ALGEBRAIC GEOMETRY - EXERCISE 4

1. Let A be a local ring with the maximal ideal $\mathfrak{m}$ and let $k = A/\mathfrak{m}$. Prove that for any finite set of generators of an A-module M there is a generating subset having $\dim_k M/\mathfrak{m}M$ elements. Does the claim hold for infinitely generated M?
2. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.
3. Let S be a multiplicative system in A, $I \subset A$ an ideal and M an A-module. Construct an isomorphism
$$S^{-1}(M/IM) \to S^{-1}M/IS^{-1}M.$$
4. Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ is reduced. Show that A is reduced.
5. A multiplicatively closed subset S of a ring A is said to be saturated if $xy \in S$ implies $x \in S$ and $y \in S$. Prove that S is saturated iff $A \setminus S$ is a union of prime ideals.
6. Verify that the set S_0 of all non-zero-divisors in A is a saturated multiplicatively closed subset of A. Hence the set D of zero-divisors in A is a union of prime ideals.