RIEMANN SURFACES

9. Week 10: Compact Riemann Surfaces

9.1. Pairing of H_1 with H^1_{DR}. Intersection pairing on H_1.

9.1.1. Given a piecewise differential path γ and a form ω on X one can integrate ω along γ and get a complex number.

Integration of a closed form along a path satisfies two properties:

1. If γ and γ' are homotopic (with fixed ends) then $\int_\gamma \omega = \int_{\gamma'} \omega$. This follows from Stokes formula.
2. Concatenation of paths corresponds to the sum of the integrals.

In particular, integrating of one-forms defines a bilinear map

$$H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)] \times Z^1 \longrightarrow \mathbb{C},$$

where Z^1 denotes the space of closed one-forms. The restriction to the harmonic forms gives a pairing

$$H_1(X, \mathbb{C}) \times H \longrightarrow \mathbb{C}. \qquad (1)$$

We claim that this pairing is nondegenerate if X is a compact Riemann surface.

This will imply, in particular, that the dimension of the space of harmonic forms on X is $2g$ where g is the genus of X.

Assume that ω is a harmonic form such that $\int_\gamma \omega = 0$ for any closed path γ. Then $\omega = df$ where f is the function defined by the formula

$$f(x) = \int_{x_0}^{x} \omega.$$

the formula defines a single-valued function. The function f is automatically harmonic since

$$d * df = d * \omega = 0.$$

Since X is compact, there are no nonconstant harmonic functions by the maximim principle. Therefore, we deduce that the pairing (1) defines an injective map

$$H \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_1(X), \mathbb{C}).$$

We will immediately check that this map is surjective.

9.1.2.

We wish now to use the standard presentation of a compact Riemann surface of genus g. In case $g = 0$ it is homeomorphic to a sphere which is simply connected

($\pi_1 = 0$). For $g > 0$ X can be obtained from a $4g$-gon by the identification of the edges defined by the word

$$x_1 y_1 x_1^{-1} y_1^{-1} \dots x_g y_g x_g^{-1} y_1^{-1}.$$

The fundamental group of X is generated by the simple loops $a_1, \dots, a_g$ and $b_1, \dots, b_g$ corresponding to the edges x_i and y_j, subject to one relation

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1.$$

The homology group $H_1(X)$ is freely generated (as an abelian group) by the classes of a_i and b_j.

Recall that for a simple closed curve γ we define a closed one-form η_γ such that

$$\int_\gamma \alpha = (\alpha, *\eta_\gamma).$$

The one-forms η_{a_i}, η_{b_j} are not harmonic; however, their orthogonal projections to H, as we will see soon, will suffice to generate the whole $\mathrm{Hom}_{\mathbb{Z}}(H_1(X), \mathbb{C})$.

9.1.3.

Define a pairing on the set of cycles on X by the formula

$$a \cdot b = \int_X \eta_a \wedge \eta_b = (\eta_a, - * \eta_b) \tag{2}$$

(recall that η_γ are real so we do not need complex conjugation).

We will prove below that the pairing so defined "counts" the number of times a intersects b. This will imply, in particular, that

$$a_i \cdot a_j = b_i \cdot b_j = 0; \quad a_i \cdot b_i = 1; \quad a_i \cdot b_j = 0 \ (i \neq j). \tag{3}$$

We will write $a_{g+1}, \dots, a_{2g}$ instead of $b_1, \dots, b_g$. Define the harmnic forms h_i, $i = 1, \dots, 2g$, as the orthogonal projections of the closed forms η_{a_i}. Since $h_i - \eta_{a_i}$ is exact, one has

$$\int_\gamma h_i = \int_\gamma \eta_{a_i},$$

so that the images of h_i form a basis in $\mathrm{Hom}_{\mathbb{Z}}(H_1(X), \mathbb{C})$ dual to $\{a_i\}$. This proves the following

9.1.4. **Theorem.** *The pairing (1) is nondegenerate for a compact Riemann surface. In particular, the dimension of the space of harmonic one-forms on X is $2g$.*

We will now study the properties of the intersection pairing (2).

9.1.5. **Proposition.** *The intersection pairing (2) satisfies the following properties.*

1. *The intersection $a\dot{b}$ depends only on the homology classes of a and b.*
2. *One has $a \cdot b = -b \cdot a$.*
3. *$a \cdot b \in \mathbb{Z}$. In case the intersection points of the curves a and b are transversal, $a \cdot b$ is the (signed) number of intersection points.*

Proof. The first property has already been explained: intergals of a closed form along homotopic paths are the same. The second property results from the anticommutativity of the multiplication of one-forms.

The third property can be checked for simple closed curves since any piecewise smooth closed curve is a finite union of simple closed curves. In this case $a \cdot b = \int_a \eta_b$ and we have to check that each intersection point of a with b contributes $+1$ or -1, depending on the orientation of the curves at the intersection point.

