LIE GROUPS, HOME ASSIGNMENT 10

1. Present the Lie algebra of $U(n)$ as a product of a commutative and a semisimple Lie algebra. Show that $U(n)$ has no compatible decomposition.
2. Given a finite dimensional representation $\rho : G \to GL(V)$, its character χ_V is the function on G defined by the formula
$$\chi_V(g) = \mathrm{Tr}(\rho(g)).$$
Prove that if V and W are nonisomorphic irreducible modules then χ_V and χ_W are orthogonal.
3. Prove that V is irreducible iff χ_V has length 1. Deduce that if $V = \oplus V_i^{d_i}$ where V_i are non-isomorphic irreducible representations, then $||\chi_V||^2 = \sum d_i^2$.
4. Can the Killing form of a Lie algebra $\mathfrak{g}$ be positively definite?