HOMOLOGICAL ALGEBRA OF HOMOTOPY ALGEBRAS

VLADIMIR HINICH

Dept. of Mathematics and Computer Science, University of Haifa, Mount Carmel, Haifa 31905 Israel

email: hinich@mathcs2.haifa.ac.il

CONTENTS

1. INTRODUCTION

1.1.   Let k be a ring and let $C(k)$ be the category of (unbounded) complexes of k-modules. As we know from Spaltenstein [Sp] (see also [AFH] and [BL]) one can do a homological algebra in $C(k)$ using the appropriate notions for $\mathcal{K}$-projective and $\mathcal{K}$-injective complexes.

The present paper started from the observation that this homological algebra in $C(k)$ (or, more generally, in the category of dg modules over an associative dg algebra) can be described using Quillen's language of closed model categories (see [Q1, Q2]). For, if we take quasi-isomorphisms in $C(k)$ to be weak equivalences and componentwise surjective maps of complexes to be fibrations, then a closed model category structure on $C(k)$ is defined, and cofibrant objects in it are precisely the $\mathcal{K}$-projectives of [Sp].

The possibility of working with unbounded complexes is very important if we wish to work with "weak algebras" — the ones satisfying the standard identities (associativity, commutativity, or Jacobi identity, for example) up to some higher homotopies. An appropriate language to describe these objects is that of operads (see [KM] and references therein) and one feels extremely uncomfortable when restricted to, say, non-negatively graded world (for instance non-negatively

graded commutative dg algebras do not admit semi-free resolutions; enveloping algebras of operad algebras are very often infinite in both directions).

In this paper we use Quillen's machinery of closed model categories to describe homological algebra connected to operads and operad algebras.

In Sections 2— 6 we define the necessary structures and prove some standard comparison results. In Section 7 we study the notion of cotangent complex of a morphism of dg operad algebras. In the last Section 8 we define a canonical structure of homotopy Lie algebra on the tangent complex. The latter is the main (concrete) result of the paper — see also 1.4.

Let us describe in a more detail the contents of the paper.

1.2.   Homological algebra of operad algebras has three different levels.

On the lowest level we have the category $\mathtt{Mod}(\mathcal{O}, A)$ of modules over a fixed algebra A over an operad $\mathcal{O}$. This is the category of dg modules over the enveloping algebra $U(\mathcal{O}, A)$ which is an associative dg algebra. As we mentioned above, this category admits a closed model category (CMC) structure — see 3.1; the corresponding homotopy category is the derived category of $U(\mathcal{O}, A)$-modules and it is denoted by $DU(\mathcal{O}, A)$.

Since operad algebra $(\mathcal{O}, A)$ in $C(k)$ is not just a firm collection of operations but is meant to be a realization of the idea of "algebra up to homotopy", we have to understand what happens to $DU(\mathcal{O}, A)$ when one substitutes $(\mathcal{O}, A)$ with a quasi-isomorphic algebra $(\mathcal{O}', A')$.

On the next level we have the category $\mathtt{Alg}(\mathcal{O})$ of algebras over a fixed operad $\mathcal{O}$. This category also admits a CMC structure, provided some extra hypotheses on $\mathcal{O}$ (Σ-splitness, see 4.2.4) are fulfilled. These extra hypotheses correspond more or less to the cases where one is able to use free algebra-resolutions instead of simplicial resolutions, in order to define algebra cohomology (see [Q3]). Thus, the operad $\mathtt{Ass}_k$ responsible for associative k-algebras is Σ-split for any k; any operad over k is Σ-split when $k \supseteq \mathbb{Q}$.

Finally, on the highest level we have the category $\mathtt{Op}(k)$ of (dg) operads over k. Quasi-isomorphic operads here correspond, roughly speaking, to different collections of higher homotopies used in an algebra A in order to make it "homotopy algebra". The category $\mathtt{Op}(k)$ also admits a CMC structure.

What is the connection between the different model structures?

First of all, if one has a quasi-isomorphism $\alpha : \mathcal{O} \to \mathcal{O}'$ of Σ-split operads compatible with Σ-splitting (this condition is fulfilled, e.g., when $k \supseteq \mathbb{Q}$) then the homotopy categories $\mathtt{Hoalg}(\mathcal{O})$ and $\mathtt{Hoalg}(\mathcal{O}')$ are naturally equivalent — see Theorem 4.7.4. This result implies, for instance, the representability of strong homotopy algebras (Lie or not) by strict algebras in characteristic zero.

A similar equivalence on the lower level takes place only for associative algebras: Theorem 3.3.1 claims that a quasi-isomorphism $f : A \to B$ of associative dg algebras induces an equivalence of the derived categories $D(A)$ and $D(B)$. For algebras over an arbitrary operad $\mathcal{O}$ one has such a comparison result only when A and B are cofibrant algebras (see Corollary 5.3.3), or when the operad $\mathcal{O}$ is cofibrant and A, B are flat as k-complexes. This suggests a definition of derived category $D(\mathcal{O}, A)$ which can be "calculated" either as $D(\mathcal{O}, P)$ where P is a cofibrant

$\mathcal{O}$-algebra quasi-isomorphic to A, or as $D(\tilde{\mathcal{O}}, A)$ where $\tilde{\mathcal{O}}$ is a cofibrant resolution of $\mathcal{O}$ and A is flat. This is done in 5.4 and in 6.8. The category $D(\mathcal{O}, A)$ is called the derived category of virtual A-modules and it depends functorially on $(\mathcal{O}, A)$.

1.3. For any morphism $f : A \to B$ of $\mathcal{O}$-algebras one defines in a standard way the functor Der:

$$\mathrm{Der}_{B/A} : \mathtt{Mod}(\mathcal{O}, B) \to C(k).$$

The functor is representable by the module $\Omega_{B/A} \in \mathtt{Mod}(\mathcal{O}, B)$. This is the module of differentials. If $\mathcal{O}$ is Σ-split so that $\mathtt{Alg}(\mathcal{O})$ admits a CMC structure, one defines the relative cotangent complex $L_{B/A} \in D(\mathcal{O}, B)$ as the module of differentials of a corresponding cofibrant resolution. This defines cohomology of $\mathcal{O}$-algebra A as the functor

$$M \in D(\mathcal{O}, A) \mapsto H(A, M) = \mathbf{R}\,\mathrm{Hom}(L_A, M) \in C(k).$$

1.4. The most interesting cohomology is the one with coefficients in $M = A$. No doubt, the complex $\mathrm{Der}^{\mathcal{O}}(A, A)$ admits a dg Lie algebra structure. The main result of Section 8, Theorem 8.5.3, claims that the tangent complex $T_A := H(A, A) = \mathbf{R}\,\mathrm{Hom}(L_A, A)$ admits a canonical structure of Homotopy Lie algebra. This means that T_A is defined uniquely up to a unique isomorphism as an object of the category $\mathtt{Holie}(k)$. Moreover, T_A is functorial with respect to quasi-isomorphisms, so that it defines a functor

$$T : \mathtt{Hoalg}(\mathcal{O})^{\mathrm{iso}} \to \mathtt{Holie}(k)^{\mathrm{iso}}$$

form the homotopy category of $\mathcal{O}$-algebras and isomorphisms to the homotopy category of Lie algebras and isomorphisms.

1.5. Let us indicate some relevant references.

Spaltenstein [Sp], Avramov-Foxby-Halperin [AFH] developed homological algebra for unbounded complexes.

Operads and operad algebras were invented by J.P. May in early 70-ies in a topological context; dg operads appeared explicitly in [HS] and became popular in 90-ies mainly because of their connection to quantum field theory.

M. Markl in [M] studied "minimal models" for operads — similarly to Sullivan's minimal models for commutative dg algebras over $\mathbb{Q}$. In our terms, these are cofibrant operads weakly equivalent to a given one.

In [SS] M. Schlessinger and J. Stasheff propose to define the tangent complex of a commutative algebra A as $\mathrm{Der}(\mathcal{A})$ where $\mathcal{A}$ is a "model" i.e. a commutative dg algebra quasi-isomorphic to A and free as a graded commutative algebra. This complex has an obvious Lie algebra structure which is proven to coincide sometimes (for a standard choice of $\mathcal{A}$) with the one defined by the Harrison complex of A.

It is clear "morally" that the homotopy type of the Lie algebra $\mathrm{Der}(\mathcal{A})$ should not depend on the choice of $\mathcal{A}$. Our main result of Section 8 says (in a more general setting) that this is really so.

1.6. **Notations.** For a ring k we denote by $C(k)$ the category of complexes of k-modules. If $X, Y \in C(k)$ we denote by $\mathcal{H}om_k(X, Y)$ the complex of maps form X to Y (not necessarily commuting with the differentials).

$\mathbb{N}$ is the set of non-negative integers; $\mathtt{Ens}$ is the category of sets, $\mathtt{Cat}$ is the 2-category of small categories. The rest of the notations is given in the main text.

2. Closed model categories

The main result of this Section — Theorem 2.2.1 — provides a category $\mathcal{C}$ endowed with a couple of adjoint functors

$$\# : \mathcal{C} \rightleftarrows C(k) : F$$

(F is left adjoint to $\#$) where $C(k)$ is the category of (unbounded) complexes of modules over a ring k and satisfying properties (H0), (H1) of 2.2, with a structure of closed model category (CMC) in sense of Quillen [Q1],[Q2], see also 2.1. This allows one to define a CMC structure on the category of (dg) operad algebras (Section 4), on the category of modules over an associative dg algebra (Section 3) and, more generally, on the category of modules over an operad algebra (Section 5). The CMC structure on the category of operads (Section 6) is obtained in almost the same way.

2.1. **Definition.** Recall (cf. [Q1], [Q2]) that a closed model category (CMC) structure on a category $\mathcal{C}$ is given by three collections of morphisms — *weak equivalences* ($\mathfrak{W}$), *fibrations* ($\mathfrak{F}$), *cofibrations* ($\mathfrak{C}$) in $\mathrm{Mor}(\mathcal{C})$ such that the following axioms are fulfilled:

(CM 1) $\mathcal{C}$ is closed under finite limits and colimits.

(CM 2) Let $f, g \in \mathrm{Mor}(\mathcal{C})$ such that gf is defined. If any two of f, g, gf are in $\mathfrak{W}$ than so is the third one.

(CM 3) Suppose that f is a retract of g i.e. that there exists a commutative diagram

in which the compositions of the horisontal maps are identities. Then if g belongs to $\mathfrak{W}$ (resp., $\mathfrak{F}$ or $\mathfrak{C}$) then so does f.

(CM 4) Let

be a commutative diagram with $i \in \mathfrak{C}$, $p \in \mathfrak{F}$. Then a dotted arrow α making the diagram commutative, exists if either

(i) $i \in \mathfrak{W}$

or

(ii) $p \in \mathfrak{W}$.

(CM 5) Any map $f : X \to Y$ can be decomposed in the following ways:

(i) $f = pi$, $p \in \mathfrak{F}$, $i \in \mathfrak{W} \cap \mathfrak{C}$;

(ii) $f = qj$, $q \in \mathfrak{W} \cap \mathfrak{F}$, $j \in \mathfrak{C}$.

The morphisms in $\mathfrak{W} \cap \mathfrak{F}$ are called *acyclic fibrations*; the morphisms in $\mathfrak{W} \cap \mathfrak{C}$ are *acyclic cofibrations*;

If the pair of morphisms $i : A \to B, p : X \to Y$ satisfies the condition (CM 4) we say that i satisfies the *left lifting property* (LLP) with respect to p or that p satisfies the *right lifting property* (RLP) with respect to i.

2.2.   Fix a base ring k and let $C(k)$ be the category of unbounded complexes over k.

Let $\mathcal{C}$ be a category endowed with a couple of adjoint functors

$$\# : \mathcal{C} \rightleftarrows C(k) : F$$

so that F is left adjoint to $\#$.

Suppose that

(H0) $\mathcal{C}$ admits finite limits and arbitrary colimits; the functor $\#$ commutes with filtered colimits.

(H1) Let $d \in \mathbb{Z}$ and let $M \in C(k)$ be the complex

$$\cdots \longrightarrow 0 \longrightarrow k = k \longrightarrow 0 \longrightarrow \cdots$$

concentrated in the degrees $d, d+1$. The canonical map $A \longrightarrow A \coprod F(M)$ induces a quasi-isomorphism $A^{\#} \longrightarrow (A \coprod F(M))^{\#}$.

We define the three classes of morphisms in $\mathcal{C}$ as follows:

— $f \in \mathrm{Mor}(\mathcal{C})$ belongs to $\mathfrak{W}$ if $f^{\#}$ is a quasi-isomorphism;

— $f \in \mathrm{Mor}(\mathcal{C})$ belongs to $\mathfrak{F}$ if $f^{\#}$ is (componentwise) surjective;

— $f \in \mathrm{Mor}(\mathcal{C})$ belongs to $\mathfrak{C}$ if it satisfies the LLP with respect to all acyclic fibrations.

2.2.1.   **Theorem.**   *Let a category $\mathcal{C}$ be endowed with a couple of adjoint functors*

$$F : \mathcal{C} \rightleftarrows C(k) : \#$$

so that the conditions (H0),(H1) are fulfilled. Then the classes $\mathfrak{W}, \mathfrak{F}, \mathfrak{C}$ of morphisms in $\mathcal{C}$ described above define on $\mathcal{C}$ a CMC structure.

The proof of Theorem 2.2.1 will be given in 2.2.4.

2.2.2. *Adding a variable to kill a cycle.* Let $A \in \mathcal{C}$, $M \in C(k)$ and let $\alpha : M \to A^{\#}$ be a map in $C(k)$ (in particular, α commutes with the differentials).

Define a functor

$$h_{A,\alpha} : \mathcal{C} \longrightarrow \mathtt{Ens}$$

by the formula

$$h_{A,\alpha}(B) = \{(f,t) | f : A \to B \in \mathrm{Mor}(\mathcal{C}), t \in \mathrm{Hom}^{-1}(M, B^{\#}) : d(t) = f^{\#} \circ \alpha\}.$$

Since $\mathcal{C}$ admits colimits, the functor $h_{A,\alpha}$ is represented as follows. Put $C = \mathrm{cone}(\alpha)$. One has a couple of maps $f : A^{\#} \to C$ and $t \in \mathcal{H}om^{-1}(M, C)$ satisfying $d(t) = f^{\#} \circ \alpha$. Let now B be a colimit of the diagram

$$A \longleftarrow F(A^{\#}) \longrightarrow F(C).$$

One sees immediately that the couple of maps $A \to B$, $M \to F(C)^{\#} \to B^{\#}$ represents the functor $h_{A,\alpha}$.

The object of $\mathcal{C}$ representing $h_{A,\alpha}$, will be denoted by $A\langle M, \alpha \rangle$.

When $M = k[n]$ and $\alpha : M \to A^{\#}$ takes the generator of M to a cycle $a \in Z^n(A)$, the representing object is obtained by "adding a variable to kill the cycle $a \in Z^n(A^{\#})$". In this case we will write $A\langle T; dT = a \rangle$ instead of $A\langle M, \alpha \rangle$.

