DESCENT OF DELIGNE GROUPOIDS

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1. Introduction

1.1. A formal deformation problem (over a base field k of characteristic zero) can be described as a (2-)functor

$$\mathcal{C} : \mathtt{art}/k \to \mathtt{Grp}$$

from the category of artinian local k-algebras with residue field k to the category of groupoids which assigns to any $A \in \mathtt{art}/k$ a groupoid $\mathcal{C}(A)$ whose objects are deformations with the base $\mathrm{Spec}(A)$ and whose morphisms are equivalences of these deformations.

On the other hand, any non-negatively graded dg Lie algebra $\mathfrak{g}$ over k gives rise to such a functor

$$\mathcal{C}_{\mathfrak{g}} : \mathtt{art}/k \to \mathtt{Grp}.$$

This functor assigns to $(A, \mathfrak{m}) \in \mathtt{art}/k$ the *Deligne groupoid* $\mathcal{C}_{\mathfrak{g}}(A) = \mathcal{C}(\mathfrak{m} \otimes \mathfrak{g})$ of the nilpotent Lie dg algebra $\mathfrak{m} \otimes \mathfrak{g}$ — see 2.2 or [GM1], sect. 2.

A common belief is that any "reasonable" formal deformation problem can be described by the Deligne groupoid $\mathcal{C}_{\mathfrak{g}}$ where $\mathfrak{g}$ is an appropriate "Lie algebra of infinitesimal automorphisms". This would imply, for instance, that if $H^0(\mathfrak{g}) = 0$ (i.e. if the automorphism group of the deformed object is discrete) then the completion of the local ring of a moduli space at a given point is isomorphic to the completion of the 0-th cohomology group of $\mathfrak{g}$.

If we are dealing with the deformations of algebraic structures (associative, commutative or Lie algebras or so), the Lie algebra $\mathfrak{g}$ is just the standard complex calculating the cohomology of the appropriate type.

In this paper we prove the following claim conjectured by V. Schechtman in [S1, S2, HS3]. It allows one to construct a dg Lie algebra which governs various formal deformations in the non-affine case.

Let X be a topological space and let $\mathfrak{g}$ be a sheaf of dg Lie k-algebras. Let $\mathcal{U} = \{U_i\}$ be a locally finite open covering of X, $\mathcal{D}_i = \mathcal{C}_{\Gamma(U_i,\mathfrak{g})}$ be the Deligne groupoids corresponding to the dg Lie algebras $\Gamma(U_i, \mathfrak{g})$ and let $\mathcal{D}$ be the groupoid of "descent data" for the collection $\mathcal{D}_i$ (see 3.2.1,2). Then $\mathcal{D}$ is naturally equivalent to the Deligne groupoid $\mathcal{C}_L$ where L is a dg Lie algebra representing the Cech complex $\check{C}(\mathcal{U}, \mathfrak{g})$.

This result implies, in particular, Corollary 6.2 which claims that Theorem 8.3 of [HS2] remains valid without the assumption of formal smoothness.

1.2. Our proof is based upon a very important notion of *nerve* $\Sigma(\mathfrak{g})$ of a nilpotent dg Lie algebra $\mathfrak{g}$ which we define in Section 2.

The nerve $\Sigma(\mathfrak{g})$ is a Kan simplicial set (cf. [GZ]); Deligne groupoid $\mathcal{C}(\mathfrak{g})$ is homotopically equivalent to $\Sigma(\mathfrak{g})$ and it identifies with the Poincaré groupoid $\Pi(\Sigma(\mathfrak{g}))$ (cf. [GZ]).

In Section 3 we present a (almost standard) definition of the total space functor in different categories. Now the claim of 1.1 follows from the main Theorem 4.1 saying that the nerve functor Σ commutes with the Tot functors up to homotopy.

The natural map $\Sigma \circ \mathrm{Tot} \to \mathrm{Tot} \circ \Sigma$ is constructed immediately; it takes some pain to check that this is a homotopy equivalence.

Section 5 is technical. We use its Proposition 5.2.8 in the proof of the main Theorem 4.1. In the last Section 6 we deduce from Theorem 4.1 an application to formal deformation theory.

The idea that the generalization of the result of [HS2] to the non-smooth case should follow from a descent property for Deligne groupoids belongs to V. Schechtman. I am very grateful to him for helpful discussions on the subject. Proposition 2.2.3 claiming that the nerve $\Sigma(\mathfrak{g})$ and the Deligne groupoid $\mathcal{C}(\mathfrak{g})$ are homotopy equivalent seems to have something common with the Main Homotopy Theorem of Schlessinger–Stasheff, cf. [SS]. I am greatly indebted to J. Stasheff who read the first draft of the manuscript and made some important remarks. I am also grateful to J. Bernstein for his valuable recommendations.

1.3. **Notations.** Throughout this paper k is a fixed field of characteristic zero.

$\mathtt{art}/k$ denotes the category of commutative local artinian k-algebras having the residue field k.

$\mathtt{dglie}(k)$ (resp., $\mathtt{cdga}(k)$) is the category of non-negatively graded dg Lie (resp., commutative) algebras over k. Its full subcategory $\mathtt{ndglie}(k)$ consists of nilpotent dg Lie algebras.

Δ is the category of ordered sets $[n] = \{0, \dots, n\}$, $n \geq 0$ and monotone maps; $\Delta^0\mathtt{Ens}$ is the category of simplicial sets; $\Delta^n \in \Delta^0\mathtt{Ens}$ are the standard n-simplices.

$\mathtt{Kan} \subseteq \Delta^0\mathtt{Ens}$ is the full subcategory of Kan simplicial sets.

$\mathtt{Ab}$ is the category of abelian groups. $C(\mathcal{A})$ (sometimes $C(R)$) is the category of complexes over an abelian category $\mathcal{A}$ (over the category of R-modules). $C^{\geq 0}$ denotes the full subcategory of non-negatively graded complexes.

