9.8] (which is now the theorem below) for the ordering (by inclusion) of orbital variety closures can be expressed as a description of the right geometric cones. This goes as follows. Given any subset $C \subset H_{4n}(S)$, let [C] denote the smallest subset of W such that

$$C \subset \sum_{w \in [C]} \mathbb{Q}[\bar{S}_w].$$

It is sometimes convenient to view [C] as the right hand side above. Our main result is the following

Theorem. *For all* $w \in W$

$$y \in [[\bar{S}_w]W] \Longleftrightarrow y \in \overline{C(w)}.$$

5.6. As in the case of the ordering of primitive ideals, the implication $\Longrightarrow$ is much easier, though we did not realize this at the time. Indeed an intelligent reading of [KL2, Sect. 6] using in particular the short exact sequence of [KL2, (2.2)] gives the implication $y \in [W[\bar{S}_w]W] \Longrightarrow y \in \overline{DC(w)}$. Replacing the Steinberg variety S by the conormal variety Y gives by a similar analysis using the rather messy deformation retraction arguments of Kazhdan and Lusztig the implication $y \in [[\bar{S}_w]W] \Longrightarrow y \in \overline{C(w)}$. We shall not give this in detail as we obtain a second proof below.

The opposite implication in the primitive ideal set-up was obtained by Vogan [Vo] in his immediate response to our conjecture. He first noted that if $I \subset J$ is an inclusion of primitive ideals of $U(\mathfrak{g})$, then one has a surjection of $U(\mathfrak{g})/I \twoheadrightarrow U(\mathfrak{g})/J$ of quotient algebras. Of course there is a corresponding surjection of the algebras of regular functions on the orbital variety closures. At first this approach appears fruitless in the geometric context. Indeed unlike the primitive ideal case these algebras are not Artinian. Moreover one can interpret Vogan's second step as the rebuilding of $U(\mathfrak{g})/I$ from its socle using the Enright functor [J2, 5.7]. This itself has no obvious analogue in the geometric context. However the functorial process itself can be interpreted as convolution in Borel-Moore homology. Then the key observation was that the analogue of building up from the socle is just 3.7 combined with 4.2. We shall see how this works out below.

5.7. Let $A \subset S$, $C \subset Y$ be closed subvarieties of top dimension. Then as in [CG, (2.7.9)] there is a bilinear map

$$H_{4n}(A) \times H_{2n}(C) \longrightarrow H_{2n}(A \circ C).$$

This makes $H_{2n}(Y)$ a left convolution module for the convolution algebra $H_{4n}(S)$.

5.8. Following 3.7 and 4.2 we know that the components of Y are relevant to the description of inclusion relation of orbit variety closures.

The following observation allows one to avoid the appearence of $H(Y)$ in the formulation of the main result 5.5.

Lemma. *The map* $[\bar{S}_w] \mapsto [\bar{Y}_{w^{-1}}]$ *extends linearly to an isomorphism* $H_{4n}(S) \longrightarrow H_{2n}(Y)$ *of left* W *modules.*

Proof. This is a formal consequence of the base change for convolution product, see Appendix, A2. In fact, the embedding $\mathfrak{n} = \mathfrak{n} \times \{B\} \to T$ induces a map

$$T \times T \times \mathfrak{n} \longrightarrow T \times T \times T$$

satisfying the conditions of Proposition A2.3. Therefore, a homomorphism $H_{4n}(S) \longrightarrow H_{2n}(Y)$ of $H_{4n}(S)$ modules is induced. It remains to check that it carries the class $[\bar{S}_w]$ to the class $[\bar{Y}_{w^{-1}}]$. Lemma A2.2 (the base change for BM homology) applied to the cartesian square

$$\begin{tikzcd}
Y_{w^{-1}} \arrow[r] \arrow[d] & Y \arrow[d] \\
S_w \arrow[r] & S
\end{tikzcd}$$

implies that the image of $[\bar{S}_w]$ is a multiple of $[\bar{Y}_{w^{-1}}]$. The coefficient is one since the map $Y_{w^{-1}} \longrightarrow S_w$ looks locally as the zero section of a complex vector bundle. $\square$

5.9. We may now give a second proof of the implication $\Longrightarrow$ of theorem 5.5. Indeed

$$\begin{aligned}
\mathbb{Q}W[\bar{Y}_w] &= H_{4n}(S)[\bar{Y}_w], \quad \text{by 5.3} \\
&\subset H_{2n}(S \circ \bar{Y}_w), \quad \text{by 5.7} \\
&= H_{2n}(\pi^{-1}\pi(\bar{Y}_w)), \quad \text{by 4.2}(ii) \\
&= \bigoplus_{y \in \overline{C(w)}} [\bar{Y}_y], \quad \text{by 3.6}(i) \ \text{ and } \ 3.7.
\end{aligned}$$

Since $\mathbb{Q}W[\bar{Y}_w] \subset [W[\bar{Y}_w]]$, the required assertion follows from 5.8 and 5.3(*).