2.2.3. *Standard cofibrations and standard acyclic cofibrations.* Let M be a complex of free k-modules with zero differential. For any $A \in \mathcal{C}$ and any map $\alpha : M \to A^{\#}$ the map

$$A \to A\langle M, \alpha \rangle \tag{1}$$

is a cofibration.

Definition. A map $A \to B$ is called *a standard cofibration* if it is a direct limit of a sequence

$$A = A_0 \to A_1 \to \ldots \to B$$

where each map $A_i \to A_{i+1}$ is as in (1).

Let M be a contractible complex of free k-modules. Then

$$A \to A\langle M, \alpha \rangle \tag{2}$$

is an acyclic cofibration.

Definition. A map $A \to B$ is called *a standard acyclic cofibration* if it is a direct limit of a sequence

$$A = A_0 \to A_1 \to \ldots \to B$$

where each map $A_i \to A_{i+1}$ is as in (2).

2.2.4. *The proof of Theorem 2.2.1.* The axioms (CM 1)–(CM 3) are obvious. Also (CM 4)(ii) is immediate. Let us check (CM 5)(i).

Let $f : A \to B \in \mathrm{Mor}(\mathcal{C})$. For each $b \in B^{\#}$ define $C_b = A\langle T_b, S_b; dT_b = S_b\rangle$ and let the map $g_b : C_b \to B$ be defined by the conditions

$$g_b^{\#}(T_b) = b; \ g_b^{\#}(S_b) = db.$$

Put C to be the coproduct of C_b under A and let $g : C \to B$ be the corresponding morphism. The map $A \to C$ is a standard acyclic cofibration and $g^{\#}$ is surjective.

Now, let us check (CM 5)(ii). For this we will construct for a given map $f : A \to B$ a sequence

$$A \to C_0 \to \ldots \to C_i \to C_{i+1} \to \ldots \to B$$

of standard cofibrations such that

(1) the maps $g_i^{\#} : C_i^{\#} \to B^{\#}$ are surjective;

(2) $Z(g_i^{\#}) : ZC_i^{\#} \to ZB^{\#}$ are surjective as well (Z denotes the set of cycles);

(3) if $z \in ZC_i^{\#}$ and $g_i^{\#}(z)$ is a boundary in $B^{\#}$ then the image of z in $C_{i+1}^{\#}$ is a boundary.

Then if we put $C = \lim_{\to} C_i$ and $g : C \to B$ is defined by g_i, the map $A \to C$ is a cofibration and $g^{\#}$ is clearly a surjective quasi-isomorphism since the forgetful functor commutes with filtered colimits.

The object C_0 is constructed exactly as in the proof of (CM 5)(i): one has to join a pair (T_b, S_b) for each element $b \in B^{\#}$ and after that to join a cycle corresponding to each cycle in $B^{\#}$.

In order to get C_{i+1} from C_i one has to join to C_i a variable T for each pair (z, u) with $z \in ZC_i^{\#}, u \in B^{\#}$, such that $g_i^{\#}(z) = du$. One has to put $dT = z$, $g_{i+1}^{\#}(T) = u$.

Let us prove now (CM 4)(i). The proof of the property (CM 5)(i) implies that if $f : A \to B$ is a weak equivalence then there exists a decomposition $f = pi$ where p is an acyclic fibration and i is a standard acyclic cofibration. If f is also a cofibration then according to (CM 4)(ii) there exists $j : B \to C$ making the diagram

commutative. This proves that any acyclic cofibration is a retract of a standard one and this immediately implies (CM 4)(i).

Theorem is proven.

Note that the proof is essentially the one given in [Q2] for DG Lie algebras or in [BoG] for commutative DG algebras.

2.2.5. *Remark.* The proof of the Theorem implies the following:

Any acyclic cofibration is a retract of a standard acyclic cofibration.

Any cofibration is a retract of a standard cofibration.

3. Differential homological algebra

In this Section k is a fixed commutative base ring.

The first application of Theorem 2.2.1 provides a CMC structure on the category of modules $\mathrm{Mod}(A)$ over a dg k-algebra A.

The constructions of this Section will be generalized in Section 4 to the category of algebras over any k-operad. However, even the case $A = k$ is not absolutely well-known: it provides the category $C(k)$ of unbounded complexes over k with a CMC structure.

The category $\mathrm{Mod}(A)$ admits another, somewhat dual CMC structure. These two structures are closely related to a homological algebra developed in [Sp] for the category of sheaves (of modules over a sheaf of commutative rings) and in [AFH] for the category of modules over a dg algebra. Another description of the results of this Section can be found in [BL].

3.1. **Models.** The obvious forgetful functor $\# : \mathrm{Mod}(A) \to C(k)$ admits the left adjoint $F = A \otimes_k$. All limits exist in $\mathrm{Mod}(A)$. Let $\alpha : M \to X^{\#}$ be a morphism in $C(k)$. Then the morphism $\alpha' : A \otimes_k M \to X$ is defined and we have

$$X\langle M, \alpha \rangle = \mathrm{cone}(\alpha').$$

The condition (H1) is trivially fulfilled.

Cofibrant objects in $\mathrm{Mod}(A)$ are exactly direct summands of semi-free A-modules, see [AFH].

The homotopy category of $\mathrm{Mod}(A)$ will be denoted by $D(A)$. Parallelly, the category $\mathrm{Mod}^r(A)$ of right dg A-modules admits the same CMC structure and the corresponding homotopy category will be denoted by $D^r(A)$.

Note that $D(A)$ is also triangulated, the shift functor and the exact triangles being defined in a standard way.

In the special case $A = k$ cofibrant objects of $\mathrm{Mod}(A)$ are exactly $\mathcal{K}$-*projective complexes* of Spaltenstein — see [Sp].

3.2. **Tensor product.** The functor

$$\otimes_A : \mathrm{Mod}^r(A) \times \mathrm{Mod}(A) \to \mathrm{Mod}(k)$$

is defined as usual: for $M, N \in \mathrm{Mod}(A)$ the tensor product $M \otimes_A N$ is the colimit of the diagram

$$M \otimes A \otimes N \rightrightarrows M \otimes N$$

where $\otimes = \otimes_k$, and the arrows take $m \otimes a \otimes n$ to $ma \otimes n$ and to $m \otimes an$ respectively. Since $\otimes_A$ takes homotopy equivalences to homotopy equivalences, and a quasi-isomorphism of cofibrant objects is a homotopy equivalence, it admits a left derived functor

$$\otimes_A^{\mathbf{L}} : D^r(A) \times D(A) \to D(k).$$

This is the functor defined actually in [AFH]. It can be calculated using semi-free resolutions with respect to either of the arguments.

3.3. **Base change.** Let now $f : A \to B$ be a morphism of dg k-algebras. There is a pair of adjoint functors

$$f^* : \mathrm{Mod}(A) \rightleftarrows \mathrm{Mod}(B) : f_*$$

where f_* is just the forgetful functor and $f^*(M) = B \otimes_A M$. Since the functor f_* is exact and f^* preserves cofibrations, one has a pair of adjoint functors

$$\mathbf{L}f^* : D(A) \rightleftarrows D(B) : f_* = \mathbf{R}f_*.$$

Note that the functor $\mathbf{L}f^*$ commutes with $\otimes^{\mathbf{L}}$.

3.3.1. **Theorem.** *Let $f : A \to B$ be a quasi-isomorphism of dg algebras. Then the functors $(\mathbf{L}f^*, f_*)$ establish an equivalence of the derived categories $D(A)$ and $D(B)$.*

Proof. According to [Q1], §4, thm. 3, we have to check that if M is a cofibrant A-module then the map $M \to f_* f^*(M)$ is a quasi-isomorphism. Since any cofibrant module is a direct summand of a semi-free module, and the functor f^* commutes with taking cones (i.e., for any $\alpha : M \to N$ $\mathrm{cone}(f^*\alpha) = f^*(\mathrm{cone}(\alpha))$) the result immediately follows. $\square$

As an immediate consequence of Theorem 3.3.1 we get the following comparison result which we firstly knew from L. Avramov (see [HS], thm. 3.6.7)

3.3.2. **Corollary.** *Let $f : A \to A'$ be a quasi-isomorphism of dg algebras, $g : M \to f_*(M')$ and $h : N \to f_*(N')$ be quasi-isomorphisms in $\mathrm{Mod}(A)$. Then the induced map $M \otimes_A^{\mathbf{L}} N \to M' \otimes_{A'}^{\mathbf{L}} N'$ is a quasi-isomorphism.*

Theorem 3.3.1 will be generalized in Section 4 to the case of operad algebras — see Theorem 4.7.4.

3.4. **Flat modules.** Let A be a DG algebra over k, M be a A-module. We will call M flat (in the terminology of [AFH] - π-flat) if the functor $\otimes_A M$ carries quasi-isomorphisms into quasi-isomorphisms.

3.4.1. **Lemma.** *1. Any cofibrant A-module is A-flat.*

2. A filtered colimit of flat A-modules is A-flat.

3. Let $f : X \to Y$ be a map of flat A-modules. Then the cone $\mathrm{cone}(f)$ is also flat.

Proof. For the claims 1,2 see [AFH], 6.1, 6.2, 6.6. The tensor product commutes with taking cone — this implies the third claim. $\square$

3.4.2. **Lemma.** *Let $\alpha : M \to M'$ be a quasi-isomorphism of flat A-modules. Then for each $N \in \mathtt{Mod}^r(A)$ the map $1 \otimes \alpha : N \otimes_A M \to N \otimes_A M'$ is a quasi-isomorphism.*

Proof. See [AFH], 6.8. □

Thus, the functor $\otimes_A^{\mathbf{L}}$ can be calculated using flat resolutions.

4. Algebras over an operad

4.1. **Introduction.** In this Section we define, using Theorem 2.2.1, a CMC structure on the category $\mathtt{Alg}(\mathcal{O})$ of algebras over an operad $\mathcal{O}$ which is Σ-*split* (see Definition 4.2.4 below). The base tensor category is always the category of complexes $C(k)$ over a fixed commutative ring k. All necessary definitions can be found in [HS], §2,3.

Recall that the forgetful functor $\# : \mathtt{Alg}(\mathcal{O}) \to C(k)$ admits a left adjoint *free $\mathcal{O}$-algebra functor* $F : C(k) \to \mathtt{Alg}(\mathcal{O})$ which takes a complex V to the $\mathcal{O}$-algebra

$$FV = \bigoplus_{n \geq 0}(\mathcal{O}(n) \otimes V^{\otimes n})/\Sigma_n,$$

Σ_n being the symmetric group.

4.1.1. **Theorem.** *Let $\mathcal{O}$ be a Σ-split operad in $C(k)$. The category $\mathtt{Alg}(\mathcal{O})$ endowed with the couple of adjoint functors*

$$\# : \mathtt{Alg}(\mathcal{O}) \rightleftarrows C(k) : F$$

satisfies the conditions (H0), (H1). Thus, $\mathtt{Alg}(\mathcal{O})$ admits a CMC structure in which $f : A \to B$ is a weak equivalence if $f^\#$ is a quasi-isomorphism and is a fibration if $f^\#$ is surjective.

The proof of Theorem 4.1.1 will be given in 4.4.

4.2. **Σ-split operads.** In this subsection we define a class of operads for which Theorem 4.1.1 is applicable. Let us just mention two important examples of a Σ-split operad:

— Any operad in $C(k)$ is Σ-split if $k \supseteq \mathbb{Q}$.

— The operad $\mathtt{Ass}_k$ of associative k-algebras is Σ-split for any k.

4.2.1. *Asymmetric operads.* We will call an asymmetric operad in $C(k)$ "an operad without the symmetric group": it consists of a collection $\mathcal{T}(n) \in C(k)$, $n \geq 0$, of unit $1 : k \to \mathcal{T}(1)$, of associative multiplications, but with no symmetric group action required.

There is a couple of adjoint functors

$$\Sigma : \mathtt{Asop}(k) \rightleftarrows \mathtt{Op}(k) : \#$$

between the category of asymmetric operads in $C(k)$ and that of operads. Here $\#$ is the forgetful functor and Σ is defined as follows.

Let $\mathcal{T}$ be an asymmetric operad. We define $\mathcal{T}^\Sigma(n) = \mathcal{T}(n) \otimes k\Sigma_n$; multiplication is defined uniquely by the multiplication in $\mathcal{T}$ in order to be Σ-invariant.

For an operad $\mathcal{O}$ the adjunction map $\pi : \mathcal{O}^{\#\Sigma} \to \mathcal{O}$ is given by the obvious formula

$$\pi(u \otimes \sigma) = u\sigma \tag{3}$$

where $u \in \mathcal{O}(n), \sigma \in \Sigma_n$.

4.2.2. *Notations: symmetric groups.* In this subsection we denote by $\langle n \rangle$ the ordered set $\{1, \dots, n\}$. Let $f : \langle s \rangle \longrightarrow \langle n \rangle$ be an injective monotone map. This defines a monomorphism $\iota_f : \Sigma_s \to \Sigma_n$ in the obvious way: for $\rho \in \Sigma_s$

$$\iota_f(\rho)(i) = \begin{cases} i \text{ if } i \notin f(\langle s \rangle) \\ f(\rho(j)) \text{ if } i = f(j) \end{cases} \tag{4}$$

Define a map (not a homomorphism) $\rho_f : \Sigma_n \to \Sigma_s$ by the condition

$$\rho = \rho_f(\sigma) \text{ iff } \rho(i) < \rho(j) \Leftrightarrow \sigma(f(i)) < \sigma(f(j)) \tag{5}$$

Define a set $T_f \subseteq \Sigma_n$ by

$$T_f = \{\sigma \in \Sigma_n | \sigma \circ f : \langle s \rangle \to \langle n \rangle \text{ is monotone }\}.$$

Lemma.   *For $\sigma \in \Sigma_n$ there is a unique presentation*

$$\sigma = \tau\iota_f(\rho)$$

with $\tau \in T_f$ and $\rho \in \Sigma_s$. In this presentation $\rho = \rho_f(\sigma)$.

Proof. Obvious. □

For $M \in C(k)$ we will write $M \otimes \Sigma_n$ instead of $M \otimes_k k\Sigma_n$. Also if M is a right Σ_n-module, N is a left Σ_n-module and Σ is a subgroup in Σ_n then we write $M \otimes_\Sigma N$ instead of $M \otimes_{k\Sigma} N = (M \otimes N)/\Sigma$.

If M admits a right Σ_n-action and $f : \langle s \rangle \to \langle n \rangle$ is monotone injective, the map ρ_f defined above induces a map

$$M \otimes \Sigma_n \longrightarrow M \otimes \Sigma_s$$

carrying the element $m \otimes \sigma$ to $m\tau \otimes \rho_f(\sigma)$ where, as in Lemma above, $\sigma = \tau\iota_f(\rho_f(\sigma))$. We denote this map also by ρ_f.

Note that the map ρ_f is equivariant with respect to right Σ_s-action.

