BASIC ALGEBRA - EXERCISE 5

1. Let $f : A \to B$ be a ring homomorphism. Prove that if J is a prime ideal in B, $f^{-1}(J)$ is a prime ideal in A.
2. Recall that for an ideal I of a commutative ring A its radical $\sqrt{I}$ is defined as the set of $x \in A$ whose certain power x^n belongs to I. Prove that $I = \sqrt{I}$ iff I is an intersection of prime ideals.
3. Let N be the nilradical of A. Prove that the following properties are equivalent.
   - A has only one prime ideal.
   - Every element of A is either invertible or nilpotent.
   - A/N is a field.
4. Let A be a local ring and M a f.g. A-module. A collection of elements $x_1, \ldots, x_n$ is called a minimal system of generators if
   - The set $x_1, \ldots, x_n$ generates M.
   - Any proper subset of $x_1, \ldots, x_n$ does not generate M.

   Prove that any two minimal systems of generators have the same number of elements.
5. Prove that if $T \supset S$ are two multiplicative systems, then $T^{-1}M$ is the module of fractions of $S^{-1}M$ with respect to T.
6. Prove that if A has no zero divisors and $0 \notin S$ then $S^{-1}A$ has no zero divisors.