5.10. To obtain the implication $\Longleftarrow$ of our theorem it would be enough to show that $H_{4n}(S)[\bar{Y}_w] = H_{2n}(S \circ \bar{Y}_w)$. Of course this is too much to expect and indeed even in the better understood primitive ideal set-up the existence of enough cyclic elements of W (namely those satisfying $\mathbb{Q}W[\bar{Y}_w] = [W\bar{Y}_w])$ is unproven. (The longest element in each Weyl subgroup is cyclic; but these fail to exhaust all the cones.) Already it is a deep fact

[J4] that the Duflo involutions provide cyclic elements in the left cells (that is to say after one has factored out the cones corresponding to lower dimension).

On the other hand it is natural to ask if the inclusion $[H_{4n}(S)[\bar{Y}_w]] \subset H_{2n}[S \circ \bar{Y}_w]$ is an equality. This would follow if we could show that $[[\bar{S}_y][\bar{Y}_w]] = H_{2n}(\bar{S}_y \circ \bar{Y}_w), \forall\, y \in W$. This amounts to showing that the fundamental class of every irreducible component of $\bar{S}_y \circ \bar{Y}_w$ of complex dimension n occurs with non-zero coefficient in the convolution product $[\bar{S}_y][\bar{Y}_w]$. By a theorem of Serre it would be enough to show that $\bar{S}_y o \bar{Y}_w$ is proper in the sense of 4.8, see Appendix A1. Of course we have already seen that properness fails and so may its anticipated conclusion; but we show that the required conclusion is sufficiently close to being true (see Proposition 5.14) by analyzing the products when y is a generating reflection. This will prove the theorem.

5.11. Let us first consider the "bad" case when $y = s_\alpha : \alpha \in \pi$ and $ws_\alpha < w$. Here $\bar{S}_{s_\alpha} \circ \bar{Y}_w = \bar{Y}_w$. Properness would imply that $[\bar{S}_{s_\alpha}][\bar{Y}_w]$ is a strictly positive multiple of $[\bar{Y}_w]$. Rather we have the

Lemma. *Suppose $\alpha \in \pi,\ w \in W$, satisfy $ws_\alpha < w$. Then*

$$[\bar{S}_{s_\alpha}][\bar{Y}_w] = -2[\bar{Y}_w].$$

Proof. By 4.7 we must have

$$[\bar{S}_{s_\alpha}][\bar{S}_{w^{-1}}] = c[\bar{S}_{w^{-1}}]$$

for some $c \in \mathbb{Q}$. Yet $[\bar{S}_{s_\alpha}]^2 = (s_\alpha - 1)^2 = -2(s_\alpha - 1) = -2[\bar{S}_{s_\alpha}]$, by 5.3. This forces $c = -2$. Then 5.8 gives the required assertion. $\square$

5.12. Recall the notation of 4.10.

Proposition. *Suppose $\alpha \in \pi$ and $w \in W$ satisfy $ws_\alpha > w$. Then*

$$[\bar{S}_{s_\alpha}][\bar{Y}_w] = [\bar{Y}_{ws_\alpha}] + \sum_{z \in F_{s_\alpha, w^{-1}}} c_z[\bar{Y}_{z^{-1}}],$$

where the c_z are strictly positive integers.

Proof. The formula follows from properness in 4.10(ii) and Lemma A1.2 from the appendix. The coefficient of $[\bar{Y}_{ws_\alpha}]$ is one by virtue of 4.4 and A1.1(5). $\square$

Remark. One may define a second action of $H_{4n}(S)$ on $H_{2n}(Y)$ so that s_α comes out on the left in the formula describing the action of $[\bar{S}_{s_\alpha}]$ on $[\bar{Y}_w]$. This action changes the resulting orbital varieties and is essentially what occurs in [KL2, Sect. 7], [J3, Sect. 4], [H, Thm. 1], [BBM, 4.14].

The above situation is the precise analogue of the enveloping algebra one. Here the coherent continuation functors give a right action of W on the $\mathcal{O}$ category; but one which changes annihilators. Via the equivalence of categories in [BG] they are transformed to the Enright functors which do not alter annihilators and implement a left action of the Weyl group [J3]. That the roles of left and right are interchanged on passing from representation theory to geometry is natural from [J2, Sect. 5] and from 6.2($*$).

5.13. We see from 5.11, 5.12 that the convolution product by $[\bar{S}_{s_\alpha}]$ recovers all the components of the set theoretic convolution except those of codimension 1.