4.2.3. *Notations: operads.* An operad $\mathcal{O}$ is defined by a collection of multiplication maps

$$\gamma : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \longrightarrow \mathcal{O}(\textstyle\sum m_i).$$

One defines the operations $\circ_k : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(m + n - 1)$ to be the compositions

$$\mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n) \otimes \mathcal{O}(1)^{\otimes k-1} \otimes \mathcal{O}(m) \otimes \mathcal{O}(1)^{\otimes n-k} \xrightarrow{\gamma} \mathcal{O}(m + n - 1)$$

with the first map induced by the units $1 : k \to \mathcal{O}(1)$. The multiplications $\circ_k$ can be described one through another for different k using the symmetric group action on $\mathcal{O}(n)$.

4.2.4. Σ-*split operads.*

Σ-splitting of an operad $\mathcal{O}$ is a collection of maps $\mathcal{O}(n) \to \mathcal{O}^{\#\Sigma}(n)$ which splits the adjunction map $\pi : \mathcal{O}^{\#\Sigma} \to \mathcal{O}$ from (3). Of course, there is some condition describing a compatibility of these maps with the multiplications maps. Here is the definition.

Definition. (1) Let $\mathcal{O}$ be an operad in $C(k)$. Σ-*splitting* of $\mathcal{O}$ is a collection of maps of complexes $t(n) : \mathcal{O}(n) \to \mathcal{O}^{\#\Sigma}(n)$ such that

(EQU) $t(n)$ is Σ_n-equivariant

(SPL) $\pi \circ t(n) = \mathrm{id} : \mathcal{O}(n) \to \mathcal{O}(n)$ and

(COM) for any $m, n > 0$ and $1 \le k \le n$ the diagram

$$
\begin{array}{ccc}
\mathcal{O}(n) \otimes \mathcal{O}(m) & \xrightarrow{\;\circ_k\;} & \mathcal{O}(n+m-1) \\
\downarrow{\scriptstyle t\otimes\mathrm{id}} & & \downarrow{\scriptstyle t} \\
\mathcal{O}(n) \otimes \Sigma_n \otimes \mathcal{O}(m) & & \\
\downarrow{\scriptstyle \rho_f\otimes\mathrm{id}} & & \mathcal{O}(n+m-1) \otimes \Sigma_{n+m-1} \\
\mathcal{O}(n) \otimes \Sigma_{n-1} \otimes \mathcal{O}(m) & & \\
\downarrow{\scriptstyle \sigma_{23}} & & \downarrow{\scriptstyle \rho_g} \\
\mathcal{O}(n) \otimes \mathcal{O}(m) \otimes \Sigma_{n-1} & \xrightarrow{\;\circ_k\otimes\mathrm{id}\;} & \mathcal{O}(n+m-1) \otimes \Sigma_{n-1}
\end{array}
$$

is commutative.

Here $f : \langle n-1 \rangle \to \langle n \rangle$ is the map omitting the value k, $g : \langle n-1 \rangle \to \langle m+n-1 \rangle$ omits the values $k, \dots, k+m-1$ and σ_{23} is the standard twist interchanging the second and the third factors.

(2) An operad $\mathcal{O}$ is Σ-split if it admits a Σ-splitting.

Remark. It is sufficient to require the validity of (COM) only for, say, $k = 1$. The compatibility of the map t with other multiplications $\circ_k$ then follows immediately since $\circ_k$ can be expressed through $\circ_1$ using the symmetric group action on the components of the operad.

4.2.5. *Examples.* There are two very important examples of Σ-split operads.

1. Let $\mathcal{T}$ be an asymmetric operad and $\mathcal{O} = \mathcal{T}^\Sigma$. Then the composition

$$
\mathcal{O}(n) = \mathcal{T}^\Sigma(n) \longrightarrow \mathcal{O}^{\#\Sigma}(n)
$$

defines a Σ-splitting of $\mathcal{O}$.

Let $\mathtt{Com}_k$ be the operad given by $\mathtt{Com}_k(n) = k$ for all n. The action of Σ_n on k is supposed to be trivial. The algebras over $\mathtt{Com}_k$ are just commutative dg k-algebras: $\mathtt{Alg}(\mathtt{Com}_k) = \mathtt{DGC}(k)$. Put $\mathtt{Ass}_k = \mathtt{Com}_k^{\#\Sigma}$. One has $\mathtt{Alg}(\mathtt{Ass}_k) = \mathtt{DGA}(k)$, the category of associative dg algebras, and $\mathtt{Ass}_k$ is naturally Σ-split.

2. Suppose $k \supseteq \mathbb{Q}$. Then any operad in $C(k)$ is Σ-split: the splitting is defined by the formula

$$t(u) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} u\sigma^{-1} \otimes \sigma.$$

The operad $\mathtt{Com}_k$ is Σ-split only when $k \supseteq \mathbb{Q}$. The same is true for the operad $\mathtt{Lie}_k$ such that the algebras over $\mathtt{Lie}_k$ are just dg Lie k-algebras. We denote in the sequel by $\mathtt{DGL}(k) = \mathtt{Alg}(\mathtt{Lie}_k)$ the category of dg Lie algebras over k.

4.3. **Extension of a homotopy on free algebras.**

Let $V \in C(k)$. Let $\alpha : V \to V$ be a map of complexes of degree zero and let $h \in \mathrm{Hom}(V, V[-1])$ satisfy the property

$$d(h) = \mathrm{id}_V - \alpha.$$

The endomorphism α induces the endomorphism $F(\alpha)$ of the free $\mathcal{O}$-algebra $F(V)$ by the obvious formula

$$F(\alpha)(u \otimes x_1 \otimes \cdots \otimes x_n) = u \otimes \alpha(x_1) \otimes \cdots \otimes \alpha(x_n).$$

We will describe now a nice homotopy H connecting $\mathrm{id}_{F(V)}$ with $F(\alpha)$. This is a sort of "skew derivation" on $F(V)$ defined by h.

The restriction of H on $F_n(V) = \mathcal{O}(n) \otimes_{\Sigma_n} V^{\otimes n}$ is given by the composition

$$\mathcal{O}(n) \otimes_{\Sigma_n} V^{\otimes n} \xrightarrow{t} \mathcal{O}(n) \otimes \Sigma_n \otimes_{\Sigma_n} V^{\otimes n} = \mathcal{O}(n) \otimes V^{\otimes n} \xrightarrow{\sum_{p+q=n-1} \mathrm{id} \otimes \alpha^{\otimes p} \otimes h \otimes \mathrm{id}^{\otimes q}} \mathcal{O}(n) \otimes_{\Sigma_n} V^{\otimes n}. \tag{6}$$

The property $dH = \mathrm{id}_{F(V)} - F(\alpha)$ is verified immediately.

4.4. **Proof of Theorem 4.1.1.**

The property (H0) is obvious.

Let us prove (H1). Let A be a $\mathcal{O}$-algebra and let $X \in C(k)$ be a contractible complex. Put $V = A^{\#} \oplus X$. The complex V is homotopy equivalent to $A^{\#}$, the maps between $A^{\#}$ and V being obvious and the homotopy equivalence being defined by a map $h : V \to V$ of degree -1 which vanishes on $A^{\#}$. One has $dh = \alpha$ where $\alpha : V \to V$ is the composition

$$V = A^{\#} \oplus X \to A^{\#} \to V.$$

According to 4.3, h defines a homotopy

$$H : F(V) \longrightarrow F(V)$$

of degree -1 extending h.

Let now I be the kernel of the natural projection $F(A^{\#}) \to A$. Let J be the ideal in $F(V)$ generated by I. We will prove now that the homotopy H satisfies the property

$$H(J) \subseteq J. \tag{7}$$

Then H induces a homotopy on $F(V)/J = A \coprod F(X)$ which proves the theorem.

To prove the property (7) let us consider the restriction of H to $\mathcal{O}(n) \otimes A^{\otimes r} \otimes V^{\otimes s}$ with $n = r + s$.

An easy calculation using the properties $\alpha|_A = \mathrm{id}$, $h|_A = 0$ shows that this restriction of H can be calculated as the composition

$$\mathcal{O}(n) \otimes A^{\otimes r} \otimes V^{\otimes s} \to \mathcal{O}(n) \otimes \Sigma_n \otimes A^{\otimes r} \otimes V^{\otimes s} \xrightarrow{\rho}$$

$$\mathcal{O}(n) \otimes \Sigma_s \otimes A^{\otimes r} \otimes V^{\otimes s} \xrightarrow{T_{23}} \mathcal{O}(n) \otimes A^{\otimes r} \otimes \Sigma_s \otimes V^{\otimes s} \to$$

$$\mathcal{O}(n) \otimes A^{\otimes r} \otimes V^{\otimes s} \xrightarrow{\sum_{p+q=s-1} \mathrm{id} \otimes \mathrm{id} \otimes \alpha^{\otimes p} \otimes h \otimes \mathrm{id}^{\otimes q}} \mathcal{O}(n) \otimes A^{\otimes r} \otimes V^{\otimes s} \to \mathcal{O}(n) \otimes_{\Sigma_n} V^{\otimes n}. \quad (8)$$

To check (7) note that the ideal $I \subseteq F(A^{\#})$ is generated over k by the expressions

$$b \otimes x_1 \otimes \cdots \otimes x_m - \mu(b \otimes x_1 \otimes \cdots \otimes x_m)$$

with $b \in \mathcal{O}(m), x_i \in A$. Therefore the ideal $J \subseteq F(V)$ is generated over k by the expressions

$$a \circ_1 b \otimes x_1 \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_{n-1} - a \otimes \mu(b \otimes x_1 \otimes \cdots \otimes x_m) \otimes y_1 \otimes \cdots \otimes y_{n-1}. \quad (9)$$

Hence, we have to check that H transforms an element of form (9) into an element of J. This easily follows from the axiom (COM) and formula (8).

Theorem is proven.

4.5. Notations. The homotopy category of $\mathsf{Alg}(\mathcal{O})$ is denoted by $\mathsf{Hoalg}(\mathcal{O})$. For the special values of $\mathcal{O}$ we denote $\mathsf{Hoass}(k) = \mathsf{Hoalg}(\mathsf{Ass}_k)$, $\mathsf{Hocom}(k) = \mathsf{Hoalg}(\mathsf{Com}_k)$, $\mathsf{Holie}(k) = \mathsf{Hoalg}(\mathsf{Lie}_k)$.

4.6. Base change. Consider now a map $\alpha : \mathcal{O} \to \mathcal{O}'$ of operads. We will study direct and inverse image functors between the categories $\mathsf{Alg}(\mathcal{O})$ and $\mathsf{Alg}(\mathcal{O}')$.

This generalizes the considerations of 3.3.1 to the case of operad algebras.

4.6.1. *Direct image.* Let A be a $\mathcal{O}'$-algebra. Its direct image $\alpha_*(A)$ is just the $\mathcal{O}$-algebra obtained from A by forgetting "the part of structure": the multiplication map is given by the composition

$$\mathcal{O}(n) \otimes A^{\otimes n} \xrightarrow{\alpha \otimes 1} \mathcal{O}'(n) \otimes A^{\otimes n} \longrightarrow A.$$

This functor is obviously exact.

4.6.2. *Inverse image.* The inverse image $\alpha^* : \mathsf{Alg}(\mathcal{O}) \to \mathsf{Alg}(\mathcal{O}')$ is by definition the functor left adjoint to α_*.

Let us explicitly construct α^*. Let F and F' be the free $\mathcal{O}$-algebra and free $\mathcal{O}'$-algebra functors respectively. For $A \in \mathsf{Mod}(\mathcal{O})$ let I_A be the kernel of the natural map $F(A^{\#}) \to A$. Then, if $F(\alpha) : F(A^{\#}) \to F'(A^{\#})$ is the map induced by α, one defines

$$\alpha^*(A) = F'(A^{\#})/(F(\alpha)(I_A)).$$

4.6.3. *Derived functors.* We wish now to construct an adjoint pair of derived functors

$$\mathbf{L}\alpha^* : \mathtt{Hoalg}(\mathcal{O}) \rightleftarrows \mathtt{Hoalg}(\mathcal{O}') : \mathbf{R}\alpha_* = \alpha_*.$$

Let us check the conditions of [Q1], §4, thm. 3.

Let $M \in C(k), A \in \mathtt{Alg}(\mathcal{O})$. Any map $f : M \to A^\#$ defines a map $f' : M \to (\alpha^* A)^\#$. Then one immediately sees that there is a canonical isomorphism

$$\alpha^*(A\langle M, f\rangle) = \alpha^*(A)\langle M, f\rangle$$

since these two $\mathcal{O}'$-algebras just represent isomorphic functors. This immediately implies that α^* carries standard cofibrations to standard cofibration and standard acyclic cofibrations to standard acyclic cofibrations. Then Remark 2.2.5 implies that α^* preserves cofibrations and acyclic cofibrations.

Let us check that α^* carries fibrations to fibrations. In fact, if $f : A \to B$ is a fibration, then $f^\# : A^\# \to B^\#$ is surjective. Thus the induced map $F'(A^\#)^\# \to F'(B^\#)^\#$ is also surjective which ensures that the induced map of the quotients

$$\alpha^*(A)^\# \to \alpha^*(B)^\#$$

is also surjective.

Let us check that α^* carries homotopy equivalences to weak equivalences.

In fact, if in the diagram

$$X \xrightarrow{i} X^I \xrightarrow{p} X \times X$$

i is a trivial cofibration and p is a fibration, the functor α^* gives rise to the diagram

$$\alpha^*(X) \xrightarrow{\alpha^*(i)} \alpha^*(X^I) \xrightarrow{\alpha^*(p)} \alpha^*(X \times X) \xrightarrow{j} \alpha^*(X) \times \alpha^*(X).$$

The map $\alpha^*(i)$ is already known to be acyclic cofibration. Thus, any pair of maps from somewhere to $\alpha^*(X)$ defined by a map to $\alpha^*(X^I)$, induce the same map in homology. This implies that if $f : X \to Y$ is a homotopy equivalence in $\mathtt{Alg}(\mathcal{O})$ then $\alpha^*(f)$ is an isomorphism in $\mathtt{Hoalg}(\mathcal{O}')$.

Since weak equivalences of cofibrant objects in $\mathtt{Alg}(\mathcal{O})$ are homotopy equivalences so α^* carries them to weak equivalences in $\mathtt{Alg}(\mathcal{O}')$.

This proves the following

4.6.4.   **Theorem.**   *Inverse and direct image functors define a pair of adjoint derived functors*

$$\mathbf{L}\alpha^* : \mathtt{Hoalg}(\mathcal{O}) \rightleftarrows \mathtt{Hoalg}(\mathcal{O}') : \mathbf{R}\alpha_* = \alpha_*.$$

4.7. **Equivalence.** Suppose that $\alpha : \mathcal{O} \to \mathcal{O}'$ is a quasi-isomorphism of Σ-split operads *compatible with the splittings*. We shall prove that $\mathbf{L}\alpha^*$ and α_* establish an equivalence of the homotopy categories $\mathtt{Hoalg}(\mathcal{O})$ and $\mathtt{Hoalg}(\mathcal{O}')$.

In order to do this, one has to check that the adjunction map

$$\eta_A : A \longrightarrow \alpha_*(\alpha^*(A))$$

is a weak equivalence for any cofibrant A.

