LIE GROUPS, HOME ASSIGNMENT 3

1. Describe the Lie algebra of the linear group $U \subset GL(n, \mathbb{R})$ of upper-triangular matrices with the units at the diagonal. And for the group B of all invertible upper-triangular matrices?
2. Let $\mathfrak{g}$ be a Lie algebra (or a commutatative algebra, or an associative algebra...) A linear endomorphism $d : \mathfrak{g} \to \mathfrak{g}$ is called a derivation if it satisfies the Leibniz rule
$$d([x, y]) = [d(x), y] + [x, d(y)].$$
The space of derivations of $\mathfrak{g}$ is denoted by $\mathrm{Der}(\mathfrak{g})$. Verify that $\mathrm{Der}(\mathfrak{g})$ a Lie algebra with respect to the operation $[f, g] = f \circ g - g \circ f$. (Note that the composition of two derivations is usually not a derivation).
3. Let $\mathfrak{g}$ be a Lie algebra. Prove that the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ carrying $x \in \mathfrak{g}$ to $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_x(y) := [x, y]$, defines a homomorphism of Lie algebras $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$.
4. Prove that $\mathrm{ad} : \mathfrak{sl}_2 \to \mathrm{Der}(\mathfrak{sl}_2)$ is an isomorphism.
5. Let $\mathfrak{g}$ be a Lie algebra. Prove that the group of automorphisms of $\mathfrak{g}$ is a linear Lie group. Prove that the Lie algebra of this group is the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of derivations.