However by 4.10(iii) in a sense these are eliminated when we replace $[\bar{S}_{s_\alpha}]$ by $[\bar{S}_{s_\alpha}]+[\bar{S}_1]$, which by 5.8 represents s_α. This will be made precise below.

We remark that whereas the behaviour of the lower dimensional components in 4.10 is crucial to our work, it was of no concern to the authors referred to in Remark 5.12.

Proof of 5.5. The implication $\Longrightarrow$ in 5.5 was already proven in 5.9. The proof of the other implication is given in 5.13–5.14.

Recall some notions from [J1]. Let V be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{Q}$ having a fixed basis $\{v_1, \dots, v_n\}$.

Definition.

(a) An element $v \in V$ is called positive *(convex in [J1]) if $v = \sum a_i v_i$ with $a_i \geq 0$.*

(b) A vector subspace $U \subset V$ is called positive if it is generated by the set U_{pos} of its positive elements.

For a positive vector $v \in V,\ v = \sum a_i v_i$, one defines it support as

$$\text{Supp}(v) = \{i | a_i > 0\}.$$

One has clearly $\text{Supp}(v_1 + v_2) = \text{Supp}(v_1) \cup \text{Supp}(v_2)$. Furthermore, one defines

$$\text{Supp } U = \bigcup_{v \in U_{\text{pos}}} \text{Supp}(v).$$

It is clear for any positive U there exists $x \in U_{\text{pos}}$ such that $\text{Supp } U = \text{Supp}(x)$.

We are interested in the vector space $H_{2n}(Y) = \oplus_{w \in W} \mathbb{Q}[\bar{Y}_w]$ with the basis $\{[\bar{Y}_w]\}$. Recall that $H_{2n}(Y)$ is a $H_{4n}(S)$ module. We fix $w \in W$.

For each $y \in W$ define

$$V(y) = \sum_{x \leq y} \mathbb{Q}[\bar{S}_x][\bar{Y}_w].$$

Lemma. *$V(y)$ is a positive subspace of $H_{2n}(Y)$ for all $y \in W$.*

Proof. Induction on $\ell(y)$. The case $y = 1$ is obvious. Suppose $V(y)$ is positive and let $y' = sy > y$ for a simple reflection s. Then

$$V(y') = \sum_{x \leq sy} \mathbb{Q}[\bar{S}_x][\bar{Y}_w] = \sum_{x \leq y} \mathbb{Q}[\bar{S}_x][\bar{Y}_w] + \sum_{\substack{x \leq y \\ x \leq sx}} \mathbb{Q}[\bar{S}_{sx}][\bar{Y}_w].$$

But $\mathbb{Q}[\bar{S}_{sx}] = \mathbb{Q}[\bar{S}_s][\bar{S}_x]$ modulo $\sum_{z<x} \mathbb{Q}[\bar{S}_z]$ for x satisfying $sx > x$. On the other hand, if $sx < x$, $\mathbb{Q}[\bar{S}_x][\bar{Y}_w] \subset V(y)$. Thus,

$$V(y) \subset V(y') \subset V(y) + [\bar{S}_s]V(y).$$

Since on the other hand $V(y')$ is $[\bar{S}_s]$ invariant, we have

$$V(y') = V(y) + [\bar{S}_s]V(y).$$

This latter is positive since $[\bar{S}_s]$ can be equally substituted by $[\bar{S}_s]+2$ which preserves positivity. $\square$

5.14. For any closed subvariety Y' of Y let $\mathrm{Comp}(Y')$ denote its set of components of dimension $n = \dim\, Y$.

Proposition. *Suppose* $\bar{Y}_z \in \mathrm{Comp}(\bar{S}_y \circ \bar{Y}_w)$. *Then* $[\bar{Y}_z] \in \mathrm{Supp}\ V(y)$.

Proof. Induction on $\ell(y)$. The case $y = 1$ is clear. Now replace y by $y' = sy > y$ where s is a simple reflection. Then

$$\bar{Y}_z \in \mathrm{Comp}(\bar{S}_{y'} \circ \bar{Y}_w) \subset \mathrm{Comp}(\bar{S}_s \circ \bar{S}_y \circ \bar{Y}_w).$$

Three cases can occur.

(1) $\bar{Y}_z \in \mathrm{Comp}(\bar{S}_y \circ \bar{Y}_w)$ with $sz < z$. In this case the result follows from 5.11 and the induction hypothesis.

(2) $\bar{Y}_t \in \mathrm{Comp}(\bar{S}_y \circ \bar{Y}_w)$, $\bar{S}_s \circ \bar{Y}_t$ is proper, and $z^{-1} \in \{st^{-1}\} \cup F_{s,t^{-1}}$.

(3) C is a component of a smaller dimension in $\bar{S}_y \circ \bar{Y}_w$ and $\bar{Y}_z$ is a component of $\bar{S}_s \circ C$.