4.7.1. *1st reduction.* Since retract of a weak equivalence is a weak equivalence, it suffices to prove the assertion for A standard cofibrant.

4.7.2. *2nd reduction.* Since any standard cofibrant object is a filtered colimit of finitely generated ones, and the functors α^* and α_* commute with filtered colimits, it suffices to prove that η_A is a weak equivalence when A is a finitely generated standard cofibrant algebra.

4.7.3. Let now A be a finitely generated standard cofibrant algebra. Let $\{x_i\}_{i\in I}$ be a finite set of (graded free) generators of A. Choose a full order on the set I of generators in order that for any $i \in I$ the differential $d(x_i)$ belongs to the algebra generated by $x_j, j < i$.

For any multi-index $m : I \to \mathbb{N}$ denote $|m| = \sum m_i$. Denote by M the set of all multi-indices. The set M is well-ordered with respect to "inverse lexicographic" order:

$$m > m' \text{ if there exists } i \in I \text{ so that } m_j = m'_j \text{ for } j > i \text{ and } m_i > m'_i.$$

Then the algebra A as a graded k-module is a direct sum indexed by M of the components

$$\mathcal{O}(|m|) \otimes_{\Sigma_m} \bigotimes_{i\in I} x_i^{\otimes m_i}.$$

Here $\Sigma_m = \prod_{i\in I} \Sigma_{m_i}$.

This defines an increasing filtration of A

$$\mathcal{F}_d(A) = \sum_{m<d} \mathcal{O}(|m|) \otimes_{\Sigma_m} \bigotimes_{i\in I} x_i^{\otimes m_i}$$

indexed by M which is obviously a filtration by subcomplexes.

The functor $\alpha^* : \mathrm{Alg}(\mathcal{O}) \to \mathrm{Alg}(\mathcal{O}')$ commutes with the functor forgetting the differentials. Thus, $A' = \alpha^*(A)$ admits the filtration analogous to $\{\mathcal{F}_d(A)\}_{d\in M}$.

In order to prove that the map η_A is a weak equivalence, we will prove by induction that the map

$$\mathcal{F}_d(A) \longrightarrow \mathcal{F}_d(A')$$

is quasi-isomorphism. For this one has to check that the maps

$$\mathrm{gr}_d(\eta) : \mathrm{gr}_d(A) \longrightarrow \mathrm{gr}_d(A')$$

are quasi-isomorphisms where

$$\mathrm{gr}_d(A) = \mathcal{F}_{d+1}(A)/\mathcal{F}_d(A) \approx \mathcal{O}(|d|) \otimes_{\Sigma_m} k$$

and similarly $\mathrm{gr}_d(A') \approx \mathcal{O}'(|d|) \otimes_{\Sigma_m} k$.

Now we will use that the map $\alpha : \mathcal{O} \to \mathcal{O}'$ is compatible with Σ-splittings. In fact, in this case the map $\mathrm{gr}_d(\eta)$ is a retract of the map

$$\mathcal{O}^\Sigma(|d|) \otimes_{\Sigma_d} k \longrightarrow \mathcal{O}'^\Sigma(|d|) \otimes_{\Sigma_d} k$$

which is obviously a quasi-isomorphism.

Thus we have proven the following

4.7.4. **Theorem.** *Let $\alpha : \mathcal{O} \to \mathcal{O}'$ be a quasi-isomorphism of Σ-split operads compatible with splittings. Then the functors*

$$\mathbf{L}\alpha^* : \mathtt{Hoalg}(\mathcal{O}) \rightleftarrows \mathtt{Hoalg}(\mathcal{O}') : \mathbf{R}\alpha_* = \alpha_*$$

are equivalences of the homotopy categories.

4.8. **Simplicial structure on $\mathtt{Alg}(\mathcal{O})$.** From now on the base ring k is supposed to contain the rationals. We define on $\mathtt{Alg}(\mathcal{O})$ the structure of simplicial category which is a direct generalization of the definitions [BoG], Ch. 5.

4.8.1. *Polynomial differential forms.* Recall (cf. [BoG], [HDT], ch. 6) the definition of simplicial commutative dg algebra $\Omega = \{\Omega(n)\}_{n \geq 0}$.

For any $n \geq 0$ the dg algebra Ω_n is the algebra of polynomial differential forms on the standard n-simplex $\Delta(n)$.

Thus, one has

$$\Omega_n = k[t_0, \dots, t_n, dt_0, \dots, dt_n]/(\textstyle\sum t_i - 1, \sum dt_i).$$

The algebras Ω_n form a simplicial commutative dg algebra: a map $u : [p] \to [q]$ induces the map $\Omega(u) : \Omega_q \to \Omega_p$ defined by the formula $\Omega(u)(t_i) = \sum_{u(j)=i} t_j$.

4.8.2. *Functional spaces for $\mathcal{O}$-algebras.* Let $A, B \in \mathtt{Alg}(\mathcal{O})$. We define $\mathrm{Hom}^{\Delta}(A, B) \in \Delta^0\mathtt{Ens}$ to be the simplicial set whose n simplices are

$$\mathrm{Hom}_n^{\Delta}(A, B) = \mathrm{Hom}(A, \Omega_n \otimes B).$$

Note that Ω_n being a commutative dg algebra over k, the tensor product admits a natural $\mathcal{O}$-algebra structure.

4.8.3. **Lemma.** *(cf. [BoG], Lemma 5.2) There is a natural morphism*

$$\Phi(W) : \mathrm{Hom}(A, \Omega(W) \otimes B) \longrightarrow \mathrm{Hom}_{\Delta^0\mathtt{Ens}}(W, \mathrm{Hom}^{\Delta}(A, B))$$

which is a bijection provided W is finite.

Proof. The map Φ is defined in a standard way. One has obviously that $\Phi(\Delta(n))$ is a bijection for any n. Now, the contravariant functor $\Omega : \Delta^0\mathtt{Ens} \to \mathtt{DGC}(k)$ carries colimits to limits; the functor

$$\_ \otimes B : \mathtt{DGC}(k) \to \mathtt{Alg}(\mathcal{O})$$

preserves finite limits. This proves that $\Phi(W)$ is bijection for any finite simplicial set W. $\qquad\square$

4.8.4.   **Lemma.**   *Let $i : A \to B$ be a cofibration and $p : X \to Y$ be a fibration in $\mathtt{Alg}(\mathcal{O})$. Then the canonical map*

$$(i^*, p_*) : \operatorname{Hom}^\Delta(B, X) \to \operatorname{Hom}^\Delta(A, X) \times_{\operatorname{Hom}^\Delta(A,Y)} \operatorname{Hom}^\Delta(B, Y)$$

is a Kan fibration. It is acyclic if i or p is acyclic.

Proof. See the proof of [BoG], Prop. 5.3. $\square$

The assertions below immediately follow from Lemma 4.8.4, see also [BoG], Ch. 5.

4.8.5.   **Corollary.**   *Let $i : A \to B$ be a cofibration and $C \in \mathtt{Alg}(\mathcal{O})$. Then*

$$i^* : \operatorname{Hom}^\Delta(B, X) \to \operatorname{Hom}^\Delta(A, X)$$

is a Kan fibration. It is acyclic if i is acyclic.

4.8.6.   **Corollary.**   *If A is cofibrant then $\operatorname{Hom}^\Delta(A, X)$ is Kan for every X.*

4.8.7.   **Corollary.**   *If A is cofibrant and $p : X \to Y$ is fibrant then $p_* : \operatorname{Hom}^\Delta(A, X) \to \operatorname{Hom}^\Delta(A, Y)$ is Kan fibration. It is acyclic if p is acyclic.*

4.8.8.   **Corollary.**   *Let A be cofibrant and let $f : X \to Y$ be weak equivalence. Then*

$$f_* : \operatorname{Hom}^\Delta(A, X) \to \operatorname{Hom}^\Delta(A, Y)$$

is a weak equivalence.

4.8.9.   *Remark.*   Note that the canonical map $A \to A^I$ is not usually a cofibration: take, for instance, A to be the trivial (one-dimensional) Lie algebra. Then A^I is commutative and of course is not cofibrant.

4.8.10.   *Simplicial homotopy.*

Definition.   Two maps $f, g : A \to B$ in $\mathtt{Alg}(\mathcal{O})$ are called *simplicially homotopic* if there exists $F \in \operatorname{Hom}^\Delta_1(A, B)$ such that $d_0 F = f$, $d_1 F = g$.

All the assertions of [BoG], Ch. 6, are valid in our case. In particular, simplicial homotopy is an equivalence relation provided A is cofibrant. In this case simplicial homotopy coincides with both right and left homotopy relations defined in [Q1]. This allows one to realize the homotopy category $\mathtt{Hoalg}(\mathcal{O})$ as the category having the cofibrant $\mathcal{O}$-algebras as the objects and the set $\pi_0 \operatorname{Hom}^\Delta(A, B)$ as the set of morphisms from A to B.

It seems however that the simplicial category $\mathtt{Alg}^\Delta(\mathcal{O})$ defined by

— $\operatorname{Ob} \mathtt{Alg}^\Delta(\mathcal{O})$ is the collection of cofibrant $\mathcal{O}$-algebras;

— $A, B \mapsto \operatorname{Hom}^\Delta(A, B)$

is more useful then the homotopy category $\mathtt{Hoalg}(\mathcal{O})$.

In this Section we study the category of modules over an operad algebra $(\mathcal{O}, A)$. This can be described as the category of modules over the universal enveloping algebra $U(\mathcal{O}, A)$. The corresponding derived category $DU(\mathcal{O}, A)$ can be different for quasi-isomorphic operad algebras, so one has to "derive" this construction to get an invariant depending only on the isomorphism class of $(\mathcal{O}, A)$ in the homotopy category $\mathrm{Hoalg}(\mathcal{O})$. To get this, one should substitute the algebra A with its cofibrant resolution — thus substituting A-modules with "virtual A-modules" and the enveloping algebra of A — with the "derived enveloping algebra".

Starting from 5.3 we suppose that the operad $\mathcal{O}$ is Σ-split.

5.1. Modules. Enveloping algebra.
We refer to [HS], ch. 3, for the definition of $(\mathcal{O}, A)$-modules, $(\mathcal{O}, A)$-tensor algebra $T(\mathcal{O}, A)$ and the universal enveloping algebra $U(\mathcal{O}, A)$.

5.1.1. Lemma.
The functor $U(\mathcal{O}, \_): \mathrm{Alg}(\mathcal{O}) \to \mathrm{DGA}(k)$ commutes with filtered colimits.

Proof. Recall that the enveloping algebra $U(\mathcal{O}, A)$ coincides with the colimit (both in $\mathrm{DGA}(k)$ and in $C(k)$) of the diagram

$$T(\mathcal{O}, \#F(\#A)) \rightrightarrows T(\mathcal{O}, \#A)$$

where $T(\mathcal{O}, \_)$ is the $\mathcal{O}$-tensor algebra functor and $F(\_)$ is the free $\mathcal{O}$-algebra functor — see [HS], ch.4. Now the lemma immediately follows from the fact that the functors $F, T, \#$ commute with filtered colimits. $\qquad\square$

5.2. Functoriality.
Let $f = (\alpha, \phi): (\mathcal{O}, A) \to (\mathcal{O}', A')$ be a map of operad algebras, where $\alpha: \mathcal{O} \to \mathcal{O}'$ is a map of operads and $\phi: A \to \alpha_*(A')$ is a map of $\mathcal{O}$-algebras. This induces a map $U(f): U(\mathcal{O}, A) \to U(\mathcal{O}', A')$ of the corresponding enveloping algebras and so by 3.3 one has the following pairs of adjoint functors

$$f^*: \mathrm{Mod}(\mathcal{O}, A) \; \rightleftarrows \; \mathrm{Mod}(\mathcal{O}', A'): f_* \tag{10}$$

$$\mathbf{L}f^*: DU(\mathcal{O}, A) \; \rightleftarrows \; DU(\mathcal{O}', A'): f_* = \mathbf{R}f_*. \tag{11}$$

The adjoint functors (11) are equivalences provided $f: U(\mathcal{O}, A) \to U(\mathcal{O}', A')$ is a quasi-isomorphism. Unfortunately, this is not always the case even when α and ϕ are quasi-isomorphisms.

5.3. Derived enveloping algebra.
Fix a Σ-split operad $\mathcal{O}$ in $C(k)$; we will write $U(A)$ instead of $U(\mathcal{O}, A)$ and $T(V)$ instead of $T(\mathcal{O}, V)$.

5.3.1. Proposition.
Let $A \in \mathrm{Alg}(\mathcal{O})$ be cofibrant, $X \in C(k)$ be contractible, $A' = A \coprod F(X)$. Then the natural map $U(A) \longrightarrow U(A')$ is a quasi-isomorphism.

Proof. The proof is similar to that of 4.7.

1st reduction. It suffices to prove the claim for standard cofibrant A since a retract of quasi-isomorphism is quasi-isomorphism.

2nd reduction. We can suppose that A is finitely generated since filtered colimit of quasi-isomorphisms is quasi-isomorphism and the functor U commutes with filtered colimits — see 5.1.1.

3rd step. (compare with 4.7). Let $\{x_i\}_{i \in I}$ be a set of homogeneous generators of A with I ordered as in 4.7. Let M be the set of multi-indices $m : I \to \mathbb{N}$ with the "opposite-to-lexicographic" order. Then $U(A)$ as a graded k-module takes form

$$U(A) = \bigoplus_{m \in M} \mathcal{O}(|m| + 1) \otimes_{\Sigma_m} \bigotimes_i x_i^{\otimes m_i}$$

where, as in 4.7, $\Sigma_m = \prod_{i \in I} \Sigma_{m_i}$. This defines a filtration of $U(A)$ by subcomplexes

$$\mathcal{F}_d(U(A)) = \sum_{m < d} \mathcal{O}(|m| + 1) \otimes_{\Sigma_m} \bigotimes_i x_i^{\otimes m_i}. \tag{12}$$

In a similar way, $U(A')$ as a graded k-module is isomorphic to a tensor algebra; it admits a direct sum decomposition as follows

$$U(A') = \bigoplus_{m \in M} (\bigoplus_{k \geq 0} \mathcal{O}(|m| + k + 1) \otimes_{\Sigma_k \times \Sigma_m} X^{\otimes k} \otimes \bigotimes_i x_i^{\otimes m_i}).$$

This defines a filtration of $U(A')$ by subcomplexes

$$\mathcal{F}(U(A')) = \sum_{m < d} (\bigoplus_{k \geq 0} \mathcal{O}(|m| + k + 1) \otimes_{\Sigma_k \times \Sigma_m} X^{\otimes k} \otimes \bigotimes_i x_i^{\otimes m_i}).$$

The associated graded complexes take form

$$\mathcal{O}(|d| + 1) \otimes_{\Sigma_d} k \text{ for } \mathrm{gr}_d \mathcal{F}(U(A))$$

and

$$\sum_{k \geq 0} \mathcal{O}(|d| + k + 1) \otimes_{\Sigma_k \times \Sigma_d} X^{\otimes k} \otimes k \text{ for } \mathrm{gr}_d \mathcal{F}(U(A')).$$

We have to check that the summands corresponding to $k > 0$ are contractible. This immediately follows from the contractibility of X and Σ-splitness of $\mathcal{O}$. $\qquad\square$

5.3.2. Corollary. *Let $f : A \to B$ be an acyclic cofibration in $\mathtt{Alg}(\mathcal{O})$ with A (and hence B) cofibrant. Then $U(f)$ is quasi-isomorphism.*

Proof. Any acyclic cofibration is a retract of a standard one; since everything commutes with filtered colimits, we immediately get the assertion. $\qquad\square$

5.3.3. Corollary. *Let $f : A \to B$ be a weak equivalence of cofibrant algebras in $\mathtt{Alg}(\mathcal{O})$. then $U(f)$ is quasi-isomorphism.*

Proof. Let $B \xrightarrow{\alpha} B^I \rightrightarrows B$ be a path diagram for B so that α is an acyclic cofibration. By Lemma above $U(\alpha)$ is quasi-isomorphism. This immediately implies that if $f, g : A \to B$ are homotopic then $U(f), U(g)$ induce the same map in cohomology.

