ALGEBRAIC GEOMETRY - EXERCISE 10

1. Let A be an integrally closed domain and let G be a finite group of automorphisms of A. Denote $B = A^G$ the subring of invariant elements of A. Prove that
   a. A is an integral extension of B.
   b. B is integrally closed.
2. (continuation) In the special case $A = k[x, y]$ and the group $\mathbb{Z}_2$ whose nontrivial element acts by $x \mapsto -x, y \mapsto -y$, prove that the resulting invariant subring B is not smooth. (Note that B is normal with $\dim(B) = 2$; this is the simplest Kleinian singularity).
3. Prove that the mormalization of $k[x, y]/(y^2 - x^3)$ is isomorphic to $k[z]$.
4. The same for $k[x, y]/(y^2 - x^2(x - 1))$.
5. Prove that $\mathbb{Z}[\sqrt{-5}]$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-5})$. Show that this ring is not UFD.