LIE GROUPS, HOME ASSIGNMENT 1

1. Prove that if G is a topological group and $H \subset G$ a subgroup, H is a topological group in the induced topology. Prove as well that the closure $\bar{H}$ is a closed (topological) subgroup of G.
2. Let G be a topological group and $U \subset G$ an open neighborhood of 1. Then there exists an open neighborhood V of 1 such that $V \cdot V \subset U$.
3. Describe all closed subgroups of $(\mathbb{R}, +)$.
4. Write down an explicit atlas for a sphere S^n, consisting of two charts.
5. Let M be a manifold of dimension n, $U \subset M$ an open subset. A chart $a : U \to \mathbb{R}^n$ is compatible with the atlas on M iff for each $V \subset U$ it establishes a bijection between $C^\infty(V)$ and $C^\infty(a(V))$, that is, if $f : a(V) \to \mathbb{R}$ is smooth iff the composition $f \circ d$ belongs to $C^\infty(V)$.
6. A map $f : X \to Y$ between two smooth manifolds is smooth iff for any open $U \subset Y$ and any $\phi \in C^\infty(U)$ the composition $f^{-1}(U) \xrightarrow{f} U \xrightarrow{\phi} \mathbb{R}$ is in $C^\infty(f^{-1}(U))$.
7. Prove that the groups $GL(n, \mathbb{R})$ and $U(n)$ are closed subgroup of $GL(n, \mathbb{C})$.
8. Construct isomorphisms (of topological groups) $\mathbb{R}^* \to \mathbb{R}^*_0 \times \mathbb{Z}_2$, $\mathbb{C}^* \to \mathbb{R}^*_0 \times SO(2)$ where $\mathbb{R}^*_0$ is the connected component of $1 \in \mathbb{R}^*$.
9. Prove that the group $SL(n, \mathbb{R})$ is a closed subgroup of $GL(n, \mathbb{R})$.
10. Recall that the algebra of quaternions $\mathbb{H} = \mathrm{Span}_{\mathbb{R}}\{1, i, j, k\}$ is defined by the formulas

    $$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

    $\mathbb{H}$ admits an involution carrying $z = a + bi + cj + dk$ to $\bar{z} = a - bi - cj - dk$. We define $|z|^2 = z\bar{z}$. Verify that this is a multiplicative norm and therefore that $\{z \in \mathbb{H} |\ |z| = 1\}$ is a topological group. Verify that it is isomorphic to $SU(2)$.