ALGEBRAIC GEOMETRY - EXERCISE 1

1. Denote by $V(I)$, $I \subset k[x_1, \ldots, x_n]$, the affine variety defined by the equations $f = 0$ for all $f \in I$. Prove that $V(I) \cup V(J) = V(IJ)$ where $IJ = \{fg | f \in I,\ f \in J\}$. Prove that $\cap V(I_i) = V(\cup I_i)$.
2. Let A be a commutative ring. Prove that all nilpotent elements of A form an ideal (it is called the nilradical of A and denoted $N(A)$). Give an example of noncommutative ring where nilpotent elements do not form an ideal.
3. For an ideal $I \subset A$ in a commutative ring its radical $\sqrt{I}$ is defined as $\{x \in A | \exists\, n : x^n \in I\}$. Verify that if $\rho : A \to A/I$ is the obvious homomorphism, $\sqrt{I} = \rho^{-1}(N(A/I))$. Prove that $\sqrt{I}$ is the smallest among the ideals $J \supset I$ such that A/J is reduced.
4. Describe precisely the correspondence between the points of the circle X given by the equation $x^2 + y^2 = 1$ and the points of $\mathbb{A}^1$:
   - what subsets of X and of $\mathbb{A}^1$ correspond to each other?
   - Does the above analysis work when k is a field of characteristic 2?
5. Prove that the only regular functions on $\mathbb{P}^n$ are the constants.
6. Given $f \in k[x, y]$ a polynomial of degree n, $X = V(f)$ and $\bar{X} \subset \mathbb{P}^2$ the projectivisation of X, what is the cardinality of $\bar{X} \setminus X$?
7. Let A be factorial (UFD) and $a \in A$. Prove that the ideal (a) is prime iff a is irreducible.