2. Nerve of a nilpotent Lie dg algebra

Throughout Sections 2–4 all dg Lie algebras will be nilpotent.

2.1. For any $n \geq 0$ denote by Ω_n the k-algebra of polynomial differential forms on the standard n-simplex Δ^n — see [BG].

One has

$$\Omega_n = k[t_0, \dots, t_n, dt_0, \dots, dt_n]/(\sum t_i - 1, \sum dt_i).$$

The algebras Ω_n form a simplicial commutative dg algebra: a map $u : [p] \to [q]$ induces the map $\Omega(u) : \Omega_q \to \Omega_p$ defined by the formula $\Omega(u)(t_i) = \sum_{u(j)=i} t_j$.

If $\mathfrak{g}$ is a dg Lie k-algebra and A is a commutative dg k-algebra then the tensor product $A\otimes\mathfrak{g}$ is also a dg Lie k-algebra. Thus, any dg Lie algebra $\mathfrak{g}$ gives rise to a simplicial dg Lie algebra

$$\mathfrak{g}_\bullet = \{\mathfrak{g}_n = \Omega_n \otimes \mathfrak{g}\}_{n\geq 0}.$$

For any dg Lie algebra $\mathfrak{g}$ denote by $\mathrm{MC}(\mathfrak{g})$ the set of elements $x\in\mathfrak{g}^1$ satisfying the Maurer-Cartan equation:

$$dx+\frac{1}{2}[x,x]=0.$$

2.1.1. **Definition.** Let $\mathfrak{g}\in\mathtt{ndglie}(k)$. Its *nerve* $\Sigma(\mathfrak{g})\in\Delta^0\mathtt{Ens}$ is defined as

$$\Sigma(\mathfrak{g})=\mathrm{MC}(\mathfrak{g}_\bullet).$$

2.1.2. Recall (see [BG]) that the collection of commutative dg algebras Ω_n defines a contravariant functor

$$\Omega:\Delta^0\mathtt{Ens}\to\mathtt{cdga}(k)$$

so that $\Omega(\Delta^n)=\Omega_n$ and Ω carries direct limits in $\Delta^0\mathtt{Ens}$ to inverse limits.

Lemma. *Let $S\in\Delta^0\mathtt{Ens}$. There is a natural map*

$$\mathrm{MC}(\Omega(S)\otimes\mathfrak{g})\to\mathrm{Hom}(S,\Sigma(\mathfrak{g}))$$

which is bijective provided S is finite (i.e., has a finite number of non-degenerate simplices).

Proof. This is because tensoring by $\mathfrak{g}$ commutes with finite limits — compare to [BG], 5.2. □

2.1.3. **Definition.** A map $f:\mathfrak{g}\to\mathfrak{h}$ in $\mathtt{ndglie}(k)$ will be called an *acyclic fibration*(this is an ad hoc definition!) if it is surjective and induces a quasi-isomorphism of the corresponding lower central series.

2.1.4. **Lemma.** *Let $f:\mathfrak{g}\to\mathfrak{h}$ be an acyclic fibration in $\mathtt{ndglie}(k)$. Then the induced map $\mathrm{MC}(f):\mathrm{MC}(\mathfrak{g})\to\mathrm{MC}(\mathfrak{h})$ is surjective.*

Proof. Induction by the nilpotence degree of $\mathfrak{g}$ — similarly to [GM1], Th. 2.4. □

2.1.5. **Lemma.** *Let $f:A\to B$ be a surjective map in $\mathtt{cdga}(k)$ and $g:\mathfrak{g}\to\mathfrak{h}$ be a surjective map in $\mathtt{ndglie}(k)$. Then the map*

$$A\otimes\mathfrak{g}\to(A\otimes\mathfrak{h})\times_{(B\otimes\mathfrak{h})}(B\otimes\mathfrak{g})$$

is an acyclic fibration provided either (a) f is quasi-isomorphism or (b) g is acyclic fibration.

Proof. Since for any commutative dg algebra A (with 1) the functor $A\otimes\_$ transforms the lower central series of $\mathfrak{g}$ into the lower central series of $A\otimes\mathfrak{g}$, it suffices to check that the above map is a surjective quasi-isomorphism. This is fairly standard. □

2.1.6. **Proposition.** *Let $f : \mathfrak{g} \to \mathfrak{h}$ be an acyclic fibration. Then $\Sigma(f) : \Sigma(\mathfrak{g}) \to \Sigma(\mathfrak{h})$ is an acyclic fibration of simplicial sets.*

Proof. Lemmas 2.1.2 and 2.1.4 reduce the question to the following. Let $K \to L$ be an injective map of finite simplicial sets. Then one has to show that the induced map of nilpotent Lie algebras

$$\Omega(L) \otimes \mathfrak{g} \to \Omega(L) \otimes \mathfrak{h} \times_{\Omega(K)\otimes\mathfrak{h}} \Omega(K) \otimes \mathfrak{g}$$

is an acyclic fibration. This follows from 2.1.5(b). $\square$

2.2. **Deligne groupoid.** Recall (cf. [GM1]) that given a nilpotent dg Lie algebra $\mathfrak{g} \in \mathtt{ndglie}(k)$ one defines the *Deligne groupoid* $\mathcal{C}(\mathfrak{g})$ as follows.

