RIEMANN SURFACES

10. Weeks 11–12: Riemann-Roch theorem and applications

10.1. **Divisors.** The notion of a divisor looks very simple. Let X be a compact Riemann surface. A divisor is an expression

$$\sum_{x\in X} a_x x$$

where the coefficients a_x are integers, and only a finite number of them are nonzero.

The set of all divisors on X forms an abelian group denoted $Div(X)$. This is a free abelian group generated by the points of X.

Let $\mathcal{M}$ denote the field of meromorphic functions on X. Any $0 \neq f \in \mathcal{M}$ defines a divisor

$$(f) = \sum_x \operatorname{ord}_x(f) \cdot x$$

where $\operatorname{ord}_x(f)$ is the degree of zero of f at x or the negative degree of pole at x. Such a divisor is called *principal*.

The group of classes of divisors $Cl(X)$ is defined as the quotient of $Div(X)$ by the subgroup of principal divisors. One has a short exact sequence

$$0 \longrightarrow \mathcal{M}^* \longrightarrow Div(X) \longrightarrow Cl(X) \longrightarrow 0.$$

A degree of a divisor $\sum a_x x$ is $\sum_x a_x \in \mathbb{Z}$. The degree of a principal divisor is zero since any nonvanishing meromorphic function has any value the same number of times (taking into account multiplicities).

Therefore, one has a canonical map

$$\deg : Cl(X) \longrightarrow \mathbb{Z}.$$

A divisor of zeroes $f^{-1}(0)$ of $f \in \mathcal{M} - \mathbb{C}$ is defined as

$$\sum_x \max(0, \deg_x f)x.$$

The class of $f^{-1}(0)$ in $Cl(X)$ coincides with the class of the divisor of any other value $f^{-1}(c)$ including $c = \infty$.

Let ω be a nonzero meromorphic differential. We define its class

$$(\omega) = \sum \operatorname{ord}_x(\omega)x.$$

since any two meromorphic differentials differ by a meromorphic function, one immediately deduces that the class of (ω) in $Cl(X)$ does not depend on ω. It is called *the canonical class of* X and is denoted $K \in Cl(X)$.

We say that a divisor $D \geq 0$ if $D = \sum a_x x$ with $a_x \geq 0$. This defines a partial order on $Div(X)$.

Let $D \in Div(X)$. We define

$$L(D) = \{f \in \mathcal{M} | (f) \geq D\}.$$

(We assume for convenience that (0) is greater than any divisor.) The dimension of $L(D)$ is called *the dimension of* D; it is denoted $r(D)$.

The following properties of the function r are easy.

10.1.1. **Proposition.** 1. $r(0) = 1$.
2. *If* $D \geq D'$ *then* $L(D) \subset L(D')$ *so that* $r(D) \leq r(D')$.
3. $\deg(D) > 0$ *implies* $r(D) = 0$.

Proof. $L(0)$ is the space of holomorphic functions. Since X is compact, $L(0)$ consists of constants.

The second claim is obvious. If $D > 0$, $L(D)$ consists of holomorphic functions having at least one zero. Thus, $L(D) = 0$. □

10.1.2. **Definition.** Index of specialty $i(D)$ of a divisor D is the dimension of the space

$$\Omega(D) = \{\omega \text{ meromorphic } |(\omega) \geq D\}.$$

10.1.3. **Proposition.** *The numbers* $r(D)$ *and* $i(D)$ *depend only on the class of* D *in* $Cl(X)$. *Moreover, one has*

$$i(D) = r(D - K)$$

where K *is the canonical class of* X.

Proof. If D and D' are equivalent, there exists $f \in \mathcal{M}^*$ such that $D' = D + (f)$. Then the multiplication by f defines an isomorphism

$$L(D) \longrightarrow L(D').$$

To check the second claim, let ω be a nonzero meromorphic differential on X.

