ALGEBRAIC GEOMETRY - EXERCISE 2

1. Let A be an algebra of finite type over $k = \bar{k}$, not necessarily reduced. Define $X = \mathtt{spec}_k(A)$ as for the reduced algebras: this is $\mathrm{Hom}_{\mathtt{Alg}_k}(A, k)$ as a set, with the topology defined by the open sets $D(S)$ and the algebras of functions $\mathcal{O}(U)$ defined as in the lecture. What is $\mathcal{O}_X(X)$?
2. Give an example of a reduced algebra A over $k = \bar{k}$ for which the construction above does not yield an isomorphism $A \to \mathcal{O}_X(X)$ (of course, A will not be a finitely generated algebra over k).
3. The same for a finitely generated algebra A over a field that is not algebraically closed.
4. Let $U = \mathbb{A}^1 \setminus \{0\}$. Define a topological space X as obtained by gluing two copies of $\mathbb{A}^1$ along the common part U. Define on X a structure of a space with functions so that the two embeddings of $\mathbb{A}^1$ into X are open subvarieties.
5. Prove that the map $\pi : \mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is a map of spaces with functions that identifies the regular functions on $U \subset \mathbb{P}^{n-1}$ with the regular functions on $\pi^{-1}(U)$ invariant with respect to the homothety for each $\lambda \neq 0$
$$(x_0, \ldots, x_n) \mapsto (\lambda x_0, \ldots, \lambda x_n)$$
6. Describe an initial object in the following category.
   - Its objects are pairs (f, x) where $f : A \to B$ is a homomorphism of commutative rings (A is fixed, B is any), and $x \in B$.
   - A morphism from (f, x) with $f : A \to B$, to (g, y) with $g : A \to C$ is given by a homomorphism $h : B \to C$ such that $g = h \circ f$ and $h(x) = y$.