INTRODUCTION TO LIE GROUPS

Contents

1. Part 1: Introduction. Examples. Topological groups. $GL(n, \mathbb{R})$

1.1. **Historical remarks.** Sophus Lie (1842–1899) developed the theory of continuous transformation groups (now: Lie groups) in the end of 19 century.

The theory of finite groups has been developed approximately at the same time (slightly earlier). Sylow was Lie's friend that taught Galois theory at Christiania (Oslo) university and proved foundational Sylow theorems.

The mathematical notion of a group is a way to describe symmetries. Its earliest appearence is due to Galois theory. If $f \in \mathbb{Q}[x]$ is a polynomial and $t_1, \ldots, t_n \in \mathbb{C}$ its roots, The Galois group of f is defined as the group of automorphisms of the field $\mathbb{Q}(t_1, \ldots, t_n)$. Any such automorphism carries the roots t_i to roots, so the Galois group is a subgroup of permutations of t_i. Studying this group allows one to deduce properties of t_i (solvability).

Sophus Lie was interested in symmetries of differential equations where the symmetry groups are have often "continuous parameters". This leads to the notion of Lie group.

1.2. **Examples.** A Lie group is a group that has some extra geometric / topological properties. It takes time to grasp the formal definition — we start with some easy examples.

- $(\mathbb{R}^n, +)$.
- $GL(n, \mathbb{R})$ is a subset of matrices, that is a subset of $\mathbb{R}^{n^2}$. So this is not just a group but also a topological space.

- $SO(2,\mathbb{R})$ is a group of rotations of the 2-dimensional euclidean space, but also a circle.
- $SU(2)$, the collection of unitary matrices of determinant 1. A general element of $SU(2)$ has form
$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}$$
with $a,b,c,d\in\mathbb{R}$ satisfying the condition $a^2+b^2+c^2+d^2=1$. Thus, $SU(2)$ is a three-dimensional sphere.

1.3. **Topological groups.** We will start with a somewhat easier notion of a topological group.

A topological group is a topological space G endowed with a structure of group, that is a map $m: G\times G\to G$ satisfying the following properties (describing compatibility of two structures).

1. m is a continuous map. Recall that to understand this we should remember that $G\times G$ is also a topological space where open subsets are unions of the subsets $U\times V$ where U,V are open in G.
2. The map $G\to G$ carrying g to g^{-1} is also continuous.

1.3.1. Let us show that $GL(n,\mathbb{R})$ is a topological group. The multiplication is given by matric multiplication which is given by simple formulas $\sum_k a_{ik}b_{kj}$. This is obviously a continuous function.

The passage to inverse has also an explicit formula including $1/det(A)$ and the adjoint matrix A' whose entries are the minors. Since the determinant is a continuos function and $x\mapsto 1/x$ is continuous when $x\neq 0$, this proves continuity of the inverse for $GL(n,\mathbb{R})$.

1.3.2. *Exercise.* Prove that if G is a topological group and H is a subgroup of G then H is a topological group in the induced topology. Prove that the closure $\bar{H}$ is a subgroup in G.

1.3.3. *Exercise.* Let G be a topological group and $U\subset G$ an open neighborhood of $1\in G$. Prove that there exists an open neighborhood V of 1 such that $V\cdot V\subset U$.

1.3.4. Similarly to the definition of a topological group, one can define many other relatives of this notion, all of them would describe a group endowed with a certain type of geometrical structure.

For instance, instead of topological spaces and continuous maps, one can work with smooth manifolds and smooth maps (see details below). This will lead to the notion of a Lie group. Similarly, one can use algebraic varieties and algebraic maps; this leads to the notion of algebraic group.

We present below a formal definition that uses the language of categories. Let $\mathcal{C}$ be a category having finite products.

One defines a group object G in $\mathcal{C}$ as an object G in $\mathcal{C}$ together with a morphism $m : G \times G \to G$ (product), $e : t \to G$ (the unit, where t is the final object of $\mathcal{C}$ that exists as the product of the empty set of objects) and $i : G \to G$ (inverse) satisfying the following properties:

- m is associative that is two maps $G \times G \times G \to G$, $m \circ (\mathrm{id} \times m)$ and $m \circ (m \times \mathrm{id})$, coincide.
- The compositions $G \to G \times G \xrightarrow{m} G$ where the first arrow is $\mathrm{id} \times e$ or $e \times \mathrm{id}$, is identity. This is the unit axiom.
- The composition $G \to G \times G \xrightarrow{m} G$ coincides with $G \to t \xrightarrow{e} G$. Here the first arrow is either $\mathrm{id} \times i$ or $i \times \mathrm{id}$.