In the second case choose $v \in V(y)_{\mathrm{pos}}$ such that $\mathrm{Supp}\ V(y) = \mathrm{Supp}(v)$. Let $v = \sum c_i[\bar{Y}_{t_i}]$ with $c_i > 0,\ t_1 = t$. Then

$$([\bar{S}_s]+3)v = \sum c_i([\bar{S}_s]+3)[\bar{Y}_{t_i}].$$

The right hand side of the equality contains $[\bar{Y}_z]$ with a strictly positive coefficient since $[\bar{S}_s][\bar{Y}_t]$ contains it and no cancellations are possible. This implies that

$$[\bar{Y}_z] \in \mathrm{Supp}(v) \cup \mathrm{Supp}([\bar{S}_s]v) \subset \mathrm{Supp}\ V(y) \cup \mathrm{Supp}\ [\bar{S}_s]V(y) = \mathrm{Supp}\ V(y').$$

In the third case C is a component of $\bar{S}_y \circ \bar{Y}_w$ of a complex dimension smaller than n, and $\bar{Y}_z$ is a component of $\bar{S}_s \circ C$. In this case C is a component of

$$\bar{S}_{s_1} \circ \cdots \bar{S}_{s_n} \circ \bar{Y}_w$$

where $y = s_1 \cdots s_n$ is a reduced decomposition. By 4.10(iii) this implies that C lies in an n-dimensional component $\bar{Y}_t$ of $\bar{S}_x \circ \bar{Y}_w$ for some $x < y$. Then $\bar{Y}_z$ has to be a component of $\bar{S}_s \circ \bar{Y}_t$ as well and the induction hypothesis implies the result.

Theorem 5.5 is proved. $\square$

Remark 1. As a consequence $\sum_{y \in C(w)} \mathbb{Q}[\bar{S}_y]$ is a right ideal of $\mathbb{Q}W$. This can be shown directly using the positivity in 5.12 and 5.11 by an argument similar to that in [J1, 4.5]. Yet the essence of the latter has already been incorporated in the above proof.

Remark 2. Combining the last paragraph of 3.7, 4.2 and 4.10 gives a second proof of equidimensionality in 3.6(i). Yet in essence both turn on equidimensionality of hypersurface intersection.

5.15. A similar argument based on 4.2(i) gives the

Theorem. For all $w \in W$

$$y \in [W[\bar{S}_w]W] \iff y \in \overline{DC(w)}.$$

6. The Work of A. Melnikov

6.1. A. Melnikov [M1-M3] has made a detailed study of the inclusion of orbital variety closures in type A. We examine some features of this below in the light of theorem 5.5.

6.2. Take $\lambda \in \mathfrak{h}^*$ and let $M(\lambda)$ (resp. $L(\lambda)$) denote the Verma (resp. simple highest weight) module with highest weight $\lambda \in \mathfrak{h}^*$. Let ρ be the half sum of the positive roots and set $w.\lambda = w(\lambda+\rho)-\rho$, $\forall\, w \in W$. Let $\mathcal{O}_0$ denote the full subcategory of modules in $\mathcal{O}$ annihilated by a power of the augmentation ideal Z_+ of the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. Set $M_w = M(w.(-2\rho)), L_w = L(w.(-2\rho)) : w \in W$. The corresponding classes $[M_w], [L_w] : w \in W$ both form $\mathbb{Z}$ bases of the Grothendieck group of $\mathcal{O}_0$. Identify $[M_w]$ with $w \in W \subset \mathbb{Z}W$. Then $[L_w]$ identifies with an element $a(w) \in \mathbb{Z}W$. The resulting basis $\{a(w) : w \in W\}$ of

$\mathbb{Q}W$ is the one mentioned in 1.2 and described in [J1]. It has now been determined by the truth [BB,BK] of the Kazhdan-Lusztig conjectures [KL1].

Given any finitely generated $U(\mathfrak{g})$ module L one may define its associated variety $\mathcal{V}(L) \subset \mathfrak{g}^*$. As is well-known $\mathcal{V}(E \otimes L) \subset \mathcal{V}(L)$ for any finite dimensional $U(\mathfrak{g})$ module E. Recall that Jantzen translation (or coherent continuation) on $\mathcal{O}_0$ is just tensoring by some finite dimensional $U(\mathfrak{g})$ module E followed by projection onto $\mathcal{O}_0$ defined by primary decomposition with respect to $Z(\mathfrak{g})$. Through the above identifications it implements a right action of W on $\mathbb{Q}W$.

Set $\mathcal{V}_0(w) = \mathcal{V}(L_w)$. It is a union of orbital varieties closures (see [J3, 4.4] for example) but is not necessarily irreducible. Through the remark above it follows exactly as in [J1] that

$$(*) \qquad y \in [a(w)W] \Longrightarrow \mathcal{V}_0(y) \subset \mathcal{V}_0(w).$$

We are unable to show that $\Longleftarrow$ holds.