Now, if $f : A \to B$ is a weak equivalence and A, B are cofibrant then f is homotopy equivalence, i.e. there exist $g : B \to A$ such that the compositions are homotopic to appropriate identity

maps. This implies that $U(f)$ and $U(g)$ induce mutually inverse maps in the cohomology. In particular, $U(f)$ is quasi-isomorphism. $\qquad\square$

Corollary 5.3.3 allows one to define the left derived functor

$$\mathbf{L}U : \mathtt{Hoalg}(\mathcal{O}) \to \mathtt{Hoass}(k)$$

from the homotopy category of $\mathcal{O}$-algebras to the homotopy category of associative dg k-algebras.

5.3.4. **Lemma.** *1. Let $f : A \to B$ be a weak equivalence of cofibrant algebras in $\mathtt{Alg}(\mathcal{O})$. Then the functors*

$$\mathbf{L}f^* : DU(\mathcal{O}, A) \rightleftarrows DU(\mathcal{O}, B) : f_* = \mathbf{R}f_*. \qquad (13)$$

of (11) are equivalences.

2. Let $f, g : A \to B$ be homotopic maps of cofibrant algebras. Then there is an isomorphism of functors

$$f_* \longrightarrow g_* : DU(\mathcal{O}, A) \to DU(\mathcal{O}, B).$$

This isomorphism depends only on the homotopy class of the homotopy connecting f with g.

Proof. The first part follows from Corollary 5.3.3 and 3.3.1.

Let $B \xrightarrow{\alpha} B^I \overset{p_0,p_1}{\Longrightarrow} B$ be a path diagram for B so that α is an acyclic cofibration. Since the functors p_{0*} and p_{1*} are both quasi-inverse to α_*, they are naturally isomorphic. Therefore, any homotopy $F : A \to B^I$ between f and g defines an isomorphism θ_F between f_* and g_*. Let now $F_0, F_1 : A \to B^I$ be homotopic. The homotopy can be realized by a map $h : A \to C$ where C is taken from the path diagram

$$B^I \xrightarrow{\beta} C \overset{q_0 \times q_1}{\Longrightarrow} B^I \times_{B \times B} B^I \qquad (14)$$

where β is an acyclic cofibration, $q_0 \times q_1$ is a fibration, $q_i \circ h = F_i, i = 0, 1$. Passing to the corresponding derived categories we get the functors $q_{i*} \circ p_{j*} : D(B) \to D(C)$ which are quasi-inverse to $\alpha_* \circ \beta_* : D(C) \to D(B)$. This implies that $\theta_{F_0} = \theta_{F_1}$. $\qquad\square$

5.4. Derived category of virtual modules.

5.4.1. For a $\mathcal{O}$-algebra A we define the derived category $D(\mathcal{O}, A)$ to be the derived category of modules $DU(\mathcal{O}, P)$ where $P \to A$ is a cofibrant resolution. The category $D(\mathcal{O}, A)$ is defined uniquely in a way one could expect from an object of 2-category $\mathtt{Cat}$: it is unique up to an equivalence which is unique up to a unique isomorphism.

Any map $f = (\alpha, \phi) : (\mathcal{O}, A) \to (\mathcal{O}', A')$ of operad algebras over a map α of operads defines a pair of adjoint functors

$$\mathbf{L}f^* : D(\mathcal{O}, A) \rightleftarrows D(\mathcal{O}', A') : \mathbf{R}f_*$$

To construct these functors one has to choose cofibrant resolutions $P \to A$ and $P' \to A'$ of the algebras; the $\mathcal{O}'$-algebra $\phi^*(P)$ is cofibrant and therefore one can lift the composition

$$\phi^*(P) \to \phi^*(A) \xrightarrow{\alpha} A'$$

to a map $\phi^*(P) \to P'$. The construction is unique up to a unique isomorphism.

We present below a more "canonical" construction of $D(\mathcal{O}, A)$ in terms of fibered categories. This approach follows [BL], 2.4.

The correspondence $A \mapsto DU(\mathcal{O}, A)$ together with the functors $\mathbf{R}f_* = f_*$ of (11) as "inverse image functors" define a fibered category $DU/\mathtt{Alg}(\mathcal{O})$.

Let $A \in \mathtt{Alg}(\mathcal{O})$. Denote by $\mathfrak{C}/A$ the category of maps $P \to A$ with cofibrant P and let $c_A : \mathfrak{C}/A \to \mathtt{Alg}(\mathcal{O})$ be the forgetful functor defined by $c_A(P \to A) = P$.

5.4.2. **Definition.** The derived category of virtual $(\mathcal{O}, A)$-modules is the fiber of $DU/\mathtt{Alg}(\mathcal{O})$ over c_A.

In other words, an object of $D(\mathcal{O}, A)$ consists of a collection $X_a \in D(\mathcal{O}, P)$ for each map $a : P \to A$ with cofibrant P, endowed with compatible isomorphisms $\phi_f : X_a \to f_*(X_b)$ defined for any presentation of a as a composition

$$P \xrightarrow{f} Q \xrightarrow{b} A.$$

5.4.3. Any object $\alpha : P \to A$ in $\mathfrak{C}/A$ defines an obvious functor $q_\alpha : D(\mathcal{O}, A) \to DU(\mathcal{O}, P)$. Also the functor $v_* : DU(\mathcal{O}, A) \to D(\mathcal{O}, A)$ is defined so that $q_\alpha \circ v_*$ and α_* are naturally isomorphic.

Proposition. *Let $\alpha : P \to A$ be a weak equivalence in $\mathfrak{C}/A$. Then the functor*

$$q_\alpha : D(\mathcal{O}, A) \to DU(\mathcal{O}, P)$$

is an equivalence. In part, the derived category of virtual A-modules "is just" the derived category of modules over the derived enveloping algebra $\mathbf{L}U(\mathcal{O}, A)$.

Proof. We will omit the operad $\mathcal{O}$ from the notations. Let us construct a quasi-inverse functor $q^\alpha : DU(P) \to D(A)$. For any $\beta : Q \to A$ in $\mathfrak{C}/A$ choose a map $f_\beta : Q \to P$. This map is unique up to a homotopy $F : Q \to P^I$ for an appropriate path diagram

$$P \xrightarrow{i} P^I \rightrightarrows P.$$

Moreover, the homotopy $F : Q \to P^I$ is itself unique up to a homotopy as in 5.3.4. Now, for any $X \in DU(P)$ put $X(\beta) = (f_\beta)_*(X)$. Lemma 5.3.4(2) implies that the collection of objects $\{X_\beta\}$ can be uniquely completed to an object of $D(A)$. $\qquad\square$

Proposition 5.4.3 implies that the functor v_* admits a left adjoint functor $v^* : D(A) \to DU(A)$.

5.4.4. Let now $f : A \to B$ be a morphism in $\mathtt{Alg}(\mathcal{O})$. The functor $f_* : D(B) \to D(A)$ is induced by the obvious functor $\mathfrak{C}/A \to \mathfrak{C}/B$.

Proposition 5.4.3 implies that f_* admits a left adjoint functor $f^* : D(A) \to D(B)$. Of course, it can be defined as

$$f^* = q^\beta \circ \mathbf{R}g^* \circ q_\alpha$$

where $\alpha : P \to A$ and $\beta : Q \to B$ are cofibrant resolutions of A and B respectively and a map $g : P \to Q$ satisfies the condition $\beta \circ g = f \circ \alpha$.

5.5. **Varying the operad.** Let now $\alpha : \mathcal{O} \to \mathcal{O}'$ be a quasi-isomorphism of operads compatible with Σ-splittings.

5.5.1. **Theorem.** *There is an isomorphism of functors*

$$\mathbf{L}U \longrightarrow \mathbf{L}U \circ \mathbf{L}\alpha^*.$$

Proof. It suffices to prove that if A is a cofibrant $\mathcal{O}$-algebra, the composition

$$U(\mathcal{O}, A) \to U(\mathcal{O}, \alpha_* \alpha^* A) \to U(\mathcal{O}', \alpha^* A)$$

is a quasi-isomorphism.

The proof is similiar to that of 5.3.1. The claim immediately reduces to the case when A is standart cofibrant and finitely generated. Choose a free homogeneous base $\{x_i\}_{i \in I}$ for A; choose a total order on I so that dx_i belongs to the subalgebra generated by the elements $\{x_j\}_{j<i}$. Let M be the set of multi-indices ordered as in 4.7.

This defines filtrations on $U(\mathcal{O}, A)$ and on $U(\mathcal{O}', \alpha^* A)$ as in (12).

The associated graded complexes take form

$$\mathcal{O}(|d| + 1) \otimes_{\Sigma_d} k \text{ and } \mathcal{O}'(|d| + 1) \otimes_{\Sigma_d} k;$$

they are quasi-isomorphic since α is quasi-isomorphism preserving Σ-splittings. Theorem is proven. $\qquad\square$

Putting together Corollary 5.3.3, Theorem 3.3.1 and Theorem 5.5.1 we get immediately the following

5.5.2. **Theorem.** *Let $f = (\alpha, \phi) : (\mathcal{O}, A) \to (\mathcal{O}', A')$ be a weak equivalence of operad algebras.*

Then the pair of derived functors

$$\mathbf{L}f^* : D(\mathcal{O}, A) \; \rightleftarrows \; D(\mathcal{O}', A') : \mathbf{R}f_*$$

provides an equivalence of the derived categories.

6. Category of operads

6.1. **Introduction.** The category $\mathtt{Op}(k)$ of operads has itself "algebraic" nature: an operad is a collection of complexes endowed with a collection of operations satisfying a collection of identities. This is why one can mimic the construction of Section 2 to define a CMC structure on $\mathtt{Op}(k)$.

The aim of this Section is to prove the following

6.1.1. **Theorem.**   *The category of operads* $\mathtt{Op}(k)$ *in* $C(k)$ *admits a structure of closed module category in which*

— $\alpha : \mathcal{O} \to \mathcal{O}'$ *is a weak equivalence if for all* n α_n *is a quasi-isomorphism.*

— $\alpha : \mathcal{O} \to \mathcal{O}'$ *is a fibration if it is componentwise surjective.*

The scheme of the proof is very close to the proof of Theorem 2.2.1. In particular, a description of cofibrations in $\mathtt{Op}(k)$ similar to that of Remark 2.2.5 will be given.

The proof of Theorem 6.1.1 is given in 6.2– 6.6. In 6.7 we check that the standard Lie and commutative operads $\mathcal{S}$, $\mathcal{S}_C$ from [HS] are cofibrant operads in the sense of Theorem 6.1.1. Finally, in 6.8 we prove that the derived category $D(\mathcal{O}, A)$ can be "calculated" using a cofibrant resolution of $\mathcal{O}$ if A is a flat k-complex.

6.2. **Free operads.** The definitions below are close to [GK], 1.1,2.1.

Let $\mathrm{Col}(k)$ be the category of *collections* of complexes in $C(k)$ numbered by nonnegative integers. As a category, this is a direct product of $\mathbb{N}$ copies of $C(k)$. The obvious forgetful functor $\# : \mathtt{Op}(k) \longrightarrow \mathrm{Col}(k)$ admits a left adjoint *free operad* functor $F : \mathrm{Col}(k) \longrightarrow \mathtt{Op}(k)$ which can be described explicitly using the language of trees.

6.2.1. **Definition.**   (cf. [HS], 4.1.3) A tree is a finite directed graph with one initial (=having no ingoing edges) vertex, such that any non-initial vertex has exactly one ingoing edge.

Terminal vertices of a tree are those having no outgoing edges; internal vertices are those that are not terminal.

Notations. For a tree T the set of its terminal (resp., internal) vertices is denoted by $\mathrm{ter}(T)$ (resp., $\mathrm{int}(T)$); $t(T)$ (resp., $i(T)$) is the number of terminal (resp., internal) vertices of T. For any $v \in \mathrm{int}(T)$ the set of its outgoing vertices is denoted by $\mathrm{out}(v)$ and their number is $o(v)$.

We choose once and forever a set of representatives of isomorphism classes of trees; only these representatives will be called trees.

For instance, for each n we have a unique tree having one internal (=initial) vertex and n terminal vertices. This is called n-corolla.

6.2.2. **Definition.**   A n-tree consists of a pair (T, e) where T is a tree and $e : \langle n \rangle \longrightarrow \mathrm{ter}(T)$ is an injective map.

Denote by $\mathrm{irr}(T)$ the set of terminal vertices of T which do not belong to the image of e.

Denote by $\mathcal{T}(n)$ the set of n-trees.

The group Σ_n acts on $\mathcal{T}(n)$ on the right by the rule

$$(T, e)\sigma = (T, e\sigma).$$

The collection $\mathcal{T} = \{\mathcal{T}(n)\}$ admits a structure of operad in the category **Ens**. In fact, if $(T_0, e_0) \in \mathcal{T}(n), (T_i, e_i) \in \mathcal{T}(m_i)$ then the composition T of the trees is defined by identifying the root of T_i with the terminal vertex $e(i)$ of T. The set $\mathrm{ter}(T)$ contains the disjoint union $\cup \mathrm{ter}(T_i)$ and

the injective map $e : \Sigma_m \longrightarrow \text{ter}(T)$ for $m = \sum m_i$ is given by the formula

$$e(m_1 + \ldots + m_{i-1} + j) = e_i(j) \in \text{ter}(T_i) \subseteq \text{ter}(T).$$

6.2.3.   Here is the explicit construction of the free operad functor. Let $V = \{V_i\} \in \text{Col}(k)$. For any $T \in \mathcal{T}(n)$ define a complex V_T by the formula

$$V_T = \bigotimes_{v \in \text{int}(T)} V(o(v)) \otimes V(0)^{\otimes \text{ter}(T) - n}.$$

This should be interpreted as follows: each internal vertex v of T we mark with an element of $V(o(v))$; each non-numbered terminal vertex of T we mark with an element of $V(0)$.

Note that for any $T \in \mathcal{T}(n)$ and $\sigma \in \Sigma_n$ the complexes V_T and $V_{T\sigma}$ are tautologically isomorphic.

