BASIC ALGEBRA - EXERCISE 3

1. Let A be a ring, M an A-module, $N \subset M$ a submodule. Prove that if N and M/N are f. g., then M is also f. g.
2. Let $A = k[x_1, x_2, \ldots]$ be the ring of polynomials in a countable number of variables (each polynomial depends of a finite number of them). Find a f. g. module M over A having a submodule that is not f. g.
3. Describe all simple A-modules, where A is a PID. Describe all simple $k[x]$-modules where $k = \mathbb{C}$ or $k = \mathbb{R}$.
4. An A-module M is called indecomposable if it cannot be presented as a nontrivial direct sum of its submodules. Describe all indecomposable modules over $\mathbb{C}[x]$ that are finite dimensional vector spaces.
5. Give an example of an indecomposable $\mathbb{R}[x]$-module V of dimension 4 such that x considered as an endomorphism of V, has no eigenvectors.
6. Let A be a PID, M an injective finitely generated module. Prove that $M = 0$.
7. Let A be a PID, K the field of fractions, $K = \{\frac{a}{b} | a, b \in A,\ b \neq 0\}$. Prove that K and K/A are injective modules. Prove that any f.g. A-module can be embedded into a finite direct sum of a number of copies of K and of K/A.