6.3. Now assume $\mathfrak{g}$ simple of type A. Then Melnikov has shown [M2] that $\mathcal{V}_0(w) : w \in W$ is irreducible. (Incidentally $\mathcal{V}(L(\lambda))$ may fail to be irreducible even in type A if λ is not integral. Examples may be read off from [JM, 4.7].) Consequently by say [J3, 8.15] one has $\mathcal{V}_0(w) = \overline{\mathcal{V}(w)}$. Thus Melnikov obtains the

Proposition. *Assume $\mathfrak{g}$ simple of type A. Then*

$$y \in [a(w)W] \Longrightarrow \mathcal{V}(y) \subset \overline{\mathcal{V}(w)}.$$

6.4. It is already clear from say [J5, 4.9] that $a(w) \neq A(w)$ in general. However in type A, Kazhdan and Lusztig conjectured [KL2, Sect. 7] that equality holds. This early optimism was crushed by Kashiwara and Saito [KS]. So we still do not know if $\Longleftarrow$ holds in 6.3.

6.5. Assume $\mathfrak{g}$ simple of type A. Although it is not yet known if the right cones defined by the $a(w) : w \in W$ coincide with those defined by the $A(w)$, the corresponding cells do coincide and indeed are given explicitly through the Robinson-Schensted correspondence. Melnikov [M1,M3] has attempted to describe right cones in terms of this correspondence, more specifically in terms of moving numbered boxes in standard tableaux of size n (which naturally enumerate the $\mathcal{V}(w) : w \in W$) according to a Robinson-Schensted type algorithm.

6.6. Suppose $\alpha \in \pi, w \in W$ satisfy $ws_\alpha > w$. Then from 5.5 and 5.12 it follows that $\mathcal{V}(ws_\alpha) \subset \overline{\mathcal{V}(w)}$. (Actually this has an elementary proof noted by N. Spaltenstein about 25 years ago.) This fact combined with the description of the right cells implements an order relation on standard tableaux called the induced Duflo order (in view of an analogous order relation on primitive ideals pointed out by Duflo [D]). Despite earlier optimism, Melnikov [M1,M3] showed that this order relation is strictly weaker than that implied by the inclusion of orbital variety closures.

6.7. After Spaltenstein, orbital varieties (in type A) may be viewed as chains of nilpotent orbits. Then the known order relations on nilpotent orbit closures imply an order relation on orbital varieties. This is described purely combinatorially as a "chain order" on standard tableaux. Melnikov [M1,M3] has shown it to be strictly stronger than that implied by orbital variety closures.

There is a way to refine by the induced Duflo order and the chain order through the Vogan calculus (cf. [J3, 9.10]). Melnikov has shown that these refined orders sandwich the order defined by orbital variety closures. Hopefully these combinatorially defined orders can be shown to coincide!

6.8. Finally let w_0 be the unique longest element of W. After Vogan [V] one has $y \in [Wa(w)] \iff Ann\ L_y \supset Ann\ L_w$. From this a symmetry property of $\{a(w) : w \in W\}$ implies that the map $Ann\ L_w \mapsto Ann\ L_{ww_0}$ is an order reversing inclusion on the set $\{P \in Prim\ U(g) \bigm| P \supset Z_+\}$. In general this map fails to be an order reversing involution even for nilpotent orbit closures, though this does hold in type A. Again (in type A) it is not known if $\overline{\mathcal{V}(w)} \mapsto \overline{\mathcal{V}(ww_0)}$ is an order reversing involution on the set of orbital variety closures.

Obviously all these considerations together with our main theorem strongly motivates the computation of the $A(w) : w \in W$.

Appendix A1: Convolution product for proper intersection

A1.1. The notation here follows [CG], 2.7. Let M_i, $i = 1, 2, 3$, be smooth complex algebraic varieties, $Z_{ij} \subseteq M_i \times M_j$, $(i, j) = (1, 2), (2, 3)$, be Zariski closed irreducible subsets and let $p_{ij} : M_1 \times M_2 \times M_3 \longrightarrow M_i \times M_j$ be the standard projections.

Denote

$$Z_{12} \circ Z_{23} = p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})).$$

We assume that the restriction of p_{13} to the intersection

$$p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) = Z_{12} \times_{M_2} Z_{23}$$

is proper so that in particular $Z_{12} \circ Z_{23}$ is closed in $M_1 \times M_3$.