The Lie algebra $\mathfrak{g}^0$ acts on $\mathrm{MC}(\mathfrak{g})$ by vector fields:

$$\rho(y)(x) = dy + [x, y] \text{ for } y \in \mathfrak{g}^0, x \in \mathfrak{g}^1.$$

This defines the action of the nilpotent group $G = \exp(\mathfrak{g}^0)$ on the set $\mathrm{MC}(\mathfrak{g})$. Then the groupoid $\mathcal{C} = \mathcal{C}(\mathfrak{g})$ is defined by the formulas

$$\mathrm{Ob}\,\mathcal{C} = \mathrm{MC}(\mathfrak{g})$$

$$\mathrm{Hom}_{\mathcal{C}}(x, x') = \{g \in G | x' = g(x)\}.$$

2.2.1. **Lemma.** *Let $\mathfrak{g} \in \mathtt{ndglie}(k)$. The natural map $\mathfrak{g} \to \mathfrak{g}_n = \Omega_n \otimes \mathfrak{g}$ induces an equivalence of groupoids $\mathcal{C}(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g}_n)$.*

Proof. It suffices to check the claim when $n = 1$. In this case an element $z = x + dt \cdot y \in \mathfrak{g}_1 = \mathfrak{g}[t, dt]$ with $x \in \mathfrak{g}^1[t], y \in \mathfrak{g}^0[t]$ satisfies MC iff $x(0) \in \mathrm{MC}(\mathfrak{g})$ and x satisfies the differential equation

$$\dot{x} = dy + [x, y].$$

This means precisely that $x = g(x(0))$ where $g \in G_1 = \exp(\mathfrak{g}_1)$ is given by the differential equation

$$\dot{g} = g(y)$$

with the initial condition $g(0) = 1$. Thus, the map $\mathcal{C}(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g}_1)$ induces a surjective map of the sets of isomorphism classes. We have also to check that if $x \in \mathrm{MC}(\mathfrak{g})$ then the map $\mathrm{Aut}_{\mathcal{C}(\mathfrak{g})}(x) \to \mathrm{Aut}_{\mathcal{C}(\mathfrak{g}_1)}(x)$ is bijective. It is obviously injective since it is split by the map $\pi : \mathrm{Aut}_{\mathcal{C}(\mathfrak{g}_1)}(x) \to \mathrm{Aut}_{\mathcal{C}(\mathfrak{g})}(x)$ sending t to zero. We will prove by induction on the nilpotence degree of $\mathfrak{g}$ that π is injeective.

Let $y \in \mathfrak{g}^0[t], y = \sum_{i=1}^n a_i t^i$ satisfy $\exp(y)(x) = x$. Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{a}] = 0$. Put $\mathfrak{h} = \mathfrak{g}/\mathfrak{a}$. By induction, we suppose that the claim is correct for the Lie algebra $\mathfrak{h}$, therefore $a_i \in \mathfrak{a}$. Then $\exp(y)(x) = x + [y, x] + dy$ which easily implies $y = 0$. $\square$

2.2.2. *Explicit description of* $\Sigma(\mathfrak{g})$. The notations are as in 2.2.

For any $n \geq 0$ let $G_n = \exp(\mathfrak{g}^0_n)$ be the group of polynomial maps from the standard n-simplex Δ_n to the group $G = G_0$. The collection $G_\bullet = \{G_n\}$ forms a simplicial nilpotent group. Right multiplication defines on $G_\bullet$ a right G-action.

Proposition. *There is a natural bijection*

$$G_\bullet \times^G \mathrm{MC}(\mathfrak{g}) \to \Sigma(\mathfrak{g})$$

of simplicial sets. Here, as usual, $X \times^G Y$ is the quotient of the cartesian product $X \times Y$ by the relation

$$(xg, y) \sim (x, gy) \text{ for } x \in X,\ y \in Y,\ g \in G.$$

Proof. This immediately follows from Lemma 2.2.1. □

Let $N\mathcal{C}(\mathfrak{g}) \in \Delta^0\mathtt{Ens}$ be the nerve of $\mathcal{C}(\mathfrak{g})$.

Define the map $\tau : \Sigma(\mathfrak{g}) \to N\mathcal{C}(\mathfrak{g})$ by the formula

$$\tau(g, x) = (g_0(x), g_1 g_0^{-1}, \ldots, g_n g_{n-1}^{-1})$$

where $g \in G_n, x \in \mathrm{MC}(\mathfrak{g}), g_i = v_i(g)$ with v_i being the i-th vertex.

Proposition 2.2.2 implies the following

2.2.3. **Proposition.** *$\Sigma(\mathfrak{g})$ is a Kan simplicial set. The map τ is an acyclic fibration identifying $\mathcal{C}(\mathfrak{g})$ with the Poincaré groupoid of $\Sigma(\mathfrak{g})$.*

More generally, if $f : \mathfrak{g} \to \mathfrak{h}$ is a surjective map of nilpotent dg Lie algebras, then the induced map

$$\Sigma(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g}) \times_{\mathcal{C}(\mathfrak{h})} \Sigma(\mathfrak{h})$$

is an acyclic fibration.

Proof. The question reduces to the following. Given a pair of polynomial maps $\alpha : \partial\Delta^n \to \mathfrak{g}^0,\ \beta : \Delta^n \to \mathfrak{h}^0$ satisfying $f\alpha = \beta|_{\partial\Delta^n}$ find a map $\gamma : \Delta^n \to \mathfrak{g}^0$ such that $\alpha = \gamma|_{\partial\Delta^n},\ \beta = f\gamma$. This is always possible since the canonical map $\Omega_n \to \Omega(\partial\Delta^n)$ is surjective. □

2.2.4. *Remark.* If one does not require $\mathfrak{g}$ to be non-negatively graded, its nerve is still a Kan simplicial set. In this case it can probably be considered as a generalization of the notion of Deligne groupoid.