The free operad $F(V)$ generated by the collection V is thus defined by the formula

$$F(V)(n) = \bigoplus_{T \in \mathcal{T}(n)} V_T. \tag{15}$$

The Σ_n-action on $F(V)(n)$ is defined as follows. Let $x \in F_T, \sigma \in \Sigma_n$. Then $x\sigma$ is "the same element as x but in $V_{T\sigma}$".

The operad multiplication is defined obviously by the multiplication in the **Ens**-operad $\mathcal{T}$.

The map $V \longrightarrow \#F(V)$ of collections carries each $V(n)$ to the direct summand V_T of $F(V)(n)$ corresponding to the n-corolla endowed with a(ny) bijective map $e : \langle n \rangle \to \text{ter}(T)$.

6.3. Ideals; limits and colimits.
Let $\mathcal{O} \in \text{Op}(k)$. An ideal $\mathcal{I}$ in $\mathcal{O}$ is a collection of Σ_n-invariant subcomplexes $\{\mathcal{I}(n) \subseteq \mathcal{O}(n)\}$ which is stable under the composition in an obvious way (if one of the factors belongs to $\mathcal{I}$ then the result belongs to $\mathcal{I}$). A kernel of a map of operads is always an ideal; if $\mathcal{I} \subseteq \mathcal{O}$ is an ideal then the quotient operad $\mathcal{O}/\mathcal{I}$ is correctly defined. If $X \subseteq \mathcal{O}^{\#}$ is a subcollection, the ideal (X) is defined as the smallest ideal containing X.

Limits in the category $\text{Op}(k)$ exist and commute with the forgetful functor $\# : \text{Op}(k) \to \text{Col}(k)$. Colimits can be constructed using the free operad construction: if $\alpha : I \to \text{Op}(k)$ is a functor, its colimit is the quotient of the free operad generated by the collection $\varinjlim \# \circ \alpha$ by an appropriately defined ideal. Note that filtered colimits commute with $\#$.

6.4. Adding a variable to kill a cycle.
Let $\mathcal{O}$ be an operad in $C(k)$, $M \in C(k)$ and let $\alpha : M \to \mathcal{O}(n)$ be a map of complexes. The operad $\mathcal{O}\langle M, n, \alpha \rangle$ is defined as in 2.2.2. If $M = k[d]$ and $\alpha : M \to \mathcal{O}(n)$ takes the generator of M to a cycle $a \in \mathcal{O}(n)^d$, the resulting operad is obtained by "adding a variable to kill the cycle a". It is denoted by $\mathcal{O}\langle T; dT = a \rangle$.

One can immediately see that a map $\mathcal{O} \to \mathcal{O}\langle T; dT = a \rangle$ satisfies the LLP with respect to any surjective quasi-isomorphism of operad. Also, if the complex $M \in C(k)$ takes form $0 \to k = k \to 0$, any map

$$\mathcal{O} \to \mathcal{O}\langle M, n, \alpha \rangle$$

satisfies the LLP with respect to any surjective map of operads.

Similarly to 2.2.3 one defines standard cofibrations and standard acyclic cofibrations as appropriate direct limits of the maps described.

6.5. **Extension of a homotopy to the free operad.** Here we repeat the construction of 4.3. In our case the construction will be even easier since the operads are similar to associative algebras and not to general operad algebras.

Let $\alpha : V \to V$ be an endomorphism of a collection V and $h : V \to V[-1]$ be a homotopy: $dh = \mathrm{id}_V - \alpha$. We wish to construct a homotopy $H : F(V) \to F(V)[-1]$ between $\mathrm{id}_{F(V)}$ and $F(\alpha)$.

For this we fix a total order on the set of terminal vertices of each corolla. This gives a lexicographic order on the set of all vertices of any tree. The restriction of H on V_T is defined as

$$H = \sum_{v \in \mathrm{int}(T) \cup \mathrm{irr}(T)} H_v$$

where

$$H_v = \bigotimes_{w \in \mathrm{int}(T) \cup \mathrm{irr}(T)} \theta_w^v$$

with

$$\theta_w^v = \begin{cases} \alpha \text{ if } w < v \\ h \text{ if } w = v \\ \mathrm{id} \text{ if } w > v \end{cases} \tag{16}$$

One immediately checks that $dH = \mathrm{id}_{F(V)} - F(\alpha)$.

6.5.1. **Lemma.** *Let an ideal $\mathcal{I}$ in the algebra $F(V)$ be generated by a set of elements $\{x_i\}$. Then, if $H(x_i) \in \mathcal{I}$ for all i, the ideal $\mathcal{I}$ is H-invariant.*

Proof. Straightforward calculation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

6.6. **Proof of Theorem 6.1.1.** The proof is close to that of Theorem 2.2.1 and Theorem 4.1.1.

Since $\mathrm{Op}(k)$ admits arbitrary limits and colimits, and the forgetful functor $\# : \mathrm{Op}(k) \to \mathrm{Col}(k)$ commutes with filtered colimits, we have only to check that for any operad $\mathcal{O}$ and contractible collection X the natural map $\mathcal{O}^\# \to (\mathcal{O} \coprod F(X))^\#$ is homotopy equivalence.

We proceed as in the proof of Theorem 4.1.1. Put $V = \mathcal{O}^\# \oplus X \in \mathrm{Col}(k)$. If I is the kernel of the natural projection $F(\mathcal{O}^\#) \longrightarrow \mathcal{O}$ and J is the ideal in $F(V)$ generated by I, then $\mathcal{O} \coprod F(X)$ is isomorphic to $F(V)/J$.

Let $\alpha : V \to V$ be the composition $V = \mathcal{O}^\# \oplus X \to \mathcal{O}^\# \to V$ and let $h : V \to V[-1]$ be the homotopy, $dh = \mathrm{id}_V - \alpha$, vanishing on $\mathcal{O}^\#$. According 6.5 a homotopy $H : F(V) \to F(V)[-1]$ is defined and by Lemma 6.5.1 the ideal J is H-invariant. Then H induces a homotopy on $F(V)/J = \mathcal{O} \coprod F(X)$ and this proves the theorem.

6.7. **Standard examples.** Suppose that k contains $\mathbb{Q}$. Recall that dg Lie algebras (resp., commutative dg agebras) are precisely algebras over an appropriate operad $\mathtt{Lie}$ (resp., over $\mathtt{Com}$). Their strong homotopy counterparts are correspondingly the algebras over the "standard" operads $\mathcal{S}$ and $\mathcal{S}_C$ see [HS], 4.1 and 4.4.

Let us show that the standard operads, $\mathcal{S}$ (the standard Lie operad) and $\mathcal{S}_C$ (the standard commutative operad) are cofibrant in $\mathtt{Op}(k)$.

These operads are constructed by consecutive "attaching a variable to kill a cycle" — as it is explained in *loc. cit.*, 4.1.1., which differs a little from our construction. However, if $k \supseteq \mathbb{Q}$, this operation also gives rise to a cofibration as shows the Lemma 6.7.1 below.

Let $\mathcal{O}$ be an operad and let $z \in \mathcal{O}(n)$ be a cycle. Let $G \subseteq \Sigma_n$ and a character $\chi : G \to k^*$ satisfy the condition $zg = \chi(g)z$ for all $g \in G$. Then the "attaching of a variable" is defined (well, is not defined) in *loc. cit.* to be the map

$$\mathcal{O} \longrightarrow \mathcal{O}' = \mathcal{O}\langle e; de = x \rangle / (eg - \chi(g)e).$$

6.7.1. **Lemma.** *In the notations above the map $\mathcal{O} \to \mathcal{O}'$ is a cofibration.*

Proof. The projection $\mathcal{O}\langle e; de = x \rangle \longrightarrow \mathcal{O}'$ is split by the map $\mathcal{O}' \to \mathcal{O}\langle e; de = x \rangle$ which sends the element e to $\frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) eg$.

Thus, the map $\mathcal{O} \to \mathcal{O}'$ is a retract of a standard cofibration and the Lemma is proven. $\square$

6.8. **More on the derived category.** In 5.4.3 we saw that the derived category $D(\mathcal{O}, A)$ of virtual A-modules can be calculated using a cofibrant resolution of A in $\mathtt{Alg}(\mathcal{O})$. Now we will show that one can take a cofibrant resolution of the operad $\mathcal{O}$ instead.

6.8.1. **Lemma.** *Let $\mathcal{O}$ be an operad over k, $\alpha : M \to \mathcal{O}(n)$ be a map of complexes. Put $\mathcal{O}' = \mathcal{O}\langle M, n, \alpha \rangle$. Let A be a $\mathcal{O}'$-algebra and let $U = U(\mathcal{O}, A)$, $U' = U(\mathcal{O}', A)$. Then one has a natural isomorphism*

$$U' \longrightarrow U\langle M \otimes A^{\otimes n-1}, \tilde{\alpha} \rangle$$

where $\tilde{\alpha}$ is the composition

$$M \otimes A^{\otimes n-1} \xrightarrow{\alpha \otimes \mathrm{id}} \mathcal{O}(n) \otimes A^{\otimes n-1} \to U(\mathcal{O}, A).$$

Proof. An $\mathcal{O}'$-algebra structure on a $\mathcal{O}$-algebra A is given by a map $f : M \otimes A^{\otimes n} \to M$ such that $d(f)$ is equal to the composition

$$M \otimes A^{\otimes n} \xrightarrow{\alpha \otimes \mathrm{id}} \mathcal{O}(n) \otimes A^{\otimes n} \to A.$$

A structure of $(\mathcal{O}', A)$-module on a $(\mathcal{O}, A)$-module X is given by a map $m : M \otimes A^{\otimes n-1} \otimes X \to X$ such that $d(m)$ is equal to the composition

$$M \otimes A^{\otimes n-1} \otimes X \xrightarrow{\alpha \otimes \mathrm{id} \otimes \mathrm{id}} \mathcal{O}(n) \otimes A^{\otimes n-1} \otimes X \to X.$$

This proves the claim. $\square$

The following Lemma 6.8.2 will be used in 6.8.3.

6.8.2. **Lemma.** *Let $A, B \in \mathtt{DGA}(k)$, $M, N \in C(k)$ and let a commutative diagram*

$$
\begin{array}{ccc}
M & \xrightarrow{\ \alpha\ } & A \\
\Big\downarrow g & & \Big\downarrow f \\
N & \xrightarrow{\ \beta\ } & B
\end{array}
$$

be given so that $f : A \to B$ is a weak equivalence and $g : M \to N$ is a quasi-isomorphism. If A, B, M, N are flat k-complexes then the induced map

$$A\langle M, \alpha\rangle \to B\langle N, \beta\rangle$$

is a weak equivalence. Moreover, the algebras $A\langle M, \alpha\rangle, B\langle N, \beta\rangle$ are also flat over k.

Proof. The associative algebra $A\langle M, \alpha\rangle$ admits a natural filtration $\{F_n\}$ defined by

$$F_n = \sum_{k=0}^{n} (A \otimes M[1])^{\otimes k} \otimes A.$$

The associated graded pieces are

$$\mathrm{gr}_n = (A \otimes M[1])^{\otimes n} \otimes A.$$

Now it is clear that the map in question induces isomorphism of the associated graded pieces and therefore is itself a quasi-isomorphism. The complex F_n can be obtained as the cone of a map $\mathrm{gr}_n \to F_{n-1}$ induced by α.

Since flatness is closed under taking cones and filtered colimits (see Lemma 3.4.1), the new algebras $A\langle M, \alpha\rangle, B\langle N, \beta\rangle$ are flat k-complexes. $\qquad\square$

6.8.3. **Corollary.** *Let $\mathcal{O}$ be a cofibrant operad and let $f : A \to A'$ be a quasi-isomorphism of $\mathcal{O}$-algebras. If A and A' are flat as complexes over k then the natural map $U(f)$ is quasi-isomorphism.*

Proof. One can suppose $\mathcal{O}$ to be standard cofibrant. Then the claim follows immediately from 6.8.1 and 6.8.2. $\qquad\square$

6.8.4. **Proposition.** *Let A be an algebra over a cofibrant operad $\mathcal{O} \in \mathtt{Op}(k)$. Then, if A is flat as a k-complex, there exists a natural equivalence*

$$D(\mathcal{O}, A) \to DU(\mathcal{O}, A).$$

Proof. The claim will immediately follow from 6.8.3 once we check that cofibrant algebras over cofibrant operads are flat.

This immediately reduces to the case of a finitely generated standard cofibrant algebra. This one admits a filtration $\{\mathcal{F}_n\}$ as in 4.7.3. Thus everything is reduced to checking that for any cofibrant operad $\mathcal{O}$ the complex $\mathcal{O}(n) \otimes_{\Sigma_n} k$ is flat. This follows from the tree description of a free operad — see 6.2.3.

$\qquad\square$

6.8.5. **Example.** (see the notations of 6.7) Let A be a flat dg Lie algebra over $k \supseteq \mathbb{Q}$. Then the derived category $DU(\mathtt{Lie}, A)$ of A-modules is equivalent to the derived category $DU(\mathcal{S}, A)$ of modules over A considered as a strong homotopy Lie algebra. In fact, the category $DU(\mathcal{S}, A)$ is equivalent to $D(\mathcal{S}, A)$ by Proposition 6.8.4 and 6.7. The category $DU(\mathtt{Lie}, A)$ is equivalent to $D(\mathtt{Lie}, A)$ by the PBW theorem for dg Lie algebras — the latter implies that enveloping algebras of quasi-isomorphic flat Lie algebras are quasi-isomorphic. Finally, the categories $D(\mathcal{S}, A)$ and $D(\mathtt{Lie}, A)$ are naturally equivalent by Theorem 5.5.2.

7. Cotangent complex; cohomology of operad algeras.

7.1. **Introduction.** Let $\mathcal{O}$ be a Σ-split operad in $C(k)$. In this Section we construct for an algebra map $B \to A$ in $\mathtt{Alg}(\mathcal{O})$ its cotangent complex $L_{A/B} \in D(\mathcal{O}, A)$ belonging to the derived category of virtual A-modules.

In the next Section we define the tangent complex T_A which is, as usual, dual to the cotangent complex. The tangent complex admits a unique (in the homotopy category) structure of dg Lie algebra. This Lie algebra must play a crucial role in the deformation theory of operad algebras.

7.2. **Derivations.**

7.2.1. **Definition.** Let $\mathcal{O}$ be an operad in $C(k)$, $\alpha : B \to A$ be a map in $\mathtt{Alg}(\mathcal{O})$ and let M be a A-module. A map $f : A \to M$ of complexes (not necessarily commuting with the differentials) is called $\mathcal{O}$-*derivation over* B if it vanishes on the image of B in A and for each $n > 0$ the following diagram

$$
\begin{array}{ccc}
\mathcal{O}(n) \otimes A^{\otimes n} & \xrightarrow{\hspace{3cm}} & A \\
\Big\downarrow{\scriptstyle \sum\limits_{a+b=n-1} 1 \otimes 1^a \otimes f \otimes 1^b} & & \Big\downarrow{f} \\
\sum\limits_{a+b=n-1} \mathcal{O}(n) \otimes A^{\otimes a} \otimes M \otimes A^{\otimes b} & \xrightarrow{\hspace{2cm}} & M
\end{array}
$$

is commutative.