Then the isomorphism $L(D - K) \to \Omega(D)$ is given by a multiplication by ω. □

Finally, the space $\Omega(0)$ is the space of holomorphic differentials $\mathcal{H}$. Therefore, $i(0) = g$.

We are now ready to formulate the famous

10.1.4. **Theorem** (Riemann-Roch for Riemann surfaces)**.** *Let* g *be the genus of a compact Riemann surface* X. *Then for any divisor* D *one has*

$$r(-D) = \deg(D) - g + 1 + i(D).$$

Let us just note that if $D=0$, the theorem just says that $i(0)=g$ which we already know.

10.2. **Some applications.**

10.2.1. **Proposition.** *Let X be compact of genus zero. Then X is isomorphic to the Riemann sphere.*

Proof. Chose a point $x \in X$ and apply RR to the divisor $D=x$. We get

$$r(-D)=2+i(D)$$

and $i(D)=0$ since there are no holomorphic forms on X. Thus, $r(-D)=2$, so that there exists a nonconstant meromorphic function on X having a simple pole at x. This function establishes an isomorphism of X with $\widehat{\mathbb{C}}$. □

10.2.2. **Proposition.** $\deg K = 2g-2$.

Proof. We do not want to give a proof based on RR since this fact will be used in the proof. We just mention it for the reference in Gap Theorems below. □

10.2.3. **Theorem** (Weierstrass gap theorem)**.** *Let X be compact of genus $g>0$ and let $x \in X$. There exist precisely g numbers*

$$1=n_1<n_2<\ldots<n_g<2g$$

such that there does not exist a holomorphic function on $X-x$ with a pole of order n_i at x.

A more general theorem will be proven. Let $x_1, x_2, \ldots$ be an infinite sequence of points of X of genus $g>0$. Define the divisors $D_k=\sum_{i=1}^k x_i$. We will say that a number k is exceptional if $L(-D_k)=L(-D_{k-1})$. Note that if $x_i=x$ a number k is exceptional if and only if there are no meromorphic functions on X with the only pole at x of order k.

10.2.4. **Theorem** (Noether gap theorem)**.** *There are precisely g exceptional integers for any sequence of points. $k=1$ is always exceptional and all exceptional integers are less than $2g$.*

Proof. The number 1 is always exceptional since $g>0$ and therefore there are no meromorphic functinos having one simple pole.

By RR

$$r(-D_k)-r(-D_{k-1})=1+i(D_k)-i(D_{k-1})\le 1.$$

Thus, the difference is always 0 or 1, so the number of non-exceptinal values $\le k$ is

$$r(-D_k)-r(-D_0)=k+i(D_k)-i(D_0)$$

or

$$r(-D_k)-1=k+i(D_k)-g.$$

If $k > 2g-2$ the degree of D_k is greater than $2g-2$ which is the degree of the canonical divisor. Thus, $i(D_k) = 0$ and we get $k-g$ non-exceptional integers $\leq k$. This means there are precisely g exceptional integers all they are all $\leq 2g-1$. $\square$

10.3. **RR in the positive case.** We will prove first that RR holds for $D \geq 0$. The case $D = 0$ being settled before, we may assume $D > 0$.

Let $D = \sum_{i=1}^m n_i x_i$ where n_i are integers > 0. Denote

$$D^+ = \sum_{i=1}^m (n_i + 1)x_i$$

and define

$$\Omega_0(-D^+)$$

as the space of meromorphic differentials of second kind (that is, with zero residues) whose divisor is $\geq -D^+$ and having zero a-periods.

The spaces $L(-D)$ and $\Omega_0(-D^+)$ are connected by the de Rham differential

$$d : L(-D) \longrightarrow \Omega_0(-D^+) \tag{1}$$

since the differential of a meromorphic function gives automatically a differential having zero residues and zero a-periods s (as well as the b-periods). The kernel of d is one-dimensional (the constants), so we can calculate the dimension of $L(-D)$ studying the image of d.