1.3.5. *Exercise.* Prove that the above definition of a group object in $\mathcal{C}$ is equivalence to the following.

1.3.6. **Definition.** A functor $G : \mathcal{C}^{\mathrm{op}} \to \mathtt{Grp}$ to groups is called a group object in $\mathcal{C}$ if the composition $\mathcal{C}^{\mathrm{op}} \xrightarrow{G} \mathtt{Grp} \to \mathtt{Set}$ with the forgetful functor $\mathtt{Grp} \to \mathtt{Set}$ is representable.

In this way one can get many meaningful notion of groups with extra structures, including Lie groups (that is, group objects in the category of smooth manifolds), group schemes (group objects in the category of affine schemes), algebraic groups (group objects in the category of algebraic varieties or, sometimes, of affine algebraic varieties), complex-analytic groups (group objects in the category of complex manifolds), etc.

1.4. **The notion of a Lie group.** Lie groups are, by definitions, groups that are simultaneously groups and smooth manifolds, the structures beling compatible in a way similar to the compatibility of the structures in the notion of topological group.

For instance, $GL(n)$ is an open subset of $Mat(n, \mathbb{R})$ that is $\mathbb{R}^{n^2}$. The maps $m : GL(n) \times GL(n) \to GL(n)$ and the inverse $GL(n) \to GL(n)$ are in fact smooth (we know what is smoothness for a map defined at an open subset of $\mathbb{R}^N$ to $\mathbb{R}^M$). This is what makes $GL(n)$ a Lie group. To define Lie groups in general, we should know well what is the natural context for the notion of smooth map. This is the context of smooth manifold. So, we will study some smooth manifold theory.

1.5. **Smooth manifolds.** A smooth manifold M is a topological space for which one can talk about smooth functions $f : U \to \mathbb{R}$ defined at an open set $U \subset M$. To talk about smoothness, it is nice to have coordinates, at least local coordinates. This leads to a definition of chart.

1.5.1. **Definition.**
- Let $U \subset M$ be an open subset. A chart on U is a continuous map $f : U \to \mathbb{R}^n$ for some n such that f defines a homeomorphism of U with $f(U)$ that is an open subset of $\mathbb{R}^n$.
- Two charts, $f : U \to \mathbb{R}^n$ and $g : V \to \mathbb{R}^m$ are compatible if the bijection $g \circ f^{-1} : f(U \cap V) \to g(U \cap V)$ is a diffeomorphism (that is is given by an invertible matrix of smooth functions, so that the inverse matrix has also smooth entries).

Note that if $U \cap V = \emptyset$ then any charts on U and V are compatible. If $U \cap V \neq \emptyset$ and the charts are compatible then necessarily $m = n$.

1.5.2. **Definition.** An atlas on a topological space M is a collection of compatible charts $f_\alpha : U_\alpha \to \mathbb{R}^n$ that covers M, $M = \cup U_\alpha$.

1.5.3. **Definition.** A smooth manifold M is a Hausdorff topological space with a countable base (of open subsets) with a chosen atlas. Two atlases on the same space M are called compatible if their union is also an atlas.

Note that if M is connected, the charts have all the same dimension called the dimension of M and denoted by $\dim(M)$. In general $\dim(M)$ is a locally constant function on M (constant on each component).

1.6. **Smooth maps. Product.** In order to define Lie groups, we still need two components. We have to define a smooth map between smooth manifolds. And we have to define the direct product of two smooth manifolds.

We do not distinguish manifold structures on the same space M defined by compatible atlases. Informally, his means that equivalent atlases determine the same manifold. More formally, we can say this in two ways: 1) We automatically add to the atlas all charts compatible to it. 2) After we define smooth maps between smooth manifolds, we will see that for a topological space endowed with two equivalent atlases, the identity map is an invertible smooth map (a diffeomorphism).

1.6.1. **Definition.** A continuous map $f : M \to N$ is smooth if for any pair of charts

$$a : U \to \mathbb{R}^m,\ b : V \to \mathbb{R}^n$$

with $U \subset M, V \subset N$, the map

$$a(U \cap f^{-1}(V)) \to \mathbb{R}^n$$

given by the composition $b \circ f \circ a^{-1}$ is smooth.

1.6.2. *Smooth functions.* In particular, given a manifold M and $U \subset M$ an open subset, we know what is a smooth function $f : U \to \mathbb{R}$. The set of smooth functions on U is a commutative ring denoted $C^\infty(U)$.

