BASIC ALGEBRA - EXERCISE 1

1. Prove that any short exact sequence of vector spaces is split.
2. Give an example of a $\mathbb{Z}$-module having no simple submodule.
3. Give an example of an epimorphism of $\mathbb{Z}$-modules that is not split.
4. Let $A = M_n(F)$ a matrix ring over a field F, $M = F^n$ the vector space of length n columns. Verify that M is a left A-module generated by one element (a cyclic module).
5. Let A and M be as in Question 4. Prove that A considered as a left A-module, is isomorphic to a direct sum of a number of coplies of M. Verify that M is a simple A-module.
6. (Weyl algebra)Let $V = F[x]$ ring of polynomials over a field F. Let $E = End_F(V)$ be the ring of linear endomorphisms of V (a sort of infinite matrices). We define two elements $X, D \in E$ as follows: X is a multiplication by x, so that $X(x^n) = x^{n+1}$. D is the derivative, that is $D(x^n) = nx^{n-1}$. We define A as the span of all monomials X^nD^m where $m, n \geq 0$.

   Prove that A is a subring of E. To do so, one has to verify that the product $X^nD^mX^{n'}D^{m'}$ can be expressed as a linear combination of X^kD^l.

   Hint. Use induction, starting from $DX = XD + 1$.
7. Let H be the quaternion algebra, that is, an algebra over $\mathbb{R}$ with the basis $\{1, i, j, k\}$ such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Prove that $H_{\mathbb{C}}$ defined as the $\mathbb{C}$ algebra with the same generators and the multiplication given by the same formulas, is isomorphic to the ring of matrices $M_2(\mathbb{C})$.