Let $d_i = \dim\ M_i$, $d_{ij} = \dim Z_{ij}$. We are interested in the convolution map

$$H_{2d_{12}}(Z_{12}) \otimes H_{2d_{23}}(Z_{23}) \longrightarrow H_{2d_{13}}(Z_{12} \circ Z_{23}) \tag{1}$$

where

$$d_{13} = d_{12} + d_{23} - d_2. \tag{2}$$

Since Z_{12} and Z_{23} are irreducible,

$$H_{2d_{12}}(Z_{12}) = \mathbb{Q}[Z_{12}], \quad H_{2d_{23}}(Z_{23}) = \mathbb{Q}[Z_{23}].$$

Thus, the convolution product (1) is defined by the coefficients c_i in the formula

$$[Z_{12}] \cdot [Z_{23}] = \sum_{i=1}^{k} c_i [C_i] \tag{3}$$

where $C_1, \ldots, C_k$ are the irreducible components of $Z_{12} \circ Z_{23}$ of dimension d_{13}.

A1.2.

Lemma. *Assume the intersection of $p_{12}^{-1}(Z_{12})$ with $p_{23}^{-1}(Z_{23})$ is* proper *that is that all components of the intersection*

$$p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})$$

have dimension d_{13} as in (2). Then all the coefficients c_i in (3) are positive integers.

Proof. According [F], Cor. 19.2(a), the product of $[p_{12}^{-1}(Z_{12})]$ with $[p_{23}^{-1}(Z_{23})]$ can be calculated in the Chow group defined in [F], 8.1. Since the intersection is proper, the result is a linear combination

$$\sum_{i=1}^{m} a_i [D_i] \tag{4}$$

where D_i are the components of $p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})$ of dimension d_{13} and a_i are positive integers (Serre's intersection numbers), see [F], 8.2. In particular, $a_i = 1$ if $p_{12}^{-1}(Z_{12})$ and $p_{23}^{-1}(Z_{23})$ intersect transversally at D_i.

Now the convolution product in (1) is the composition of the cup product with $(p_{13})_*$. Since

$$(p_{13})_*([D_i]) = \begin{cases} 0, & \dim\ p_{13}(D_i) < d_{13} \\ \deg(D_i/p_{13}(D_i))[p_{13}(D_i)], & \dim\ p_{13}(D_i) = d_{13}, \end{cases} \tag{5}$$

the assertion of the lemma follows. $\square$

Appendix A2: Base change for convolution product.

A2.1. Sheaf-theoretic formulation of BM homology. In the study of base change for Borel-Moore homology one has to use its bivariant and sheaf-theoretic nature.

In what follows we denote by $D^+(X)$ the derived category of bounded below complexes of sheaves of $\mathbb{Q}$ vector spaces on a locally compact space X. Recall that for a continuous map $f : X \longrightarrow Y$ of locally compact spaces the (derived) functor $f_! : D^+(X) \longrightarrow D^+(Y)$ of direct image with proper support is defined. Moreover, if $f_!$ has a finite cohomological dimension, a right adjoint functor

$$f^! : D^+(Y) \longrightarrow D^+(X)$$

is defined. The bivariant homology theory for locally compact spaces assigns to a map $f : X \longrightarrow Y$ of a complex (or the collection of its homology groups)

$$H(f) = \mathbf{R}\mathrm{Hom}(\mathbb{Q}_X, f^!(\mathbb{Q}_Y)),$$

where $\mathbb{Q}_X$ denotes the constant sheaf on X corresponding to $\mathbb{Q}$, see [FM], I.7.3. Borel-Moore homology $H(X)$ is a fragment of this bivariant theory, namely, it is described as $H(f)$ where $f : X \longrightarrow *$ is the map from X to a point. The functoriality of Borel-Moore homology is deduced from this interpretation:

- *Push-forward:* any proper map $f : X \longrightarrow Y$ gives rise to a degree zero map $f_* : H(X) \longrightarrow H(Y)$.

- *Pull-back:* any map $f : X \longrightarrow Y$ *together with a degree d class* $\alpha \in H(f)$ defines a degree d map $f^*_\alpha : H(Y) \longrightarrow H(X)$.

Classically considered pullbacks for Borel-Moore homology are special cases of the one described above, with the element α being defined by a "canonical orientation" of f. For example, any morphism $f : X \to Y$ of smooth complex varieties admits a canonical orientation $\alpha \in H(f)$ of degree $\dim\ Y - \dim\ X$.

Another important ingredient of bivariant theory is the following. Let

(6)
$$\begin{tikzcd} T \arrow[r, "u"] \arrow[d, "v"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

be a cartesian diagram. Then a map $f^* : H(g) \longrightarrow H(v)$ is defined.

A2.2. The following result is a direct consequence of the base change theorem for locally compact topological spaces, see [KSch], Prop. 2.5.11.

Lemma. (Base change for BM homology). *Suppose that in the cartesian diagram (6) the map f is proper. Let $\alpha \in H(g)$ and denote $\beta = f^*(\alpha)$. Then*

$$g_\alpha^* f_* = u_* v_\beta^* : H(Y) \longrightarrow H(Z). \tag{7}$$

A2.3. Application to convolution product.