3. "Total space" functor

3.1. **Generalities.**

3.1.1. *The category $\mathcal{M}$.* Here and below $\mathcal{M}$ denotes the following category of morphisms of Δ:

The objects of $\mathcal{M}$ are morphisms $[p] \to [q]$ in Δ. A morphism from $[p] \to [q]$ to $[p'] \to [q']$ is given by a commutative diagram

$$\begin{array}{ccc} [p] & \longrightarrow & [q] \\ {\scriptstyle\alpha} \uparrow & & {\scriptstyle\beta} \downarrow \\ [p'] & \longrightarrow & [q'] \end{array}$$

The morphism in $\mathcal{M}$ corresponding to $\alpha = \mathrm{id}, \beta = \sigma^i$, is denoted by σ^i; the one corresponding to $\alpha = \mathrm{id}, \beta = \partial^i$, is denoted by ∂^i. "Dually", the morphism corresponding to $\alpha = \partial^i, \beta = \mathrm{id}$, is denoted by d_i and the one with $\alpha = \sigma^i, \beta = \mathrm{id}$, is denoted by s_i.

3.1.2. *Total space.* Let $\mathcal{C}$ be a simplicial category having inverse limits and functorial *function objects* $\underline{\mathrm{hom}}(S, X) \in \mathcal{C}$ for $X \in \mathcal{C},\ S \in \Delta^0\mathtt{Ens}$ — see [BK], IX.4.5 and the examples below. The total space $\mathrm{Tot}(X)$ of a cosimplicial object $X \in \Delta\mathcal{C}$ is defined by the formula

$$\mathrm{Tot}(X) = \lim_{\leftarrow}{}_{\phi \in \mathcal{M}} \underline{\mathrm{hom}}(\Delta^p, X^q)$$

where $\phi : [p] \to [q]$.

3.2. **Examples.** We will use three instances of the described construction.

3.2.1. Let $\mathcal{C} = \Delta^0\mathtt{Ens}$. Then the above definition coincides with the standard one given in *loc. cit.*, XI.3.

Let $G \in \Delta\mathtt{Grp}$ be a (strict) cosimplicial groupoid. We will consider $\mathtt{Grp}$ as a full subcategory of $\Delta^0\mathtt{Ens}$, so the simplicial set $\mathrm{Tot}(G)$ is defined.

Lemma. *$T = \mathrm{Tot}(G)$ is a groupoid. The objects of T are collections $\{a \in \mathrm{Ob}\, G^0, \theta : \partial^1(a) \xrightarrow{\sim} \partial^0(a)\}$ with θ satisfying the cocycle condition:*

$$\sigma^0(\theta) = \mathrm{id}(a);\ \partial^1\theta = \partial^0\theta \circ \partial^2\theta.$$

A morphism in T from $\{a, \theta\}$ to $\{b, \theta'\}$ is a morphism $a \to b$ compatible with θ, θ'.

Thus, $\mathrm{Tot}(G)$ *is "the groupoid of descent data" for G.*

3.2.2. Let $\mathcal{C} = C(k)$ be the category of complexes over k. For $S \in \Delta^0\mathtt{Ens}$ and $X \in C(k)$ the complex $\underline{\mathrm{hom}}(S, X)$ is defined to be $\mathrm{Hom}(C_\bullet(S), X)$ where $C_\bullet$ is the complex of normailized chains of S with coefficients in k. The above definition of the functor Tot coincides with the standard one.

3.2.3. Let $\mathcal{C} = \mathtt{dglie}(k)$. For $S \in \Delta^0\mathtt{Ens}$ and $\mathfrak{g} \in \mathtt{dglie}(k)$ define $\underline{\mathrm{hom}}(S, \mathfrak{g}) = \Omega(S) \otimes \mathfrak{g}$. Then the functor $\mathrm{Tot} : \Delta\mathtt{dglie}(k) \to \mathtt{dglie}(k)$ coincides with the Thom-Sullivan functor described in [HS2], 5.2.4. The De Rham theorem (see, e.g., *loc. cit.*, 5.2.8) shows that the functor Tot commutes up to homotopy with the forgetful functor $\# : \mathtt{dglie}(k) \to C(k)$.

4. The main theorem

Now we come to the main result of the paper.

Let $\mathfrak{g} \in \Delta\mathtt{ndglie}(k)$ be a cosimplicial nilpotent dg Lie k-algebra. Suppose that $\mathfrak{g}$ is *finitely dimensional in the cosimplicial sense*, i.e. that the normalization

$$N^n(\mathfrak{g}) = \{x \in \mathfrak{g}^n | \sigma^i(x) = 0 \text{ for all } i\}$$

vanishes for sufficiently big n.

4.1. **Theorem.** *There is a natural homotopy equivalence*

$$\Sigma(\mathrm{Tot}(\mathfrak{g})) \longrightarrow \mathrm{Tot}(\Sigma(\mathfrak{g}))$$

in $\mathtt{Kan}$.

The proof of the theorem is given in 4.2– 5.2.

Taking into account 2.2.3, we easily get

4.1.1. **Corollary.** *Let $\mathfrak{g} \in \Delta\mathtt{ndglie}(k)$ be a nilpotent cosimplicial dg Lie k-algebra. Suppose that $\mathfrak{g}$ is finitely dimensional in the cosimplicial sense.*

Then there is a natural equivalence of groupoids

$$\mathcal{C}(\mathrm{Tot}(\mathfrak{g})) \to \mathrm{Tot}(\mathcal{C}(\mathfrak{g})).$$

Proof. One has to check that the functor Tot carries the map $\tau : \Sigma(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g})$ to a homotopy equivalence. By [BK], X.5, it suffices to check τ is a fibration in sense of *loc. cit.*

This follows from (the second claim of) Proposition 2.2.3 since for any n the natural map from $\mathfrak{g}^{n+1}$ to the n-th matching space $M^n(\mathfrak{g})$ (in the notations of *loc. cit*) is surjective. $\square$

4.2. **Proof of the theorem: a natural map.** The set of n-simplices of the left-hand side is

$$\mathrm{MC}(\Omega_n \otimes \varprojlim \Omega_p \otimes \mathfrak{g}^q)$$

and for the right-hand side:

$$\varprojlim \mathrm{MC}(\Omega(\Delta^n \times \Delta^p) \otimes \mathfrak{g}^q).$$

Here the inverse limits are taken over the category $\mathcal{M}$ defined in 3.1.1 and Lemma 2.1.2 is used to get the second formula.