Recall that η_b is defined as differential of a function f_b having a discontinuity along b. The function f_b is zero far away from b. Thus, the integral over a can be presented as a sum of the integrals over small segments of a_i of a containing the intersection points x_i of a with b.

The intergral $\int_{a_i} \eta_b$ has been already calculated once. The result was ± 1. □

As a result, we have deduce that the intersection matrix in the basis $(a_1, \ldots, a_{2g})$ of $H_1(X, \mathbb{C})$ looks like $J := \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$.

In what follows we will work with any fixed basis of $H_1(X, \mathbb{Z})$ having the same intersection matrix. We will can such basis *a canonical basis* of $H_1(X)$. We do not care whether this basis comes from a polygonal presentation of X.

9.1.6. **Corollary.** *For any canonical basis $\{a_i\}$ of H_1 its dual basis $\{\alpha_i | i = 1, \ldots, 2g\}$ of H consists of real-valued harmonic functions.*

Proof. This is the property of all our constructions. The complex conjugation is defined on $L^2(X)$ and the spaces E, E^* and H are invariant with respect to it. Thus, the orthogonal projection to H commutes with the complex conjugation. Since η_{a_i} are real, their projections to H are real. This proves the claim for the choice of a_i derived from the polygonal presentation of X. In general one has to apply a transfer matrix with real (even integral) values.

□

9.2. **Holomorphic one-forms.** We wish to understand how do holomorphic forms lie in the space of harmonic forms. This is a pure linear algebra. Recall that the space H of harmonic complex-valued forms has complex dimension $2g$. Two $\mathbb{R}$-linear operators are defined on H: a complex conjugation and the operator $*$.

9.2.1. Decomposition of H

The operator $\alpha \mapsto \frac{1}{2}\alpha + i * \alpha$ transforms any harmonic form into a holomorphic form and acts identically on holomorphic forms. Its kernel consists of antiholomorphic forms since if $\alpha + i * \alpha = 0$, one has $\bar{\alpha} - i * \bar{\alpha} = 0$ which means that $\bar{\alpha}$ is holomorphic. We denote by $\mathcal{H}$ the space of holomorphic forms.

This proves the following

Theorem. *One has a canonical decomposition*

$$H = \mathcal{H} \oplus \bar{\mathcal{H}}$$

of the space of harmonic forms into the sum of holomorphic and antiholomorphic forms. In particular, $\dim_{\mathbb{C}} \mathcal{H} = g$.

9.2.2. The matrix of the operator $*$

Recall that the space H has a basis $\alpha_1, \ldots, \alpha_{2g}$ dual to a chosen canonical basis $a_1, \ldots, a_{2g}$ of H_1. The intersection form on H_1 is dual to the inner product on H. Since the forms α_i are real, the inner product is given by a symmetric positive definite real $2g \times 2g$ matrix. Denote it $\Gamma = (\Gamma_{i,j}) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$. Note that $A = A^t,\ D = D^t,\ C + B^t$ and that $A > 0,\ D > 0$.

The $*$-operator restricted to H is a linear operator whose square is -1.

We note that $(\alpha_i, \alpha_j) = \Gamma_{i,j}$ and

$$(\alpha_i, *\alpha_j) = -\int_X \alpha_i \wedge \alpha_j = -a_i \cdot a_j = -J_{i,j}.$$

Comparing two formulas above we deduce the formula for the operator $*$. Assume that $*$ is given by a matrix $G = (G_{i,j})$. Then

$$-J_{i,j} = (\alpha_i, *\alpha_j) = (\alpha_i, \sum_k G_{k,j}\alpha_k) = \sum_k \Gamma_{i,k} G_{k,j}$$

so that $-J = \Gamma G$ or, since $G^2 = -1$, $G = J^{-1}\Gamma = -J\Gamma$. Therefore,

$$G = \left(\begin{array}{c|c} -C & -D \\ \hline A & B \end{array}\right).$$

Note that $G^2 = -1$ implies that

$$CD = DB,\ AC = BA,\ C^2 - DA + 1 = 0,\ B^2 - AD + 1 = 0.$$

9.2.3. **Proposition.** *Let* ω *and* ω' *be closed one-forms on* X. *Then*

$$\int_X \omega \wedge \omega' = \sum_{i=1}^{g} \left[\int_{a_i} \omega \int_{a_{g+i}} \omega' - \int_{a_{g+i}} \omega \int_{a_i} \omega' \right].$$

Proof. Both expressions are bilinear in ω, ω'. Both vanish of one of them is closed (Stokes). Thus, we can assume they are harmonic. Since α_i form a basis of harmonic forms, it is sufficient to check the formula from $\omega = \alpha_i,\ \omega' = \alpha_j$ where $i, j = 1, \ldots, 2g$. In this case the claim amount to the formula

$$(a_i, a_j) = J_{ij}.$$

□

In what follows we will be willing to use a generalization of the above formula for pairs ω, ω' with ω' meromorphic, see formula (4).

