SOLUTION OF PROBLEM 5, HW11

We will prove the following result.

0.0.1. **Proposition.** *A Lie group G is reductive iff*

1. $\mathtt{Lie}(G)$ *is a reductive Lie algebra.*
2. *The factor group $G/\overline{[G,G]}$ is compact.*

Proof. First of all, if G is reductive, the adjoint representation is completely reducible, so $\mathfrak{g} = \mathtt{Lie}(G)$ is reductive. The factor group $G/\overline{[G,G]}$ is commutative, so isomorphic to a product $\mathbb{R}^a \times U(1)^b$. If $a > 0$, there exist a surjective homomorphism $f : G \to \mathbb{R}$. This implies that G admits a non-semisimple module, that is G is not reductive. It remains to prove that if the conditions hold then any finite dimensional representation of G is completely reducible.

We have $\mathfrak{g} = \mathfrak{a} \times \mathfrak{s}$ where $\mathfrak{a}$ is commutative and $\mathfrak{s}$ is semisimple. Let $\tilde{G} = \mathbb{R}^a \times S$ be a simply connected Lie group with $\mathtt{Lie}(\tilde{G}) = \mathfrak{g}$, $a = \dim \mathfrak{a}$, $\mathtt{Lie}(S) = \mathfrak{s}$. Then $G = \tilde{G}/\Gamma$ for a discrete subgroup Γ of the center $Z(G) = \mathbb{R}^a \times Z(S)$.

Let $\rho : G \to GL(V)$ be a finite dimensional representation of G. We see it as a representation of $\tilde{G}$ on which Γ acts trivially.

Step 1. For any $z \in Z(S)$ $\rho(z)$ has finite order. In fact, V considered as a representation of S is a direct sum of irreducibles, $V = \oplus V_i$. The image of S in $GL(V_i)$ lies in $SL(V_i)$ as $S = [S,S]$, so the image of $Z(S)$ is in the scalar matrices of determinant 1. This is a finite group.

Step 2. The center $Z(G) = \mathbb{R}^a \times Z(S)$ acts on V semisimply. Denote by $A = \mathtt{Lie}(\rho) : \mathbb{R}^n \to \mathrm{End}(V)$ the Lie algebra action. For $(x,c) \in Z(G)$ we have $\rho(x,c) = \exp(A(x))\rho(c)$. For $(x,c) \in \Gamma$ we have $1 = \rho(x,c) = \exp(A(x))\rho(c) = 1$, so, as $\rho(c)$ has finite order, $\exp(A(x))$ has finite order, therefore, it is semisimple. Denote by X the image of Γ under the projection $\Gamma \to \mathbb{R}^a$. We see that $A(x)$ is semisimple for $x \in X$; by assumption 2 X generates $\mathbb{R}^a$, so all of the center acts semisimply.

Step 3. Now $Z(G)$ acts semisimply on V, so $V = \oplus V_\chi$ decomposes by characters $\chi : Z(G) \to \mathbb{C}^*$. Each V_χ is a G-submodule. Decomposing each V_χ as an S-module into a sum of irreducibles yields the decomposition of V.

□

0.1. Remarks and examples.

0.1.1. In the above notation, $[G,G]$ is the image of $[\tilde{G},\tilde{G}] = [S,S] = S$. However, the image of S in G is not necessarily closed. Here is an example.

$\tilde{G} = \mathbb{R} \times S$ where $S = \widetilde{SL}(2, \mathbb{R})$ so that the center of S is an infinite cyclic group generated by z. The group $\Gamma \subset \mathbb{R} \times \mathbb{Z}$ is generated by two elements $(1, 1)$ and (α, z). Thus G is the quotient of $\mathbb{R}/\mathbb{Z} \times S$ by the subgroup generated by $\alpha, z)$ where α is irrational, and sogenerates a dense subgroup of $\mathbb{R}/\mathbb{Z}$. The image of S is dense in G, so $G/\overline{[G, G]}$ is trivial.

0.1.2. It would be slightly easier to prove that if G has a compact center and $\mathfrak{g}$ is reductive then G is reductive (the idea is the same but we can present G is a quotient of $T \times S$ where T is a compact torus and then decompose V into characters of T).

The map $Z(G) \to G/\overline{[G, G]}$ is surjective, so compactness of the center implies compactness of the factor. However, the following example shows that compactness of $G/\overline{[G, G]}$ does not imply compactness of $Z(G)$.

0.1.3. *Example.* Now $\tilde{G} = \mathbb{R} \times S$ as above, and $G = \tilde{G}/\Gamma$ where Γ is generated by $(1, z)$. We have $Z(G) = \mathbb{R}$ and $G/\overline{[G, G]} = \mathbb{R}/\mathbb{Z}$.