We claim that $\dim \Omega_0(-D^+) = \sum n_i = \deg(D)$. In fact, choose disjoint coordinate neighborhoods near the points $x_i \in X$ for each $i = 1, \ldots, m$. Recall that for each $2 \leq j \leq n_i + 1$ a meromorphic differential $\tau_{x_i}^{(j)}$ was constructed, having an only pole of order j at x_i. Since the functions $\tau_{x_i}^{(j)}$ are linearly independent, we deduce that $\dim \Omega_0(-D^+) \geq \sum n_i = \deg(D)$. We will show that the functions $\tau_{x_i}^{(j)}$ actually span the whole $\Omega_0(-D^+)$. In fact, define a linear map

$$\Omega_0(-D^+) \longrightarrow \mathbb{C}^{\deg(D)} \tag{2}$$

assigning to each meromorphic differential ω its Laurent coefficients in the chosen coordinate systems at x_i (note that -1st coefficients vanish since ω is assumed to have no residues). The kernel of (2) consists of regular differentials having zero a-periods, therefore, the map (2) is injective.

The image of d in (1) coincides with the space of differentials having zero b-periods. Thus, the image is given by g linear equations in $\Omega_0(-D^+)$ which yields an inequality

$$r(-D) = \dim \operatorname{Im}(d) + 1 \geq \deg(D) - g + 1. \tag{3}$$

called *Riemann inequality* (this is a weaker form of RR theorem).

In order to get a precise formula, we have to express the rank of the matrix composed of the coefficients of the g equations defining the image of the de Rham differential in (1). If $\widetilde{g}$ is actual value of the rank, we will have

$$r(-D) = \deg D - \widetilde{g} + 1.$$

iRecall that the space $\Omega_0(-D^+)$ is spanned by the elements $\tau_{x_i}^{(j)}$ with $i = 1, \dots, m$ and $j = 2, \dots, n_i + 1$ and that the image of d is define in $\Omega_0(-D^+)$ by the conditions that the b-differentials vanish.

Recall the formula for the b-differentials of τ's.

$$\int_{b_j} \tau_{x_i}^{(k)} = \frac{2\pi i}{k-1} \alpha_{i,k-2}^{(j)}, \tag{4}$$

where the coefficients α are defined by the decomposition of the basis elements ζ_j of the space of holomorphic differentials dual to the a-basis, as follows

$$\zeta_j = (\sum_{s=0}^{\infty} \alpha_{i,s}^{(j)} z^s) dz \text{ at } x_i \in X. \tag{5}$$

Thus, the equations defining our subspace of $\Omega_0(-D^+)$ are given by the matrix with entries

$$\frac{2\pi i}{k-1} \alpha_{i,k-2}^{(j)}, \tag{6}$$

where the upper index $j = 1, \dots, g$ denotes the row number, and the pair of lower indices $(i, k-2)$, with $i = 1, \dots, m$ and $k = 2, \dots, n_i + 1$, denotes the column number.

Note that the coefficient $\frac{2\pi i}{k-1}$ has no influence on the rank of the matrix. By formula (5) the transpose of this matrix defines a linear map

$$T : \Omega(0) \to \mathbb{C}^{\deg(D)}$$

assigning to any regular differential the Taylor coefficients of its decomposition near x_i. The kernel of T is the space $\Omega(D)$ so we finally have

$$\widetilde{g} = g - i(D)$$

or

$$r(-D) = \deg D - g + 1 + i(D).$$

Riemann-Roch formula is proven for nonnegative divisors.

10.4. **Some consequences.** Note that the positive part of RR is already sufficient to deduce that a compact genus 0 Riemann surface is isomorphic to the Riemann sphere. In fact, in the proof of the assertion we applied RR to $D = x$, a single point in X. This is a positive divisor.