Taking into account that the functor MC commutes with the inverse limits, a canonical map $\Sigma \circ \mathrm{Tot}(\mathfrak{g}) \to \mathrm{Tot} \circ \Sigma(\mathfrak{g})$ is defined as the composition

$$\Omega_n \otimes \varprojlim \Omega_p \otimes \mathfrak{g}^q \to \varprojlim \Omega_n \otimes \Omega_p \otimes \mathfrak{g}^q \to \varprojlim \Omega(\Delta^n \times \Delta^p) \otimes \mathfrak{g}^q$$

the latter arrow being induced by the canonical projections of $\Delta^n \times \Delta^p$ to Δ^n and to Δ^p.

We wish to prove that the map described induces a homotopy equivalence.

4.2.1. Since $\mathfrak{g}$ is finitely dimensional, the first map in the composition is bijective — see [HS2], Thm. 6.11 where the inverse limit $\underleftarrow{\lim}\, \Omega_p \otimes X^q$ is explicitly calculated.

4.2.2. In order to prove that the second map is a homotopy equivalence, let us fix n and p and present the map

$$\alpha : \Omega_n \otimes \Omega_p \to \Omega(\Delta^n \times \Delta^p)$$

as the composition

$$\Omega_n \otimes \Omega_p \xrightarrow{\beta} \Omega(\Delta^n \times \Delta^{p+1}) \otimes \Omega_p \xrightarrow{\pi} \Omega(\Delta^n \times \Delta^p)$$

where β is induced by the projection $\Delta^n \times \Delta^{p+1} \to \Delta^n$ and π by the pair of maps $\mathrm{id} \times \partial^{p+1} : \Delta^n \times \Delta^p \to \Delta^n \times \Delta^{p+1}$, $\mathrm{pr}_2 : \Delta^n \times \Delta^p \to \Delta^p$.

We will check below that the maps β and π induce homotopy equivalences for different reasons: β induces a strong deformation retract and π induces an acyclic (Kan) fibration. This will prove the theorem.

4.3. In the sequel functors $X : \mathcal{M} \to \Delta^0\mathtt{Ens}$ will be called $\mathcal{M}$-simplicial sets.

Any bisimplicial commutative dg algebra $A \in (\Delta^0)^2\mathtt{cdga}$ defines a $\mathcal{M}$-simplicial set $\Sigma(A, \mathfrak{g})$ as follows:

For $a : [p] \to [q], n \in \mathbb{N}$ one has

$$\Sigma(A, \mathfrak{g})(a)_n = \mathrm{MC}(A_{np} \otimes \mathfrak{g}^q).$$

Define also a simplicial set $\sigma(A, \mathfrak{g})$ to be the inverse limit of $\Sigma(A, \mathfrak{g})$ as a functor from $\mathcal{M}$ to $\Delta^0\mathtt{Ens}$.

4.3.1. We apply the above to define the diagram

$$\sigma(A, \mathfrak{g}) \stackrel{\sigma(\beta,\mathfrak{g})}{\to} \sigma(C, \mathfrak{g}) \stackrel{\sigma(\pi,\mathfrak{g})}{\to} \sigma(B, \mathfrak{g})$$

as follows:

1. Put $A_{np} = \Omega_n \otimes \Omega_p$, $B_{np} = \Omega(\Delta^n \times \Delta^p)$. These are obviously bisimplicial dg algebras.

2. Define Δ^{+1} to be the cosimplicial simplicial set with $(\Delta^{+1})^n = \Delta^{n+1}$ whose cofaces and codegeneracies are the standard maps between the standard simplices (they all preserve the final vertex). Put $C_{np} = \Omega(\Delta^n \times \Delta^{p+1}) \otimes \Omega_p$. Bisimplicial structure on C in defined by the cosimplicial structure on Δ^{+1}.

Our aim is to prove that the maps $\beta : A \to C$ and $\pi : C \to B$ induce homotopy equivalences $\sigma(\beta, \mathfrak{g})$ and $\sigma(\pi, \mathfrak{g})$.

We will check immediately that $\sigma(\beta, \mathfrak{g})$ is a strong deformation retract. Afterwards, using the criterion 5.2.8 below we will get that $\sigma(\pi, \mathfrak{g})$ is an acyclic fibration.

4.3.2. *Checking β.* For $S \in \Delta^0\mathtt{Ens}$ and nilpotent $\mathfrak{g} \in \mathtt{dglie}(k)$ denote by $\Sigma^S(\mathfrak{g})$ (or just Σ^S when $\mathfrak{g}$ is one and the same) the simplicial set whose set of n-simplicies is $\mathrm{MC}(\Omega(\Delta^n \times S) \otimes \mathfrak{g})$.

Any map $f : K \times S \to T$ in $\Delta^0\mathtt{Ens}$ induces a map

$$\Sigma^f : K \times \Sigma^T \to \Sigma^S$$

as follows. Let $k \in K_n,\ x \in \Sigma^T_n = \mathrm{MC}(\Omega(\Delta^n \times T) \otimes \mathfrak{g})$. Denote by F_k the composition

$$\Delta^n \times S \xrightarrow{\mathrm{diag}\times 1} \Delta^n \times \Delta^n \times S \xrightarrow{1\times k\times 1} \Delta^n \times K \times S \xrightarrow{1\times f} \Delta^n \times T.$$

Then $\Sigma^f(k, x)$ is defined to be $(\Omega(F_k) \otimes \mathrm{id}_{\mathfrak{g}})(x)$.