Lemma A2.2 is a purely formal claim. It becomes meaningful in the case that both α and β correspond to "canonical orientations" of g and of v respectively.

Let M_i, Z_{ij} be as in A1.1. Let $f_i : M_i' \longrightarrow M_i$ be morphisms with M_i' smooth complex algebraic varieties. Define

$$Z_{ij}' = (f_i \times f_j)^{-1}(Z_{ij}).$$

We wish to compare two convolution products,

$$H(Z_{12}) \otimes H(Z_{23}) \longrightarrow H(Z_{12} \circ Z_{23}) \tag{8}$$

and

$$H(Z_{12}') \otimes H(Z_{23}') \longrightarrow H(Z_{12}' \circ Z_{23}'). \tag{9}$$

The maps $f_i \times f_j$, $(i,j) = (1,2)$ or $(2,3)$, admit a canonical orientation since these are map of smooth oriented manifolds. Thus, the induced maps

$$f_{ij} : Z_{ij}' \longrightarrow Z_{ij}$$

admit an induced orientation. Thus, one has the maps

$$f_{ij}^* : H(Z_{ij}) \longrightarrow H(Z'_{ij}), \quad (i,j) = (1,2) \text{ or } (2,3). \tag{10}$$

The inverse image maps (10) can be descibed also using the inverse image maps in cohomology, if one identifies $H_*(Z_{ij})$ with the cohomology $H^{n-*}(M_i \times M_j, M_i \times M_j \setminus Z_{ij})$ and similarly for $H(Z'_{ij})$.

Similarly, the map $f_1 \times f_3 : M'_1 \times M'_3 \longrightarrow M_1 \times M_3$ induces the map

$$f_{13}^* : H(Z_{12} \circ Z_{23}) \longrightarrow H(Z'_{12} \circ Z'_{13}),$$

where f_{13} is the restriction of $f_1 \times f_3$.

We will say that base change holds for the convolution product if the diagram

$$\begin{tikzcd} H(Z_{12}) \otimes H(Z_{23}) \arrow[r] \arrow[d, "f_{12}^*\otimes f_{23}^*"'] & H(Z_{12} \circ Z_{23}) \arrow[d, "f_{13}^*"'] \\ H(Z'_{12}) \otimes H(Z'_{23}) \arrow[r] & H(Z'_{12} \circ Z'_{23}) \end{tikzcd} \tag{11}$$

commutes.

Proposition. *Suppose $f_2 : M'_2 \longrightarrow M_2$ is an isomorphism. Then base change holds for the convolution product.*

Proof. The convolution product is defined as a composition of the inverse image $p_{12}^* \otimes p_{23}^*$ and the direct image p_{13*}. The inverse image obviously commutes with the base change. Thus it remains to check that the left square of the commutative diagram

$$\begin{tikzcd} Z'_{12} \times_{M'_2} Z'_{23} \arrow[r, "p'_{13}"] \arrow[d, "f_{12}\times f_{23}"'] & Z'_{12} \circ Z'_{23} \arrow[r] \arrow[d, "f_{13}"'] & M'_1 \times M'_3 \arrow[d, "f_1\times f_3"'] \\ Z_{12} \times_{M_2} Z_{23} \arrow[r, "p_{13}"] & Z_{12} \circ Z_{23} \arrow[r] & M_1 \times M_3 \end{tikzcd} \tag{12}$$

induces the commutativity relation

$$(p'_{13})_*(f_{12}^* \times f_{23}^*) = f_{13}^*(p_{13})_*.$$

Since f_2 is an isomorphism, the left commutative square in the diagram (12) is cartesian. Therefore, by Lemma A2.2, we have just to check that the canonical orientations used for construction of the inverse images $f_{12}^* \times f_{23}^*$ and f_{13}^*, are compatible.

This amounts to checking that in the cartesian diagram

$$\begin{tikzcd} M'_1 \times M'_2 \times M'_3 \arrow[r, "p'_{13}"] \arrow[d, "f_1 \times f_2 \times f_3"'] & M'_1 \times M'_3 \arrow[d, "f_1 \times f_3"'] \\ M_1 \times M_2 \times M_3 \arrow[r, "p_{13}"] & M_1 \times M_3 \end{tikzcd}$$

the canonical orientations of $f_1 \times f_2 \times f_3$ and of $f_1 \times f_3$ are compatible. This is obvious since f_2 is an isomorphism.

□

References

[BB] A. Beilinson and J. Bernstein, *Localisation de $\mathfrak{g}$ modules*, CRAS, Paris, Sér. A, **292** (1981), 15–18.

[B] E. Benlolo, *Sur la quantification de certaines variétés orbitales*, Bull. Sci. Math. **3** (1994), 225–241.