10.4.1. **Lemma.** *The degree of a canonical divisior is $2g - 2$.*

Proof. If $g = 0$, we have a Riemann sphere. Choose $\omega = dz$; this is a meromorphic divisor having an only pole at $z = \infty$ of degree 2.

If $g > 0$ X admits a nonzero holomorphic differential, say, ω. Since ω is holomorphic, (ω) is positive. Apply RR to $D = (\omega)$. We get

$$r(-D) = \deg D - g + 1 + i(D)$$

We know that $r(-D) = i(0)$ and $i(D) = r(0) = 1$, so

$$\deg D = 2g - 2.$$

□

10.5. **Bootstraping.** Note that the RR formula depends only on the class of D in $Cl(X)$. Therefore, we get that RR holds for any D equivalent to a nonnegative divisor.

Another idea: the RR formula is invariant with respect to substitution $D \mapsto K - D$. In fact, we know that $i(D) = r(D - K)$ so that if RR holds for D we have

$$i(K - D) = deg(D) + 1 - g + r(D - K)$$

or

$$r(-(K - D)) = -deg(D) - 1 + g + i(K - D) = deg(K - D) + 1 - g + i(K - D).$$

Thus, we deduce that RR holds as well for any divisor equivalent to $K - D$ where D is nonnegative.

Note that if $r(-D) > 0$ then there exists a meromorphic function f such that $(f) \geq -D$ which means that D is equivalent to $(f) + D \geq 0$, so in this case RR holds for D.

Symmetrically, if $i(D) > 0$ then $K - D$ is equivalent to a nonnegative divisor. Thus, the only remaining case implies $r(-D) = i(D) = 0$. It remains, therefore, to prove

10.5.1. **Proposition.** *Assume neither D nor $K - D$ is equivalent to a nonnegative divisor. Then* $\deg(D) = g - 1$.

Proof. Write $D = D_1 - D_2$ where D_i are positive and have disjoint supports.

We know $\deg(D) = \deg(D_1) - \deg(D_2)$. By RR applied to D_1 we get

$$r(-D_1) \geq \deg(D_1) + 1 - g = \deg(D) + \deg(D_2) + 1 - g.$$

Let us assume $\deg(D) > g - 1$. Then the above inequality yields

$$r(-D_1) > \deg(D_2).$$

In other words, the space of meromorphic functions f satisfying $(f) - D_1 \geq 0$ has dimension strictly greater than $\deg(D_2)$.

Such functions are holomorphic at the support of D_2 since the supports of D_i are disjoint. Therefore, the requirements $\deg_x(f) \geq \deg_x(D_2)$ give $\deg(D_2)$ linear equation on f. Therefore, we deduce that there exists a meromorphic function f belonging to $L(-D_1 + D_2) = L(-D)$. Contradiction — with the assumption $\deg(D) > g - 1$. Therefore, we deduce that

$$\deg(D) \leq g - 1.$$

Appying the same reasoning to $K - D$ instead of D, we get the opposite inequality.

Proposition (and, therefore, the general form of RR) is proven. □

10.6. **Some more applications.**

10.6.1. Genus one

Let X be a compact RS of genus one. We know that X admits a holomorphic differential. The corresponding divisor is nonnegative and has degree $2g - 2 = 0$, so it is trivial. Fix this differential ω; it is unique up to multiplicative constant and it has no zeroes. The integrals $\int_a \omega$ and $\int_b \omega$ generate a lattice L in $\mathbb{C}$ and the expression

$$f(x) = \int_{x_0}^{x} \omega$$

define a holomorphic map from X to $\mathbb{C}/L$. It is obviously invertible since ω has no zeroes, so any genus 1 Riemann surface is an elliptic curve.

10.6.2. m-differentials

Take care: here is a small terminological problem: we say differentials instead of one-forms. Now we will define m-differentials which will not be m-forms!