Define the map $\Phi_p : \Delta^1 \times \Delta^{p+1} \to \Delta^{p+1}$ as the one given on the level of posets by the formula

$$\Phi_p(i,j) = \begin{cases} j \text{ if } i = 0 \\ p+1 \text{ if } i = 1 \end{cases}$$

The maps Φ_p induce the maps $\Sigma^{\Phi_p}(\Omega_p \otimes \mathfrak{g}^q)$ which define for any $a : [p] \to [q]$ in $\mathcal{M}$ the simplicial set $\Sigma(A, \mathfrak{g})(a) = \mathrm{MC}(A_{\bullet p} \otimes \mathfrak{g}^q)$ as a strong deformation retract of $\Sigma(C, \mathfrak{g})(a)$. The retractions $\Sigma^{\Phi_p}(\Omega_p \otimes \mathfrak{g}^q)$ are functorial in $a \in \mathcal{M}$, therefore the inverse image map $\sigma(\beta, \mathfrak{g})$ is also strong deformation retract.

4.3.3. *Checking π.* The last thing to do is to prove that the map $\pi : C \to B$ of bisimplicial dg algebras defined above induces an acyclic fibration $\sigma(\pi, \mathfrak{g})$. In Section 5 below we provide a necessary machinery for this. First, we define in 5.2 the notion of acyclic fibration for bisimplicial commutative dg algebras and prove that if $f : A \to B$ in $(\Delta^0)^2\mathtt{cdga}(k)$ is an acyclic fibration then for any cosimplicial nilpotent dg Lie algebra $\mathfrak{g}$ the map $\sigma(f, \mathfrak{g})$ is an acyclic Kan fibration. Then we provide in 5.2.8 a criterion for a map in $(\Delta^0)^2\mathtt{cdga}(k)$ to be an acyclic fibration.

Thus, in order to prove the theorem we have only to check that the map $\pi : C \to B$ satisfies the conditions of 5.2.8. Let us do this.

4.3.4. *Checking the hypotheses of 5.2.8.* First of all, let us check that $\pi(S, T)$ is surjective. For this consider $D_{np} = \Omega(\Delta^n \times \Delta^{p+1})$ and the map $\rho_{np} : D_{np} \to B_{np}$ which is the composition of π with the natural embedding $D_{np} \to C_{np} = D_{np} \otimes \Omega_p$. Surely, it suffices to prove that $\rho(S, T)$ is surjective. For this we note that $D(S, T) = \Omega(S \times \widehat{T})$ where $\widehat{\ }: \Delta^0\mathtt{Ens} \to \Delta^0\mathtt{Ens}$ is the functor satisfying the condition $\widehat{\ }|_{\Delta} = \Delta^{+1}$ and preserving colimits (see [GZ],II.1.3). Then, since the map $S \times T \to S \times \widehat{T}$ is injective, the map

$$\Omega(S \times \widehat{T}) \to \Omega(S \times T)$$

is surjective.

Next, one has

$$C(S, \Delta^p) = \Omega(S \times \Delta^{p+1}) \otimes \Omega_p,$$
$$B(S, \Delta^p) = \Omega(S \times \Delta^p)$$

so the condition 5.2.8 (b) is fulfilled.

The simplicial abelian groups $C^d(S, \_)$ are contractible by the Künneth formula. The abelian groups $B^d(S, \_) = \Omega^d(S \times \_)$ are contractible since any injective map $T \to T'$ induces a surjection $\Omega(S \times T') \to \Omega(S \times T)$. The same reason proves the condition (d).

Therefore, the map $\pi : C \to B$ of bisimplicial commutative dg algebras is an acyclic fibration by Proposition 5.2.8 and then by Proposition 5.2.5 the map $\sigma(\pi, \mathfrak{g})$ is an acyclic fibration in $\Delta^0\mathtt{Ens}$.

The Theorem is proven.

5. Acyclic fibrations of different things

5.1. **$\mathcal{M}$-simplicial sets.** Recall (see 4.3) that the functors $X : \mathcal{M} \to \Delta^0\mathtt{Ens}$ are called $\mathcal{M}$-simplicial sets. We wish to find a sufficient condition for a map $f : X \to Y$ of $\mathcal{M}$-simplicial sets to induce an acyclic fibration $\varprojlim f$ of the inverse limits.

5.1.1. *Matching spaces.* Fix $n \in \mathbb{N}$. Let $\partial^i : [n-1] \to [n]$, $i = 0, \ldots, n$, be the standard face maps and let $\sigma^i : [n] \to [n-1]$, $i = 0, \ldots, n-1$, be the standard degeneracies.

The n-th matching space of a $\mathcal{M}$-simplicial set X is a simplicial subset $\mu_n(X)$ of the product

$$\prod_{i=0}^{n} X(\partial^i) \times \prod_{i=0}^{n-1} X(\sigma^i)$$

consisting of the collections $(x_i \in X(\partial^i), y^i \in X(\sigma^i))$ satisfying the following three conditions:

(d): $d_i x_j = d_{j-1} x_i$ for $i < j$.

(σ): $\sigma^j y^i = \sigma^i y^{j+1}$ for $i \leq j$.

$(d\sigma)$: $\sigma^j x_i = d_i y^j$ for all i, j.

One has a canonical map $X(\mathrm{id}_n) \to \mu_n(X)$ which sends an element $x \in X(\mathrm{id}_n)$ to the collection $(d_0 x, \ldots, d_n x, \sigma^0 x, \ldots, \sigma^{n-1} x)$.