7.2.2. The $\mathcal{O}$-derivations from A to M over B form a complex $\mathrm{Der}_B^{\mathcal{O}}(A, M)$. When $B = F(0) = \mathcal{O}(0)$ is the initial object in $\mathtt{Alg}(\mathcal{O})$ we will omit the subscript B from the notation. We will also omit the superscript $\mathcal{O}$ when it does not make a confusion.

The complex of derivations $\mathrm{Der}_B^{\mathcal{O}}(A, M)$ is a subcomplex of $\mathcal{H}om_k(A, M)$. This defines a functor
$$
\mathrm{Der}_B^{\mathcal{O}}(A, \_) : \mathtt{Mod}(A) \to C(k)
$$
which is representable in the following sense.

Proposition. *There exists a (unique up to a unique isomorphism) A-module $\Omega_{A/B}$ (called the module of relative differentials) together with a derivation $\partial : A \to \Omega_{A/B}$ inducing the natural isomorphism of complexes*
$$
\mathcal{H}om_A(\Omega_{A/B}, M) \xrightarrow{\sim} \mathrm{Der}_B^{\mathcal{O}}(A, M).
$$

Proof. 1. Consider firstly the absolute case $B = F(0)$. The functor $M \mapsto \mathcal{H}om_k(A, M)$ is obviously represented by the free A-module $U(\mathcal{O}, A) \otimes A$. Thus, the functor $\mathrm{Der}(A, \_)$ is represented by the quotient of $U(\mathcal{O}, A) \otimes A$ modulo the relations which guarantee the commutativity of the diagrams

$$
\begin{array}{ccc}
\mathcal{O}(n) \otimes A^{\otimes n} & \longrightarrow & A \\
\downarrow & & \downarrow f \\
\displaystyle\sum_{a+b=n-1} \mathcal{O}(n) \otimes A^{\otimes a} \otimes (U(\mathcal{O}, A) \otimes A) \otimes A^{\otimes b} & \longrightarrow & U(\mathcal{O}, A) \otimes A
\end{array}
$$

with left vertical arrow $\displaystyle\sum_{a+b=n-1} 1 \otimes 1^a \otimes f \otimes 1^b$.

Here $f : A \to U(\mathcal{O}, A) \otimes A$ is given by $f(a) = 1 \otimes a$.

2. In general one sees immediately that the complex $\Omega_A / \partial \circ \alpha(B)$ represents the functor $\mathrm{Der}_B^{\mathcal{O}}(A, \_)$. $\qquad\square$

7.2.3. Let $C \xrightarrow{\alpha} B \xrightarrow{f} A$ be a pair of maps in $\mathtt{Alg}(\mathcal{O})$. Any derivation $d : A \to M$ over C defines a derivation $d \circ f : B \to f_*(M)$ over C. This defines a canonical map $\Omega^f : f^* \Omega_{B/C} \to \Omega_{A/C}$.

On the other hand, any derivation $d : A \to M$ over B can be considered as a derivation over C. This defines a canonical map $\Omega_\alpha : \Omega_{A/C} \to \Omega_{A/B}$.

Proposition 7.3.7 below claiming that sometimes Ω^f and Ω_α are quasi-isomorphisms, allows one to define correctly the cotangent complex $L_{A/B} \in D(\mathcal{O}, A)$ as a (sort of) left derived functor of the functor $\Omega_{A/B}$ — see 7.4.3.

7.2.4. More generally, given a map $\alpha : \mathcal{O} \to \mathcal{O}'$ of operads and a couple of maps $f : B \to A \in$ Mor $\mathtt{Alg}(\mathcal{O})$, $f' : B' \to A' \in$ Mor $\mathtt{Alg}(\mathcal{O}')$, one defines $u : f \to f'$ over α to be the pair of maps $\phi : \alpha^* B \to B', \psi : \alpha^* A \to A'$ making the diagram

$$
\begin{array}{ccc}
\alpha^* B & \xrightarrow{\phi} & B' \\
\downarrow f & & \downarrow f' \\
\alpha^* A & \xrightarrow{\psi} & A'
\end{array}
$$

commutative.

For any A'-module M one has a canonical "restriction map"

$$
\mathrm{Der}_{B'}^{\mathcal{O}'}(A', M) \to \mathrm{Der}_B^{\mathcal{O}}(A, f_*(M))
$$

which gives a canonical map $\Omega^u : u^*(\Omega_{A/B}) \to \Omega_{A'/B'}$. Of course, Ω^u can be presented as a composition of three maps — the map $\Omega^\alpha : \Omega_{A/B} \to \Omega_{\alpha^* A/\alpha^* B}$ responsible for the change of operads ($\phi = \mathrm{id}_{\alpha^* B}, \psi = \mathrm{id}_{\alpha^* A}$); the map $\Omega^\psi : \Omega_{\alpha^* A/\alpha^* B} \to \Omega_{A'/\alpha^* B}$ and the map $\Omega_\phi : \Omega_{A'/\alpha^* B} \to \Omega_{A'/B'}$.

7.3. Module of differentials of a cofibration.

7.3.1. Let $A = F(X)$ be the free $\mathcal{O}$-algebra generated by a complex $X \in C(k)$. Then there is a natural isomorphism

$$\Omega_A = U(\mathcal{O}, A) \otimes X.$$

We wish to describe the module $\Omega_{A/B}$ when $\alpha : B \to A$ is a standard cofibration.

7.3.2. Let $\alpha : B \to A$ be a map of $\mathcal{O}$-algebras, $M \in C(k)$. Let $f : M \to A^{\#}$ be a map in $C(k)$ and let $A' = A\langle M, f \rangle$ be defined as in 2.2.2. Put $U = U(\mathcal{O}, A)$, $U' = U(\mathcal{O}, A')$. The map $\partial \circ \alpha : M \to \Omega_{A/B}$ defines $\alpha' : U' \otimes M \to U' \otimes_U \Omega_{A/B}$.

Lemma. *The A'-module $\Omega_{A'/B}$ is naturally isomorphic to the cone of α'.*

Proof. Any B-derivation from A' to a A'-module is uniquely defined by its restrictions to A and to M satisfying the obvious compatibility condition including the map f. Thus $\Omega_{A'/B}$ and $\mathrm{cone}(\alpha')$ represent the same functor. □

7.3.3. Corollary. *Let $\alpha : B \to A$ be is a cofibration. Then $\Omega_{A/B}$ is a cofibrant A-module (in the sense of 3.1).*

Proof. If α is a standard cofibration, Lemma 7.3.2 immediately implies that Ω_A is semi-free, i.e., standard cofibrant in $\mathrm{Mod}(A)$.

In the general case, let $i : A \rightleftarrows C : p$ represent α as a retract of a standard cofibration $i \circ \alpha : B \to C$. Then the maps

$$\Omega_{A/B} = p^* i^* \Omega_{A/B} \to p^* \Omega_{C/B} \to \Omega_{A/B}$$

define $\Omega_{A/B}$ as a retract of $p^* \Omega_{C/B}$. Since the inverse image functor preserves cofibrations, the claim follows. □

7.3.4. Proposition. *Let $C \xrightarrow{\alpha} B \xrightarrow{f} A$ be a pair of maps in $\mathrm{Alg}(\mathcal{O})$ so that f is a cofibration. Then the sequence*

$$0 \longrightarrow f^* \Omega_{B/C} \xrightarrow{\Omega^f} \Omega_{A/C} \xrightarrow{\Omega_\alpha} \Omega_{A/B} \longrightarrow 0 \tag{17}$$

is exact.

Proof. If f is a standard cofibration one proves the claim by induction using Lemma 7.3.2.

To prove the general case, let $A' \xrightarrow{q} A \xrightarrow{j} A'$ satisfy $q \circ j = \mathrm{id}_A$ so that $f' = j \circ f$ is a standard cofibration. If $\mathfrak{s}$ denotes the sequence (17) and $\mathfrak{s}'$ the same sequence constructed for f' instead of f, one immediately obtains that $j^*(\mathfrak{s})$ is a retract of $\mathfrak{s}'$ and is, therefore, exact. Since the A-module $\Omega_{A/B}$ is cofibrant, $j^*(\mathfrak{s})$ splits and so it remains exact after application of q^*. Therefore, $\mathfrak{s} = q^* j^*(\mathfrak{s})$ is exact. □

7.3.5. Corollary. Let $C \xrightarrow{\alpha} B \xrightarrow{f} A$ be a pair of maps in $\mathtt{Alg}(\mathcal{O})$ where f is an acyclic cofibration. Then the natural map Ω^f is a quasi-isomorphism.

Proof. If f is a standard acyclic cofibration, the first claim follows immediately from Lemma 7.3.2. If f is any acyclic cofibration, it is a retract of a standard acyclic cofibration and the claim follows from the axiom (CM 3). $\qed$

7.3.6. Proposition. Let $C \xrightarrow{\alpha} B \xrightarrow{f} A$ be a pair of maps in $\mathtt{Alg}(\mathcal{O})$. If f is a quasi-isomorphism and $\alpha, f \circ \alpha$ are cofibrations then the map Ω^f is a weak equivalence.

Proof. Since α and $f \circ \alpha$ are cofibrations, there exists a map $g : A \to B$ over C homotopy inverse to f. Let $A \xrightarrow{i} A^I \overset{p_0,p_1}{\Longrightarrow} A$ be a path diagram for A such that i is an acyclic cofibration and the homotopy between id_A and $f \circ g$ is given by a map $h : A \to A^I$.

Since i is an acyclic cofibration the map $i^*\Omega_{A/C} \to \Omega_{A^I/C}$ is a quasi-isomorphism of cofibrant A^I-modules; thus for $i = 0,1$ the maps $p_i^*\Omega_{A^I} \to \Omega_A$ are quasi-isomorphisms as well. The map $\Omega^h : h^*\Omega_{A/C} \to \Omega_{A^I/C}$ becomes therefore quasi-isomorphism of cofibrant modules after application of p_1^*. Since p_1 is a weak equivalence of cofibrant algebras, Corollary 5.3.3 implies that Ω^h itself is a quasi-isomorphism. Then $p_2^*(\Omega_h)$ is also quasi-isomorphism. Thus, the map $\Omega^{fg} : (fg)^*\Omega_{A/C} \to \Omega_{A/C}$ is a quasi-isomorphism.

In the same way, the map $\Omega^{gf} : (gf)^*\Omega_{B/C} \to \Omega_{B/C}$ is a quasi-isomorphism. This immediately implies that Ω^f is isomorphism. $\qed$

7.3.7. Proposition. Let

$$
\begin{array}{ccc}
B & \xrightarrow{\alpha} & B' \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
A & \xrightarrow{\beta} & A'
\end{array}
$$

be a commutative diagram in $\mathtt{Alg}(\mathcal{O})$ so that α, β are weak equivalences, f, f' are cofibrations and B, B' are cofibrant. Then the composition

$$\beta^*\Omega_{A/B} \xrightarrow{\Omega^\beta} \Omega_{A'/B} \xrightarrow{\Omega_\alpha} \Omega_{A'/B'}$$

is a weak equivalence.

Proof. Applying Proposition 7.3.4 thrice we get the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & \beta^*f^*\Omega_B & \to & \beta^*\Omega_A & \to & \beta^*\Omega_{A/B} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \beta^*f^*\Omega_B & \to & \Omega_{A'} & \to & \Omega_{A'/B} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & g^*\Omega_{B'} & \to & \Omega_{A'} & \to & \Omega_{A'/B'} & \to & 0
\end{array}
$$

Its rows are split exact sequences and the first two columns are quasi-isomorphisms (or even isomorphisms). Therefore the last column we need consists of quasi-isomorphisms as well. □

Lemma 7.3.2 implies easily the following

7.3.8. Corollary.   *Let* $\alpha : \mathcal{O} \to \mathcal{O}'$ *be a map of operads,* $f \in \mathrm{Mor}\, \mathtt{Alg}(\mathcal{O})$, $f' \in \mathrm{Mor}\, \mathtt{Alg}(\mathcal{O}')$, $u : f \to f'$ *be as in 7.2.4. If* α, ϕ, ψ *are weak equivalences,* B, B' *are cofibrant and* f, f' *are cofibrations then* Ω^u *is a quasi-isomorphism.*

Proof. By 7.3.7 one can suppose that $B' = \alpha^*(B), A' = \alpha^*(A)$, so that we have to check that Ω^α is a weak equivalence. Actually we shall prove that this is isomorphism.

If $f : B \to A$ is a standard cofibration, the result immediately follows from Lemma 7.3.2 by induction. Otherwise, if f is a retract of a standard cofibration $g : B \to C$, then $\Omega_{A/B}$ is a retract of $\pi^*\Omega_{C/B}$ where $\pi : C \to A$ is the corresponding projection. This proves the claim. □

7.4. **Cotangent complex.**

7.4.1. *Model structure on* $\mathrm{MOR}(\mathcal{C})$. Let $\mathcal{C}$ be a closed model category. Then the category $\mathrm{MOR}(\mathcal{C})$ of arrows in $\mathcal{C}$ admits a closed model category as follows.

A map from $g : Q \to P$ to $f : B \to A$ given by a pair of maps $\beta : Q \to B$ and $\alpha : P \to A$ satisfying the condition $f \circ \beta = \alpha \circ g$ is:

— a weak equivalence (resp., a fibration) if both α and β are weak equivalences (resp., fibrations);

— a cofibration if β and also the natural map $B \coprod^Q P \to A$ are cofibrations.

We will use this model structure for the case $\mathcal{C} = \mathtt{Alg}(\mathcal{O})$. Note that, in particular, a map $g : Q \to P$ is cofibrant if Q is a cofibrant object and g is a cofibration. The following lemma is the result of the existence of the above defined model structure on $\mathrm{MOR}(\mathtt{Alg}(\mathcal{O}))$.

7.4.2. Lemma.   *1. For any map* $f : B \to A$ *in* $\mathtt{Alg}(\mathcal{O})$ *there exists a cofibrant resolution.*

2. For any pair $g : Q \to P$, $g' : Q' \to P'$ *of cofibrant resolutions of* f *there exists a map, unique up to homotopy, from* g' *to* g.

In the definition of cotangent complex below we use the notations of 5.4.3.

7.4.3. Definition. Let $f : B \to A$ be an algebra morphism in $\mathtt{Alg}(\mathcal{O})$. The cotangent complex of f, $L_{A/B}$, is the object of $D(\mathcal{O}, A)$ defined by the formula

$$L_{A/B} = q^\alpha(\Omega_{P/Q})$$

where $g : Q \to P$ together with $\beta : Q \to B$, $\alpha : P \to A$ define a cofibrant resolution of f.

7.4.4. Proposition.   *The cotangent complex* $L_{A/B}$ *is defined uniquely up to a unique isomorphism.*

Proof. This immediately follows from Lemma 7.4.2 and Proposition 5.4.3. □

7.4.5. *Functoriality.* Let $\alpha : \mathcal{O} \to \mathcal{O}'$ be a map of operads, $f : B \to A$ and $f' : B' \to A'$ be maps in $\mathtt{Alg}(\mathcal{O})$ and in $\mathtt{Alg}(\mathcal{O}')$ respectively, and let $u : f \to f'$ be a map over α as in 7.2.4.

