ALGEBRAIC GEOMETRY - EXERCISE 9

1. Prove that for $X = \mathtt{spec}(A)$ and for $\mathfrak{m}$ the maximal ideal corresponding to $x \in X$, the ring homomorphism $A \to \mathcal{O}_{X,x}$ identifies $\mathcal{O}_{X,x}$ with the localization $A_{\mathfrak{m}}$.
2. Prove that the natural map
$$S^{-1}\Omega(A) \to \Omega(S^{-1}A)$$
is an isomorphism.
3. Prove the following *prime avoidance lemma*: in a commutative ring A, if an ideal $\mathfrak{a}$ lies in the union $\cup_{i=1}^{n}\mathfrak{p}_i$ where $\mathfrak{p}_i$ are prime for $i > 2$ then $\mathfrak{a} \subset \mathfrak{p}_i$ for some i.

   I won't grade this; you can try proving it by induction in n or, if you are very busy, just read the proof in the Wikipedia.
4. A plane curve $C \subset \mathbb{A}^2$ is defined by the equation $y^2 = x(x-a)(x-b)$. When is the closure $\bar{C}$ of C in $\mathbb{P}^2$ smooth?
5. Let X be a hypersurface in $\mathbb{A}^n$ defined by the irreducible equation $f(x_1, \ldots, x_n) = 0$. Prove that X is smooth at $a = (a_1, \ldots, a_n)$ iff for some i the partial derivative $\frac{\partial f}{\partial x_i}$ does not vanish at a. *Hint: use the proof of 8.4.3.*
6. Conversely, let $\frac{\partial f}{\partial x_i}(x) = 0$ for all i for some $x \in X = V(f)$. Is x necessarily singular?
7. Complete the proof of 8.4.3: verify that the set of smooth points is open.