ALGEBRAIC GEOMETRY - EXERCISE 5

1. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of spaces with functions. Let $Y = \cup V_i$ be an open covering and $V_i = f^{-1}(U_i)$. Prove that if $f_{|U_i} : (U_i, \mathcal{O}_{U_i}) \to (V_i, \mathcal{O}_{V_i})$ are isomorphisms then f is an isomorphism. *(this means that being isomorphism of spaces with functions is a local property).*
2. Let $(X, \mathcal{O}_X)$ be a space with functions with $A = \mathcal{O}_X(X)$. For $f \in A$ we denote by X_f the open set $\{x \in X | f(x) \neq 0\}$. Assume that $f_1, \ldots, f_n \in A$ generate A and the spaces with functions X_{f_i} are affine. Prove that X is affine. *Hint: Study the canonical map* $X \to \mathtt{spec}_k(A)$.
3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be an affine morphism. Prove that preimage of any open affine subspace of Y is affine.
4. Prove that a composition of affine morphisms is affine; that a composition of finite morphisms is finite.
5. Let $f : X \to Y$ be a finite morphism of algebraic varieties. Prove that for any affine open $U \subset Y$ the ring $\mathcal{O}_X(f^{-1}(U))$ is a finitely generated module over $\mathcal{O}_Y(U)$.
6. A morphism $f : X \to Y$ is called quasi-finite if for any $y \in Y$ the fiber $f^{-1}(y)$ is finite (or empty). Give an example of surjective quasi-finite morphism that is not finite.