ALGEBRAIC GEOMETRY - EXERCISE 3

1. Prove that a topological space is irreducible iff any its nonempty open subset is dense.
2. Let X be a noetherian topological space. Assume that X is Hausdorff (any two different points have open neighborhoods with empty intersection). Prove that X is a finite set with the discrete topology.
3. Let X be a topological space, Z an irreducible subspace (as a topological space with the induced topology). Prove that the closure $\bar{Z}$ of Z in X is also irreducible.
4. Let $Z \subset \mathbb{A}^3$ be given by the pair of equations $x^2 - yz = 0, xz - x = 0$. Show that Z has three irreducible components and find the corresponding prime ideals in the ring of regular functions.
5. Let A be a reduced finitely generated algebra over $k = \bar{k}$. Recall that the map $\texttt{spec}_k(A) \to \texttt{Spec}(A)$ carries an algebra homomorphism $x : A \to k$ to its kernel. Deduce from weak Nullstellensatz that the image of $\texttt{spec}_k(A)$ is dense in $\texttt{Spec}(A)$.
6. Let $f : A \to B$ be an algebra honmomorphism and let $f^* : \texttt{spec}_k(B) \to \texttt{spec}_k(A)$ be the corresponding morphism of affine varieties. Let $x \in \texttt{spec}_k(A)$ and let $\mathfrak{m} \subset A$ be the corresponding maximal ideal. Identify $f^{*-1}(x)$ with $\texttt{spec}_k(B/f(\mathfrak{m})B)$. What happens if $B/f(\mathfrak{m})B$ has nilpotents? Can this happen?