LIE GROUPS, HOME ASSIGNMENT 11

1. Prove that a complex non-commutative Lie algebra is not compact (when considered as a real Lie algebra).
2. Let G be a simply connected Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Construct a Lie group homomorphism $f : G \to SL(2,\mathbb{C})$ that induces a Lie algebra embedding $\mathtt{Lie}(f) : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{C})$. Calculate the kernel of f.
3. (continuation of 2) Prove that for any finite dimensional complex representation $\rho : G \to GL(n,\mathbb{C})$ the corresponding Lie algebra representation factors through $\mathtt{Lie}(f)$. Deduce that the kernel of f acts trivially on any representation of G, so that G is not linear.
4. Prove that $\phi = \sqrt{\delta}$ as at the end of the proof of 12.5.4, is a Lie algebra automorphism.
5. We call a Lie group G reductive if any its finite dimensional representation is completely reducible. Is any reductive group necessarily compact? Find a caracterisation of connected reductive Lie groups similar to Proposition 12.1.