1.6.3. *Exercise.* A chart $a : U \to \mathbb{R}^n$ is compatible with an atlas on M iff for each open $V \subset U$ it establishes a bijection between $C^\infty(V)$ and $C^\infty(a(V))$, that is $f : a(V) \to \mathbb{R}$ is smooth iff the composition $f \circ a$ belongs to $C^\infty(V)$.

This means that the smooth structure on M is uniquely defined by the assignment $U \mapsto C^\infty(U)$.

1.6.4. *Exercise.* A map $f : X \to Y$ between two smooth manifolds is smooth iff for any open $U \subset Y$ and any $\phi \in C^\infty(U)$ the composition $f^{-1}(U) \xrightarrow{f} U \xrightarrow{\phi} \mathbb{R}$ is in $C^\infty(f^{-1}(U))$.

1.6.5. Given two smooth manifolds M and N, their cartesian product $M \times N$ has a structure of manifold. The charts for $M \times N$ are pairs of charts: if $a : U \to \mathbb{R}^n$ and $b : V \to \mathbb{R}^m$ are charts for $U \subset M$ and for $V \subset N$ respectively, $a \times b : U \times V \to \mathbb{R}^{m+n}$ gives a chart for $M \times N$. If $M = \cup U_\alpha$ and $N = \cup V_\beta$, $M \times N = \cup U_\alpha \times V_\beta$. Thus, we get an atlas for $M \times N$.

The projections $M \times N \to M$ and $M \times N \to N$ are obviously smooth.

1.6.6. **Lemma.** *Let M, N, K be three manifolds. A smooth map $K \to M \times N$ is uniquely defined by the pair of it compositions with the projections.*

Proof. Exercise. ☐

1.6.7. **Remark.** Exercises 1.6.3 and 1.6.4 show that the notion of a smooth manifold can be defined in the language of *spaces with functions*, as in G. Kempf, Algebraic manifolds.

1.7. **Lie groups.** We are now ready to give the main definition of this course.

1.7.1. **Definition.** A Lie group is a manifold G together with a binary operation $m : G \times G \to G$ that is a smooth map converting G into a group, so that the map $G \to G$ carrying $x \in G$ to x^{-1}, is smooth.

Recall that all charts of a connected manifold have the same dimension. It is called the dimension of a manifold. In general, if a manifold in not connected, different components may have different dimensions.

1.7.2. **Lemma.** *All components of a Lie group G have the same dimension.*

Proof. Since the multiplication is a smooth map, for any $g \in G$ the left multiplication by g, $L_g : G \to G$ is a smooth map. It is invertible, so it is a diffeomorphism. This implies that the dimension of the component of 1 is equal to the dimension of the component of $g \in G$. ☐

Zero-dimensional Lie groups are just the discrete (countable) groups.

1.7.3. *Lie group homomorphism.* Let G, H be Lie groups. A map $f : G \to H$ is a Lie group homomorphism if it is a group homomorphism and a smooth map. It is a Lie group isomorphism if it is a group isomorphism and a diffeomorphism.

Example: $f : \mathbb{R} \to U(1)$ given by the formula $f(x) = \exp(ix)$.

1.7.4. *Direct product.* Let G, H be Lie groups. Their product has a structure of a smooth manifold, as well as of a group. The structures are obviously compatible, so define a structure of a Lie group $G \times H$.

1.8. **Some remarks.** We discussed today a pletora of notions that are close relatives: topological groups, Lie groups, real-analytic groups and complex-analytic groups.

We will now say a few words about connections between these notions.

1.8.1. *Topological groups.*

1.8.2. It is not true that all topological groups are Lie groups. A typical example is $\mathbb{Z}_p$ or $\mathbb{Q}_p$ integer or rational p-adic numbers.

1.8.3. Fifth Hilbert problem asks: whether a topological group that is locally homeomorphic to an open disc, is a Lie group. The problem has been positively solved in 50-ies. This means that every such group admits a compatible smooth structure. Moreover, E. Cartan theorem that we will study in this course, implies that such Lie group structure is unique.

1.8.4. Any Lie group has a real-analytic structure. This follows from Campbell-Hausdorff formula that establishes a local isomorphism between a neighborhood of 1 in a Lie group G and a neighborhood of 0 in the corresponding Lie algebra.

1.8.5. Some Lie groups have a structure of a complex manifold. This structure needs not be unique: the complex tori $\mathbb{C}/\Gamma$ where Γ is a lattice (a discrete rank 2 subgroup) are isomorphic as Lie groups but not as complex manifolds (these are the elliplic curves).

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