ALGEBRAIC GEOMETRY - EXERCISE 6

1. Let A, B be reduced algebras of finite type over $k = \bar{k}$. Prove that $A \times B$ is reduced. Describe $\mathtt{spec}_k(A \times B)$. Here $A \times B$ is the direct product of A and B that is the set of pairs with componentwise operations.
2. Let A be a reduced k-algebra of finite type of dimension 0. Prove that $A = k \times \ldots \times k$ (finite number of copies).
3. Prove that $\mathtt{spec}_k(A)$ is disconnected (that is can be presented as a disjoint union of two closed subsets) iff A contains a nontrivial idempotent $e \in A$.
4. Let $A = \oplus A_i$ be a reduced graded algebra of finite type over $k = \bar{k}$.
   a. Prove that $\mathtt{proj}_k(A)$ is empty iff $A = A_0$.
   b. Let $A = A_0[x]$ where x is a generator of degree 1. Prove that the natural map $\mathtt{proj}_k(A) \to \mathtt{spec}_k(A_0)$ is an isomorphism of varieties.
5. a. Let $f : A \to B$ be a (graded) homomorphism of reduced graded k-algebras such that $f_n : A_n \to B_n$ is an isomorphism for n big enough. Prove that f induces an isomorphism $\mathtt{proj}_k(B) \to \mathtt{proj}_k(A)$. (In particular, graded ring A cannot be reconstructed from $\mathtt{proj}_k(A)$).
   b. Let A be as above; define a new graded algebra $A^{(d)}$ by the formula $A^{(d)}_n = A_{dn}$. Prove that the map $A^{(d)} \to A$ induces an isomorphism of the projective spectra.
6. Let A be a reduced algebra of finite type over over $k = \bar{k}$ and let I be an ideal in A. Let $B \subset A[x]$ be the graded algebra defined by $B_k = I^k x^k$. Prove that the natural map $\pi : \mathtt{proj}_k(B) \to \mathtt{spec}_k(A)$ induces an isomorphism

$$\pi : \pi^{-1}(D(I)) \to D(I).$$