ALGEBRAIC GEOMETRY - EXERCISE 13

1. Let X be a curve and F be a coherent sheaf on X. Prove that $\mathcal{R}(F) = F \otimes \mathcal{K}$ where $\mathcal{K}$ is the constant sheaf with value $k(X)$.
2. Prove that the skyscraper sheaf $\mathtt{Sky}_x(M)$ is quasicoherent iff any element $m \in M$ is annihilated by a power of the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{x,X}$.
3. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of sheaves on a curve X. Prove that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.
4. A locally free sheaf $\mathcal{F}$ of rank n on X can be described by gluing data, similarly to an invertible sheaf (see 10.2), but with $\theta_{ij} \in GL(n, \mathcal{O}_X(U_{ij}))$ instead of $\mathcal{O}^*(U_{ij})$. We define the invertible sheaf $\det(\mathcal{F})$ by the same affine open covering and the gluing data $\det \theta_{ij} \in \mathcal{O}^*(U_{ij})$. Prove that a short exact sequence
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
gives rise to an isomorphism $\det(\mathcal{F}) = \det(\mathcal{F}') \otimes \det(F'')$.
5. Let X be a complete curve. Deduce that if $\mathcal{F}$ has filtration
$$0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$
such that $\mathcal{F}_i/\mathcal{F}_{i+1}$ are invertible sheaves, $\deg(\det(\mathcal{F})) = \sum \deg(\mathcal{F}_i/\mathcal{F}_{i-1})$.
6. Let X be a complete curve and $\mathcal{F}$ a locally free sheaf of rang n. Show that
$$\chi(\mathcal{F}) = \deg(\det(\mathcal{F})) + n(1-g).$$