[BG] I.N. Bernstein and S.I. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*, Compos. Math. **41** (1980), 245–285.

[BBM] W. Borho, J.-L. Brylinski and R. MacPherson, *Nilpotent orbits, primitive ideals and characteristic classes*, Progr. Math. 78, Birkhäuser, Boston, 1989.

[BK] J.-L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), 387–410.

[BN] J. Beck and H. Nakajima, *Crystal bases and two-sided cells of quantum affine algebras*, arXiv: math.QA/0212253, 2002.

[C] R.W. Carter, *Simple groups of Lie type*, Interscience, John Wiley, London 1972.

[CG] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston 1997.

[Dub] A. Dubson, *Formule pour les cycles évanescents*, CRAS, Paris **299**, Sér I (1984), 182–184.

[D] M. Duflo, *Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple*, Ann. Math. **105** (1977), 107–120.

[F] W. Fulton, *Intersection Theory*, Springer, Berlin 1984.

[FM] W. Fulton and R. MacPherson, *Categorical framework for the study of singular spaces,* Memoirs of AMS, 243 (1981).

[G] V. Ginzburg, $\mathfrak{g}$*-modules, Springer representations and bivariant Chern classes*, Adv. Math. **61** (1986), 1–48.

[H] R. Hotta, *On Joseph's construction of Weyl group representations*, Tôhoku Math. J., **36** (1984), 49–74.

[J1] A. Joseph, *W-module structure in the primitive spectrum of the enveloping algebra of a semisimple Lie algebra*, LN 728, Springer, Berlin 1979, 117–135.

[J2] A. Joseph, *Completion functors in the $\mathcal{O}$ category*, LN 1020, Springer, Berlin 1983, 80–106.

[J3] A. Joseph, *On the variety of a highest weight module*, J. Algebra **88** (1984), 238–278.

[J4] A. Joseph, *On the cyclicity of vectors associated with Duflo involutions*, LN 1243, Springer, Berlin 1985, 144–188.

[J5] A. Joseph, *On the characteristic polynomials of orbital varieties*, Ann. Éc. Norm. Sup. **22** (1989), 569–603.

[J6] A. Joseph, *Orbital varieties of the minimal orbit*, **31** (1998), 17–45.

[JM] A. Joseph and A. Melnikov, *Quantization of hypersurface orbital varieties*, Progr. Math. Birkhäuser, Boston 2003, 165–196.

[K] M. Kashiwara, *Systems of microdifferential equations*, Birkhäuser, Boston 1983.

[KS] M. Kashiwara and Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), 9–36.

[KSch] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Springer, Berlin 1994.

[KL1] D.A. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[KL2] D.A. Kazhdan and G. Lusztig, *A topological approach to Springer's representations*, Adv. Math. **38** (1980), 222–228.

[Mac] R. MacPherson, *Chern classes for singular varieties*, Ann. Math. **100** (1974), 423–432.

[M] A. Melnikov, *Geometric interpretation of Robinson-Schensted procedure and related orders on Young tableaux*, Thesis, Weizmann Institute, 1992.

[M1] A. Melnikov, *Robinson-Schensted procedure and combinatorial properties of geometric order* $\mathfrak{sl}(n)$, CRAS, Paris **315**, Sér. I (1992), 53–57.

[M2] A. Melnikov, *Irreducibility of the associated variety of simple highest weight modules in* $\mathfrak{sl}(n)$, CRAS, Paris **316**, Sér. I (1993), 53–57.

[M3] A. Melnikov, *On orbital variety closures in* $\mathfrak{sl}(n)$ *I, II.*

[P] E. Perelman, , Thesis, Weizmann 2003.

[Re] M. Reineke, *Quivers, desingularizations and canonical bases*, Progr. Math. **210**, Birkhäuser, Boston 2003, 325–344.

[R] W. Rossmann, *Invariant eigendistributions on a semisimple Lie algebra and homology classes on the conormal variety*, I, II, J. Funct. Anal. **96** (1991), 130–154, 155–193.

[S1] N. Spaltenstein, *On the fixed point set of a unipotent element on the variety of Borel subgroups*, Topology **16** (1977), 203–204.

[S2] N. Spaltenstein, *Classes unipotentas et sous-groupes de Borel*, LN 946, Springer, Berlin, 1982.

[T] T. Tanisaki, *Characteristic varieties of highest weight modules and primitive quotients*, Adv. Studies Pure Math., **14** (1988), 1–30.

[V] M. Vergne, *Polynômes de Joseph et représentations de Springer*, Ann. Sci. Éc. Norm. Sup. **23** (1990), 543-562.

[Vo] D. Vogan, *Ordering in the primitive spectrum of a semisimple Lie algebra*, Math. Ann. **248** (1980), 195–203.