5.1.2. **Definition.** A map $f : X \to Y$ of $\mathcal{M}$-simplicial sets is called an *acyclic fibration* if for any $n \in \mathbb{N}$ the commutative square

$$\begin{array}{ccc} X(\mathrm{id}_n) & \longrightarrow & Y(\mathrm{id}_n) \\ \downarrow & & \downarrow \\ \mu_n(X) & \longrightarrow & \mu_n(Y) \end{array}$$

defines an acyclic fibration

$$X(\mathrm{id}_n) \to Y(\mathrm{id}_n) \times_{\mu_n(Y)} \mu_n(X).$$

5.1.3. **Lemma.** *Let $f : X \to Y$ be an acyclic fibration of $\mathcal{M}$-simplicial sets. Then the induced map of the corresponding inverse limits, $\varprojlim f$, is an acyclic fibration.*

Proof. Let $\mathcal{M}_{\leq n}$ be the full subcategory of $\mathcal{M}$ consisting of morphisms $\alpha : [p] \to [q]$ with $p \leq n, q \leq n$. Put $X(n) = \varprojlim X|_{\mathcal{M}_{\leq n}}$. Then it is easy to see that $X(n) = X(\mathrm{id}_n) \times_{\mu_n(X)} X(n-1)$. This immediately proves the lemma. $\square$

5.2. **Bisimplicial algebras.** $\mathrm{Aut}_{\mathcal{C}(\mathfrak{g})}(x) \to \mathrm{Aut}_{\mathcal{C}(\mathfrak{g}_1)}(x)$

5.2.1. Fix a cosimplicial nilpotent dg Lie algebra $\mathfrak{g}$. Any bisimplicial commutative dg algebra $A \in (\Delta^0)^2\mathtt{cdga}$ defines a $\mathcal{M}$-simplicial set $\Sigma(A,\mathfrak{g})$ and a simplicial set $\sigma(A,\mathfrak{g})$ as in 4.3.

We provide now a sufficient condition for a map $f : A \to B$ of bisimplicial commutative dg algebras to induce an acyclic fibration $\Sigma(f,\mathfrak{g})$ for any cosimplicial dg Lie algebra. According to Lemma 5.1.3 this implies that $\sigma(f,\mathfrak{g})$ is also an acyclic fibration.

5.2.2. Any bisimplicial abelian group A gives rise to a functor

$$A : \Delta^0\mathtt{Ens} \times \Delta^0\mathtt{Ens} \longrightarrow \mathtt{Ab}$$

which is uniquely described by the following properties:

— $A(\Delta^m, \Delta^n) = A_{mn}$

— A carries direct limits over each one of the arguments to inverse limit.

We will identify bisimplicial abelian groups with the functors they define.

5.2.3. **Definition.** The matching space $M_{mn}(A)$ of a bisimplicial abelian group A is defined to be

$$M_{mn}(A) = A(\partial\Delta^m, \Delta^n) \times_{A(\partial\Delta^m,\partial\Delta^n)} A(\Delta^m, \partial\Delta^n)$$

where $\partial\Delta^n$ is the boundary of the n-simplex. One has a canonical map $A_{mn} \to M_{mn}(A)$.

5.2.4. **Definition.** A map $f : A \to B$ in $(\Delta^0)^2C(\mathbb{Z})$ is called an *acyclic fibration* if for any m, n the canonical map

$$A_{mn} \to B_{mn} \times_{M_{mn}(B)} M_{mn}(A)$$

is a surjective quasi-isomorphism.

5.2.5. **Proposition.** *Let $f : A \to B$ in $(\Delta^0)^2\mathtt{cdga}(k)$ be an acyclic fibration. Let $\mathfrak{g}$ be a cosimplicial nilpotent dg Lie algebra. Then the induced map $\Sigma(f,\mathfrak{g})$ is an acyclic fibration of $\mathcal{M}$-simplicial sets.*

Proof. This is a direct calculation using 2.1.4, 2.1.5(a). Here we use that the natural map from $\mathfrak{g}^{n+1}$ to the n-th matching space $M^n(\mathfrak{g})$ (see [BK], X.5) is surjective. $\square$

Now we wish to formulate a sufficient condition for $f : A \to B$ to be an acyclic fibration of bisimplicial dg algebras. The following trivial lemma will be useful.

5.2.6. **Lemma.** *Let in a commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ C & \longrightarrow & D \end{array}$$

of abelian groups the map g and the map $\mathrm{Ker}(g) \to \mathrm{Ker}(h)$ *be surjective. Then the induced map $A \to B \times_D C$ is also surjective.*

We start with a simplicial case. Recall that a simplicial abelian group $A \in \Delta^0\mathtt{Ab}$ defines a functor $A : \Delta^0\mathtt{Ens} \to \mathtt{Ab}$ by the formula

$$A(S) = \mathrm{Hom}(S, A).$$

5.2.7. **Lemma.** *Let $f : A \to B$ be a map in $\Delta^0 C^{\geq 0}(\mathbb{Z}) = C^{\geq 0}(\Delta^0\mathtt{Ab})$. Suppose that*

(a) $f_n : A_n \to B_n$ is a quasi-isomorphism in $C(\mathbb{Z})$

(b) for any $S \in \Delta^0\mathtt{Ens}$ the map $f(S) : A(S) \to B(S)$ is surjective

(c) for any $d \in \mathbb{Z}$ the d-components A^d and B^d are contractible simplicial abelian groups.

Then for any injective map $\alpha : S \to T$ the induced map

$$A(T) \to A(S) \times_{B(S)} B(T)$$

is a surjective quasi-isomorphism.

Proof. For any $S \in \Delta^0\mathtt{Ens}$ the map $f(S) : A(S) \to B(S)$ is quasi-isomorphism — this follows from [Le], remark at the end of III.2, applied to [Ha], Thm. 1.5.1. This immediately implies that for any $\alpha : S \to T$ the induced map

$$A(T) \to A(S) \times_{B(S)} B(T)$$

is a quasi-isomorphism. Let us prove that it is surjective if α is injective.

Since A^d are contractible (and Kan), the map $A(\alpha) : A(T) \to A(S)$ is surjective. Thus, by Lemma 5.2.6, it suffices to check that the map

$$\mathrm{Ker}\, A(\alpha) \to \mathrm{Ker}\, B(\alpha)$$

is surjective.