A meromorphic one-form is defined by an assignment of an expression $f(z)dz$ to each local chart so that it changes properly under the change of coordinates: $f(z)dz \mapsto f(z(w))z'_w dw$. Similarly, a meromorphic m-differntial is an expression $f(z)(dz)^m$ which changes properly with the change of coordinates, so that

$$f(z)(dz)^m \mapsto f(z(w))(z'_w)^m (dw)^m.$$

Any two m-differentials differ by a meromorphic function, so they define the same element in $Cl(X)$. It is clear that this element is just mK.

Assume $m > 1$ and $D = mK$. Assume as well that $g > 1$. We have by RR

$$r(-mK) = m(2g - 2) - g + 1 + i(mK) = (2m - 1)(g - 1)$$

since $i(mK) = 0$ — the divisor of a differential cannot be greater then the divisor of m-differential since $degK = 2g - 2$ and $\deg mK = m(2g - 2) > 2g - 2$.

10.7. **More on Weierstrass gaps.** Recall that for each $x \in X$ there are precisely g "gaps" between 1 and $2g - 1$. Thus, there are precisely g "non-gaps" between 1 and $2g$, which we will denote

$$1 < \alpha_1 < \ldots, \alpha_g = 2g.$$

10.7.1. **Proposition.** *For any $j = 1, \dots, g$ one has*

$$\alpha_j + \alpha_{g-j} \geq 2g$$

.

Proof. Recall that the number of gaps and of non-gaps between 1 and $2g$ is the same. Moreover, the sum of non-gaps is a non-gap. Thus, if $\alpha_j + \alpha_{g-j}$ were less than $2g$ the same was true for $\alpha_k + \alpha_{g-j}$ for all $k < j$ which would give at least j non-gaps strictly between α_{g-j} and α_g. This would give altogether more non-gaps than one can have: $g - j + j + 1 > g$. ☐

10.7.2. **Corollary.** *If $\alpha_1 = 2$, then $\alpha_k = 2k$ for all $k = 1, \dots, d$.*

In fact, if 2 is a non-gap, all even integers are non-gaps; since there are g even numbers between 1 and $2g$, these are all non-gaps.

10.7.3. **Lemma.** *If $\alpha_1 > 2$ than $\alpha_j + \alpha_{g-j} > 2n$ for some j.*

Proof. If $g = 2$ the only possibility is $(3, 4)$ and there is nothing to prove.

If $g = 3$ the possibilities are $(3, 4, 6)$ and $(3, 5, 6)$ and again there is nothing to prove.

Assume now $g \geq 4$. Assume that $\alpha_j + \alpha_{g-j} = 2g$ for all j. Consider the sequence $\beta_j = j \cdot \alpha_1$. The number of β_j in the range up to $2g$ is strictly less than g; therefore, there exists an non-gap between the β's. Let α be the first such non-gap, $\beta_r < \alpha < \beta_{r+1}$. Then $\alpha = \alpha_{r+1}$, $\alpha_{g-r-1} = 2g - \alpha$, $\alpha_j = j\alpha_1$ and $\alpha_{g-j} = 2g - j\alpha_1$ for $j \leq r$.

We deduce that

$$\alpha_1 + \alpha_{g-r-1} = \alpha_1 + 2g - \alpha > 2g - r\alpha_1 = \alpha_{g-r}.$$

The latter means that we have found a non-gap $\alpha_1 + \alpha_{g-r-1}$ greater than α_{g-r-1} but not among α_{g-j}, $j \leq r$. Contradiction. ☐

10.7.4. **Corollary.** *One has*

$$\sum_{i=1}^{g-1} \alpha_i \geq g(g-1),$$

with the equality iff $\alpha_1 = 2$.

Proof. The inequality follows from the inequality $\alpha_j + \alpha_{g-j} \geq 2g$. If $\alpha_1 = 2$, the inequality becomes equality. Otherwise Lemma 10.7.3 ensures the inequality is strict. ☐

10.8. **Wronskian.** Let $f_1, \ldots, f_n$ be holomorphic functions in an open subset $U \subset X$. Define the Wronskian by the formula

$$W(f_1, \ldots, f_n) = \det(f_i^{(j-1)}),$$

where $f^{(j)}$ denotes $j-$th derivative of f.