If $g : Q \to S$ is a cofibrant resolution of f, and $g' : Q \to P'$ is a cofibrant resolution for f', there is, according to 7.4.1, a map $v : \alpha^*(g) \to g'$ aking the corresponding diagram commutative; moreover, this map is unique up to homotopy. This defines a map

$$L^u : \mathbf{L}u^*(L_{A/B}) \to L_{A'/B'}.$$

Corollary 7.3.8 immediately gives the following

Proposition. Let $\alpha : \mathcal{O} \to \mathcal{O}'$ be a map of operads, $f \in \mathrm{Mor}\,\mathtt{Alg}(\mathcal{O})$, $f' \in \mathrm{Mor}\,\mathtt{Alg}(\mathcal{O}')$, $u : f \to f'$ be as in 7.2.4. If α, ϕ, ψ are weak equivalences then $L^u : \mathbf{L}u^*(L_{A/B}) \to L_{A'/B'}$ is an isomorphism in $D(\mathcal{O}', A')$.

7.5. **Cohomology.** Let A be a $\mathcal{O}$-algebra, $M \in D(\mathcal{O}, A)$ be a virtual A-module. The (absolute) cohomology of A with coefficients in M are defined to be

$$H(A, M) = \mathbf{R}\,\mathrm{Hom}_A(L_A, M) \in D(k).$$

Proposition 7.4.5 immediately implies the following comparison theorem

7.5.1. **Theorem.** Let $f = (\alpha, \phi) : (\mathcal{O}', A') \to (\mathcal{O}, A)$ be a weak equivalence of operad algebras. Let $M \in \mathtt{Mod}(\mathcal{O}, A)$, $M \in \mathtt{Mod}(\mathcal{O}', A')$ and let $g : M' \to f_*(M)$ be a quasi-isomorphism of A'-modules. Then the induced map

$$H(A', M') \to H(A, M)$$

is an isomorphism in $D(k)$.

8. Tangent Lie algebra

Let $\mathcal{O}$ be a Σ-split operad. For a cofibrant $\mathcal{O}$-algebra A its tangent Lie algebra is defined to be $T_A = \mathrm{Der}^{\mathcal{O}}(A, A)$.

The aim of this Section is to extend this correspondence to a functor from the category $\mathtt{Hoalg}(\mathcal{O})^{\mathrm{iso}}$ of homotopy $\mathcal{O}$-algebras and isomorphisms to the category $\mathtt{Holie}(k)^{\mathrm{iso}}$ of homotopy Lie algebras and isomorphisms.

8.1. For any map $\alpha : A \to B$ let $\mathrm{Der}_\alpha(A, B)$ be the complex of derivations from A to B considered as a A-module via α. One has a pair of maps

$$T_A \xrightarrow{\alpha_*} \mathrm{Der}_\alpha(A, B) \xleftarrow{\alpha^*} T_B.$$

Lemma. Let A and B be cofibrant $\mathcal{O}$-algebras and let α be a weak equivalence. Then α^* and α_* are quasi-isomorphisms.

Proof. Recall that

$$T_A = \mathcal{H}om_A(\Omega_A, A), \ T_B = \mathcal{H}om_B(\Omega_B, B), \mathrm{Der}_\alpha(A, B) = \mathcal{H}om_A(\Omega_A, B).$$

Since A and B are cofibrant, Ω_A and Ω_B are cofibrant modules over A and B respectively. The map α_* is a weak equivalence since Ω_A is cofibrant and α is a quasi-isomorphism. The map α^* is a weak equivalence since $\Omega^\alpha : \alpha^*(\Omega_A) \to \Omega_B$ is a weak equivalence of cofibrant B-modules by Proposition 7.3.6. $\qquad\square$

8.2. Acyclic fibrations.

Let $\alpha : A \to B$ be an acyclic fibration ($=$ surjective quasi-isomorphism). Put $I = \operatorname{Ker}\alpha$. Define

$$T_\alpha = \{\delta \in T_A | \delta(I) \subseteq I\}.$$

Then T_α is a dg Lie subalgebra of T_A and a natural Lie algebra map $\pi_\alpha : T_\alpha \to T_B$ is defined. Denote by $\iota_\alpha : T_\alpha \to T_A$ the natural inclusion.

8.2.1. Proposition.

The map π_α is a surjective quasi-isomorphism while ι_α is an injective quasi-isomorphism.

Proof. Step 1. Let us check that π_α is surjective. Suppose first of all that A is standard cofibrant i.e. is obtained from nothing by a successive joining of free variables. Derivation on A is uniquely defined by its values on the free generators therefore any derivation on B can be lifted to a derivation on A and it will belong automatically to T_α. For a general cofibrant A let C be a standard cofibrant algebra so that A is a retract of C. This means that there are maps $i : A \to C$ and $p : C \to A$ so that $p \circ i = \operatorname{id}_A$. Put $J = \operatorname{Ker}(\alpha \circ p)$. Then if $\delta \in T_B$ and if $\tilde\delta \in T_C$ lifts δ then the composition $p \circ \delta \circ i$ is a derivation of A which lifts δ.

Step 2. One has

$$\operatorname{Ker}\pi_\alpha = \{\delta \in T_A | \delta(A) \subseteq I\} = \operatorname{Der}(A, I).$$

Since I is contractible, $\operatorname{Ker}\pi_\alpha$ is contractible.

Taking into account Steps 1 and 2 we deduce that π_α is a surjective quasi-isomorphism.

Step 3. The diagram

$$
\begin{array}{ccc}
T_A & \xrightarrow{\alpha_*} & \operatorname{Der}_\alpha(A, B) \\
\uparrow{\scriptstyle\iota_\alpha} & & \uparrow{\scriptstyle\alpha^*} \\
T_\alpha & \xrightarrow{\pi_\alpha} & T_B
\end{array}
$$

is commutative. Since the maps $\alpha^*, \alpha_*, \pi_\alpha$ are quasi-isomorphism, ι_α is also quasi-isomorphism. Proposition is proven. $\qquad\square$

Proposition 8.2.1 allows one to define the map $T(\alpha) : T_A \to T_B$ in the homotopy category $\texttt{Holie}(k)$ as

$$T(\alpha) = \pi_\alpha \circ \iota_\alpha^{-1}.$$

8.2.2. Lemma. *Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be a pair of acyclic fibrations. Then one has*

$$T(\beta \circ \alpha) = T(\beta) \circ T(\alpha)$$

in Holie(k).

Proof. Put $T = T_\alpha \times_{T_B} T_\beta$. The map $T \to T_\beta$ is a surjective quasi-isomorphism since it is obtained by a base change from π_α. The Lie algebra T identifies with

$$\{\delta \in T_A | \delta(\mathrm{Ker}\,\alpha) \subseteq \mathrm{Ker}\,\alpha \text{ and } \delta(\mathrm{Ker}(\beta \circ \alpha)) \subseteq \mathrm{Ker}(\beta \circ \alpha)\}.$$

Therefore, T is a subalgebra of $T_{\beta \circ \alpha}$ and all the maps involved are quasi-isomorphisms. This proves the lemma. □

Note that the existence of morphism $T(\alpha)$ for any acyclic fibration α already implies that weakly equivalent algebras have weakly equivalent tangent Lie algebras.

8.3. Standard acyclic cofibrations. Let $\alpha : A \to B$ be a standard acyclic cofibration. This means that α is isomorphic to a canonical injection $A \to A \coprod F(M)$ where $M \in C(k)$ is a contractible complex of free k-modules and $F(M)$ is the corresponding free algebra. Put

$$T_\alpha = \{\delta \in T_B | \delta(A) \subseteq A\}.$$

Denote by $\kappa_\alpha : T_\alpha \to T_B$ the natural inclusion. Note that T_α is a dg Lie subalgebra and a Lie algebra map $\rho_\alpha : T_\alpha \to T_A$ is defined.

8.3.1. Proposition. *The map ρ_α is a surjective quasi-isomorphism while κ_α is an injective quasi-isomorphism.*

Proof. Step 1. Prove that ρ_α is surjective. Since α is a standard acyclic cofibration, any derivation of A can be trivially extended by zero to a derivation of B. It will automatically belong to T_α.

Step 2. One has
$$\mathrm{Ker}\,\rho_\alpha = \{\delta \in T_B | \delta(A) = 0\}.$$
Put $B = A \coprod F(M)$ as above. Then $\mathrm{Ker}\,\rho_\alpha = \mathcal{H}om(M, B)$ is contractible.

Step 3. Exactly as in Step 3 of Proposition 8.2.1 the diagram

$$
\begin{array}{ccc}
T_A & \xrightarrow{\alpha_*} & \mathrm{Der}_\alpha(A, B) \\
\big\uparrow{\rho_\alpha} & & \big\uparrow{\alpha^*} \\
T_\alpha & \xrightarrow{\kappa_\alpha} & T_B
\end{array}
$$

is commutative and therefore we get that κ_α is also a weak equivalence. □

Proposition 8.3.1 allows one to define the map $T(\alpha) : T_A \to T_B$ in the homotopy category $\mathtt{Holie}(k)$ as

$$T(\alpha) = \kappa_\alpha \circ \rho_\alpha^{-1}.$$

8.3.2. Lemma. *Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be a pair of standard acyclic cofibrations. Then one has*

$$T(\beta \circ \alpha) = T(\beta) \circ T(\alpha)$$

in $\mathtt{Holie}(k)$.

Proof. See Lemma 8.2.2. $\qquad\square$

8.4. ... and their comparison.

8.4.1. Proposition. *Let $A \xrightarrow{\alpha} B \xrightarrow{\sigma} C$ be a pair of morphisms so that α is a standard acyclic cofibrations and $\sigma, \sigma \circ \alpha$ are acyclic fibrations. Then one has*

$$T(\sigma) \circ T(\alpha) = T(\sigma \circ \alpha)$$

in the homotopy category $\mathtt{Holie}(k)$.

Proof. Consider the diagram

Here $T = T_\alpha \times_{T_B} T_\sigma$ identifies with the Lie subalgebra

$$T = \{\delta \in T_B | \delta(A) \subseteq A \text{ and } \delta(I) \subseteq I\}$$

where $I = \operatorname{Ker} \sigma$. Then the dotted map $\mu : T \to T_{\sigma \circ \alpha}$ is defined by the formula $\mu(\delta) = \delta|_A$.

Step 1. Let us check that μ is surjective. Note that $B = A \coprod F(M)$ where $M \in C(k)$ is freely generated over k by a collection of elements $m_i \in M$ and their differentials dm_i. The map $\sigma : B \to C$ is therefore uniquely defined by its restriction $\sigma \circ \alpha$ to A and by its values $c_i = \sigma(m_i) \in C$. Since $\sigma \circ \alpha$ is surjective there exist $a_i \in A$ such that $c_i = \sigma\alpha(a_i)$. Therefore,

choosing an appropriate isomorphism between B and $A \coprod F(M)$ (the one sending m_i to $m_i - a_i$) we can suppose that M belongs to $I = \operatorname{Ker} \sigma$.

If now $\delta : A \to A$ is a derivative vanishing on $\operatorname{Ker} \sigma \circ \alpha = I \cap A$ we can define $\tilde{\delta} \in T$ by the formula

$$\tilde{\delta}|_A = \delta, \ \tilde{\delta}|_M = 0.$$

Step 2. One has

$$\operatorname{Ker} \mu = \{\delta \in T_B | \delta(A) = 0 \text{ and } \delta(I) \subseteq I\} = \mathcal{H}om(M, I).$$

This is obviously contractible.

Step 3. Now we see that all the arrows in the above diagram are quasi-isomorphisms. Since the diagram commutes, this proves the claim. $\square$

8.5. **Final steps.**

8.5.1. **Lemma.** *Let $i : A \to B$ be a standard acyclic cofibration and $p : B \to A$ be left inverse to i (so that it is acyclic fibration). Then $T(i) = T(p)^{-1}$.*

8.5.2. **Lemma.** *Let $\alpha, \beta : A \to B$ be two homotopy equivalent acyclic fibrations. Then $T(\alpha) = T(\beta)$ in $\mathtt{Holie}(k)$.*

Proof. Let

$$B \xrightarrow{i} B^I \overset{p_0, p_1}{\Longrightarrow} B$$

be a path object and $h : A \to B^I$ be a homotopy connecting $\alpha = p_0 \circ f$ with $\beta = p_1 \circ f$. Since A is cofibrant we can suppose that i is a standard acyclic cofibration. The maps p_0 and p_1 are both left inverse to $i : B \to B^I$, therefore $T(p_0) = T(p_1)$. Present the map $f : A \to B^I$ as a composition $f = q \circ j$ where q is an acyclic fibration and j is a standard acyclic cofibration. According to Lemma 8.2.2 $T(p_0 \circ q) = T(p_1 \circ q)$ and then Proposition 8.4.1 ensures that $T(\alpha) = T(\beta)$. $\square$

Now the main result of this Section follows.

8.5.3. **Theorem.** *Let $\mathcal{O}$ be a Σ-split operad over k. There exists a functor $T : \mathtt{Hoalg}(\mathcal{O})^{\text{iso}} \to \mathtt{Holie}(k)^{\text{iso}}$ from the homotopy category of $\mathcal{O}$-algebras and isomorphisms to the homotopy category of dg Lie k-algebras and isomorphisms which assigns to each cofibrant $\mathcal{O}$-algebra the dg Lie algebra $T_A = \operatorname{Der}_{\mathcal{O}}(A, A)$.*

Proof. Any quasi-isomorphism $\alpha : A \to B$ can be presented as a composition $\alpha = p \circ i$ where p is an acyclic fibration and i is a standard acyclic cofibration. In this case we set

$$T(\alpha) = T(p) \circ T(i).$$

To prove the theorem we have to check that if $p \circ i$ and $q \circ j$ are homotopic with p, q acyclic fibrations and i, j standard acyclic cofibrations then $T(p) \circ T(i) = T(q) \circ T(j)$ — see the Picture below.
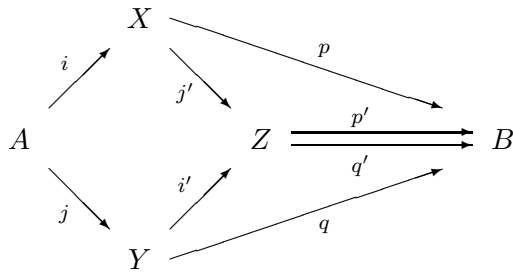
Put $Z = X \coprod^A Y$. Then the map $A \to Z$ is also a standard acyclic cofibration. Choose $p', q' : Z \to B$ so that $p = p' \circ j'$, $q = q' \circ i'$. Then obviously p' and q' are homotopic. Proposition 8.4.1 then says that $T(p) = T(p') \circ T(j')$ and $T(q) = T(q') \circ T(i')$ which implies

$$T(p) \circ T(i) = T(p') \circ T(j') \circ T(i) = T(p') \circ T(i') \circ T(j) = T(q') \circ T(i') \circ T(j) = T(q) \circ T(j).$$

Theorem is proven. □

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