Since the functors $A, B : \Delta^0\mathtt{Ens} \to C(\mathbb{Z})$ carry colimits to limits, the map $\mathrm{Ker}\, A(\alpha) \to \mathrm{Ker}\, B(\alpha)$ is a direct summand of the map $f(T/S) : A(T/S) \to B(T/S)$ which is surjective by (b). $\square$

Now we are able to prove the following criterion for a map of bisimplicial complexes of abelian groups to be an acyclic fibration.

5.2.8. **Proposition.** *Let $f : A \to B$ be a map in $(\Delta^0)^2 C^{\geq 0}(\mathbb{Z})$ satisfying:*

(a) for any $S, T \in \Delta^0\mathtt{Ens}$ the map $f(S, T)$ is surjective

(b) for any $S \in \Delta^0\mathtt{Ens}$, $p \in \mathbb{N}$ the map $f(S, \Delta^p) : A(S, \Delta^p) \to B(S, \Delta^p)$ is a quasi-isomorphism in $C(\mathbb{Z})$.

(c) for any $S \in \Delta^0\mathtt{Ens}$ and any d the simplicial abelian groups $A^d(S, \_)$, $B^d(S, \_)$ are contractible.

(d) for any p, d the simplicial abelian group $A^d(\_, \Delta^p)$ is contractible.

Then f is an acyclic fibration.

Proof. Apply Lemma 5.2.7 to the map $f(S, \_) : A(S, \_) \to B(S\_)$. We immediately get that for any $S \in \Delta^0\mathtt{Ens}$ and any injective map $\alpha : T \to T'$ the induced map

$$A(S, T') \to A(S, T) \times_{B(S,T)} B(S, T')$$

is a surjective quasi-isomorphism. Put $T' = \Delta^p,\ T = \partial\Delta^p$. The map $A_{np} \to B_{np} \times_{M_{np}(B)} M_{np}(A)$ is then automatically quasi-isomorphism, an we have only to check it is surjective. According to (d) the map $A(S', T') \to A(S, T')$ is surjective for $T' = \Delta^p$. Define

$$X(T) = \mathrm{Ker}(A(S', T) \to A(S, T)),\ Y(T) = \mathrm{Ker}(B(S', T) \to B(S, T)).$$

The groups $X(T)$ and $Y(T)$ are direct summands of $A(S'/S, T)$ and of $B(S'/S, T)$ respectively. Hence the map $X \to Y$ satisfies the hypotheses of Lemma 5.2.7. Therefore the map

$$X(T') \to X(T) \times_{Y(T)} Y(T')$$

is surjective by Lemma 5.2.6. Proposition is proven. □

6. Application to Deformation theory

In this Section we describe how to deduce from Corollary 4.1.1 the description of universal formal deformations for some typical deformation problems.

Consider, for example, three deformation problems which have been studied in [HS2].

Let X be a smooth separated scheme X over a field k of characteristic 0, G an algebraic group over k and $p : P \longrightarrow X$ a G-torsor over X. Consider the following deformation problems.

Problem 1. Flat deformations of X.

Problem 2. Flat deformations of the pair (X, P).

Problem 3. Deformations of P (X being fixed).

To each problem one can assign a sheaf of k-Lie algebras $\mathfrak{g}_i$ on X (these are the sheaves $\mathcal{A}_i,\ i = 1, 2, 3$ from *loc. cit.*, Section 8).

According to Grothendieck, to each problem corresponds a (2-)functor of infinitesimal deformations

$$F_i : \mathtt{art}/k \longrightarrow \mathtt{Grp}$$

from the category of local artinian k-algebras with the residue field k to the (2-)category of groupoids.

In each case, $\mathfrak{g}_i$ is "a sheaf of infinitesimal automorphisms" corresponding to F_i (in the sense of [SGA1], Exp.III, 5, especially Cor. 5.2 for Problem 1; for the other problems the meaning is analogous).

If X is affine, the functor F_i is equivalent to the Deligne groupoid $\mathcal{C}_L,\ L = \Gamma(X, \mathfrak{g}_i)$ defined as the functor

$$\mathcal{C}_L : \mathtt{art}/k \to \mathtt{Grp}$$

which is given by the formula

$$\mathcal{C}_L(A) = \mathcal{C}(\mathfrak{m} \otimes L)$$

where $\mathfrak{m}$ is the maximal ideal of A.

The deformation functor F_i defines a stack $\mathcal{F}_i$ in the Zariski topology of X. The Deligne functor defines a fibered category $\mathcal{C}_{\mathfrak{g}_i}$ which assigns a groupoid $\mathcal{C}_{\Gamma(U, \mathfrak{g}_i)}(A)$ to each Zariski open set U and to each $A \in \mathtt{art}/k$.

We have a canonical map of fibered categories $\mathcal{C}_{\mathfrak{g}_i} \to \mathcal{F}_i$ so that $\mathcal{F}_i$ is equivalent to the stack associated to the fibered category $\mathcal{C}_{\mathfrak{g}_i}$.

Using Corollary 4.1.1 we immediately get

6.1. **Corollary.** *The deformation groupoid F_i is naturally equivalent to the Deligne groupoid associated with the dg Lie algebra $\mathbf{R}\Gamma^{\mathrm{Lie}}(X, \mathfrak{g}_i)$.*

In particular, the following generalization of [HS2], Thm. 8.3 takes place.

6.2. **Corollary.** *Suppose that $H^0(X, \mathfrak{g}_i) = 0$; let $\mathfrak{S} = \mathrm{Spf}(R)$ be the base of the universal formal deformation for Problem i. Then we have a canonical isomorphism*

$$R^* = H_0^{Lie}(\mathbf{R}\Gamma^{Lie}(X, \mathfrak{g}_i))$$

where R^* denotes the space of continuous k-linear maps $R \longrightarrow k$ (k considered in the discrete topology) and H_0^{Lie} denotes the 0-th Lie homology.

Recall that this result has been proven in *loc. cit.* for $\mathfrak{S}$ formally smooth.

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