Since f_i are holomorphic, their Wronskian is holomorphic as well. We want to know the degree of W at $x \in U$. It can be calculated as follows.

Let V be the vector space of functions generated by f_i. We define a sequence of nonnegative integers $d_1 < d_2 < \ldots < d_n$, depending on x, as follows.

10.8.1. **Definition.** The sequence $d_1 < \ldots < d_n$ is defined uniquely by the condition

$$\{d_1, \ldots, d_n\} = \{\deg_x f | f \in V\}.$$

Note that the above definition contains a claim which needs to be proven. In fact, it is claimed that the number of different values of $\deg_x(f)$ coicides with $\dim V$. This is actually an easy exercise: choose $f_1 \in V$ to have the minimal degree and let V^+ be the space of functions in V whose degree is strictly greater than $\deg_x f_1$. Then obviously $V = \mathbb{C} \cdot f_1 \oplus V^+$ so $\dim V^+ = \dim V - 1$ and we get the claim by induction.

10.8.2. **Lemma.** *One has*

$$\deg_x W = \sum_{i=1}^{n} (d_i - i + 1).$$

Proof. Linear transformation of f_i multiplies W by a constant, therefore, it does not change the degree of W. Therefore, we can assume that $\deg_x f_i = d_i$. Choose a coordinate near x so that x corresponds to $z = 0$. Let

$$f_1 = z^{d_1} + az^{\delta} + \text{ higher terms}$$

where $\delta > d_1$.

We will prove the lemma by induction in n. We have

$$W = -\sum_{i=1}^{n} (-1)^i f_i W(f_1', \ldots, \widehat{f_i'}, \ldots, f_n').$$

The degree of i-th summand in the right-hand side is

$$d_i + \delta - 1 + \sum_{j \neq 1, i} (d_j - 1) - \sum_{j=1}^{n-2} j = \delta + \sum_{j=2}^{n} d_j - \sum_{j=1}^{n-1} j$$

for $i \neq 1$ and

$$d_1 + \sum_{j=2}^{n} d_j - \sum_{j=1}^{n-1} j$$

for $i = 1$. We see that component corresponding to $i = 1$ has degree strictly less than the rest of the summands, so it will give the degree of the sum. □

10.8.3. **Corollary.** *If f_i are linearly independent, $W(f_1, \ldots, f_n)$ is not identically zero.*

In what follows we will call *the weight of V at x* the degree of the Wronskian at x. As we saw above it is given by the formula

$$\mathrm{wt}_x V = \sum_{i=1}^{n} (d_i - i + 1).$$

10.9. **Weierstrass points.** We apply the constructions of the previous subsection to the space $\mathcal{H}$ of holomorphic differentials.

In a coordinate neighrorhood U holomorphic one-forms can be written as $f(z)dz$ so one can directly apply the above constructions. Note that if one makes a change of coordinates $z = z(w)$, the differentials $\zeta_i = f_i(z)dz$ become $f_i(z(w)z'_w dw$ and one can easily see that the Wronskian changes to

$$W(z(w))(z'_w)^m$$

where $m = \frac{g(g+1)}{2}$. Thus, the wronskian of a sytem of g one-forms is an m-differential. That is the only difference.

10.9.1. **Definition.** A point $x \in X$ of a compact Riemann surface of genus g is called Weierstrass point if $\mathrm{wt}_x \mathcal{H} \neq 0$.

There are no Weierstrass points for $g \leq 1$: for $g = 0$ there are no holomorphic differentials and for $g = 1$ the only holomorphic differential has no zeroes.

In what follows we assume $g > 1$.

10.9.2. **Proposition.** *The point x is Weierstrass iff one of the following equivalent conditions holds.*

- $i(g \cdot x) > 0$.
- $r(-g \cdot x) \geq 2$.

Proof. $\mathrm{wt}_x \mathcal{H} > 0$ means that there is a holomorphic differential ω with $\deg_x \omega > g$. This means the first condition. The equivalence of the two conditions follows from RR applied to $D = g \cdot x$. □

Note that the second condition is equivalent to saying that $\alpha_1 \leq g$. Thus, the usage of Wronskian immediately implies that for the non-Weierstrass points one has $\alpha_i = g + i,\ i = 1, \ldots, g$.

Let W be the Wronskian of a basis of holomorphic differentials. We have explained that W is an m-differential where $m = \frac{g(g+1)}{2}$. It is holomorphic, therefore it has precisely $\deg W = m(2g - 2)$ zeroes. Thus, we have

10.9.3. **Proposition.**

$$\sum_{x\in X}\deg_x W=(g-1)g(g+1).$$

10.9.4. **Corollary.** *Let $g\geq 2$. There exist Weierstrass ponts.*

10.9.5. **Theorem.** *For each $x\in X$ one has $\mathrm{wt}_x\,\mathcal{H}\leq\frac{g(g-1)}{2}$. The equality happens only when the gap sequence starts with 2.*

Proof. Fix $x\in X$. Let $1<\alpha_1<\ldots<\alpha_g=2g$ be the non-gap sequence and $1=n_1<\ldots<n_g$ be the gap sequence.

Recall that by RR we have

$$r(-nx)-r(-(n-1)x)=1+i(nx)-i((n-1)x)$$

which implies that n is a gap iff there are holomorphic differentials having degree $n-1$ at x. Thus, in the above notation

$$\mathrm{wt}_x\,\mathcal{H}=\sum_{j=1}^{g}(d_j-(j-1))=\sum_{j=1}^{g}(n_j-j).$$

Furthermore,

$$\mathrm{wt}_x\,\mathcal{H}=\sum_{j=1}^{g}(n_j-j)=\sum_{j=g+1}^{2g-1}j-\sum_{j=1}^{j-1}\alpha_j\leq\frac{3g(g-1)}{2}-g(g-1)=\frac{g(g-1)}{2}$$

by Corollary 10.7.4. By the same Corollary the equality holds only when $\alpha_1=2$. ∎

10.9.6. **Corollary.** *The number of Weierstrass points of a compact surface of genus g is at least $2g+2$.*

Home assignment.

1. A compact RS X is called *hyperelliptic curve* if it has a meromorphic function f having two poles. Let B be the number of branch points of the covering $f:X\to\widehat{\mathbb{C}}$. Express B through the genus g of X using Riemann-Hurwitz formula.

2 (cont. of 1.) Let $x\in X$ be a branch point for f. Find a meromorphic function on X having a double pole at x. Deduce, using Corollary 10.7.2, that x is a Weierstrass point of X with the non-gap sequence $2,4,\ldots$. Deduce from this that branch points are the only Weierstrass points in the hyperelliptic case.

3. Deduce that if the number of Weierstrass points on X is $2g+2$ then X is hyperelliptic.

4. Let $F:X\longrightarrow X$ be a nontrivial automorphism of X. Choose $x\neq F(x)$ and let f be a meromorphic function with an only pole at x and $r:=\deg_x f\leq g+1$. Thus, the polar divisor of f is $r\cdot x$. Deduce that the function $h:=f-f\circ F$ has

at most $2g+2$ zeroes. Deduce from this that F cannot have more than $2g+2$ fixed points.

5. Assume now that X is *not* hyperelliptic. Prove that any automorphism of X is uniquely defined by the permutation of Weierstrass points it defines. This proves that the group of automorphisms of a non-hyperelliptic curve is finite.

Note. It is finite even for hyperelliptic curves of genus > 1*. Thus, however, requires a more careful analysis.*