9.2.4. **Corollary.** *If $\omega \in H$, one has*

$$||\omega||^2 = \sum_{i=1}^{g} \left[\int_{a_i} \omega \int_{a_{g+i}} *\bar{\omega} - \int_{a_{g+i}} \omega \int_{a_i} *\bar{\omega} \right].$$

Let now $\omega \in \mathcal{H}$. Denote for $i = 1, \ldots, g$

$$A_i = \int_{a_i} \omega,\ \ B_i = \int_{a_{g+i}} \omega.$$

The numbers A_i and B_i are called *a-periods and b-periods* of ω.

Then one has

9.2.5. **Corollary.**

$$||\omega||^2 = i \sum_{i=1}^{g} (A_i \bar{B}_i - B_i \bar{A}_i)$$

The latter result implies that if a holomorphic one-form ω has vanishing a-periods then $\omega = 0$. The same is true for a holomorphic form having real all a- and b-periods.

9.2.6. Period matrix of X

Choose an arbitrary basis $\zeta_1, \ldots, \zeta_g$ of $\mathcal{H}$. We claim that the matrix

$$i, j \mapsto \int_{a_i} \zeta_j$$

is nondegenerate.

In fact, if this were not true, a certain linear combination $\theta = \sum c_i \zeta_i$ of the basic holomorphic one-forms would have vanishing a-periods.

Therefore, there exists a unique basis of $\mathcal{H}$ satisfying the condition

$$\int_{a_i} \zeta_j = \delta_{i,j}, i = 1, \ldots, g.$$

We define the *period matrix* B of a compact Riemann surface X by the formulas

$$B_{i,j} = \int_{b_i} \zeta_j.$$

Applying Proposition 9.2.3 to the forms ζ_i, ζ_j one gets $B_{i,j} = B_{j,i}$. Thus, the period matrix is symmetric.

Finally, applying Corollary 9.2.5 to $\theta = \sum c_i \zeta_i$, we get its periods

$$A_i = c_i,\ B_i = \sum B_{i,j} c_j,$$

so that

$$||\theta||^2 = i \sum_{j,k=1}^{g} (c_j \bar{B}_{j,k} \bar{c}_k - \bar{c}_j B_{j,k} c_k)$$

which implies that $\operatorname{Im} B > 0$.

We have therefore proven that the period matrix B is symmetric and its imaginary part is positively definite.

Note that our construction of B depends on the choice of a canonical basis $a_1, \ldots, a_{2g}$. Another choice of canonical basis, $a'_1, \ldots, a'_{2g}$, is described by a transfer matrix $C \in GL(2g, \mathbb{Z})$ (more precisely, $C \in Sp(2g, \mathbb{Z})$). In the same way the dual basis $\alpha_1, \ldots, \alpha_{2g}$ is being changed. It is, however, difficult to describe what happens with the basis $\zeta_1, \ldots, \zeta_g$ of holomorphic forms defined by the conditions $\int_{a_i} \zeta_j = \delta_{i,j},\ i, j = 1, \ldots, g$.

Therefore, the Period matrix of a compact Riemann surface is not uniquely defined — it depends on the choice of a canonical basis in H_1.

9.3. **Periods of meromorphic differentials.** Let τ be a meromorphic differential and let γ be a closed loop wich does not pass through the poles of τ. The integral $\int_\gamma \tau$ is still defined, but the result may change if one replaces γ with a homotopic curve γ'.

If τ has zero residues, this problem does not appear. Otherwise one can use integration along specific paths, but it is forbidden to replace carelessly a path with a homotopic one.

A typical problem one can ask is as follows.

Let τ be a fixed meromorphic differential. We can add a linear combination of ζ_i to make sure that the a-periods of τ vanish. If τ were holomorphic, it would have to be zero, so that its b-periods would also vanish. However, since τ is merely meromorphic, the b-periods have some interesting values.

In order to be able to apply the bilinear relations to τ, we will present another approach to this formula.

Let θ and τ be closed one-forms.

We present the Riemann surface X as the quotient of the polygon $\mathcal{P}$ by the relation identifying its edges. Then the integral

$$\int_X \theta \wedge \tau$$

can be calculated as the integral over $\mathcal{P}$ of the inverse image of the corresponding two-form. We will denote it by the same letter as the original forms.

Now θ is a closed form on a contractible surface $\mathcal{P}$, so it is exact, $\theta = df$ where f is a smooth function on $\mathcal{P}$ (having in general different values at the boundary points of $\mathcal{P}$ which are identified in X.

We have

$$\int_X \theta\wedge\tau = \int_{\mathcal{P}} df\wedge\tau = \int_{\mathcal{P}} d(f\tau) = \int_{\partial\mathcal{P}} f\tau = \sum_{j=1}^{g}(\int_{a_j} f\tau + \int_{b_j} f\tau + \int_{a_j^{-1}} f\tau + \int_{b_j^{-1}} f\tau).$$

Now the sum

$$\int_{a_j} f\tau + \int_{a_j^{-1}} f\tau$$

can be easily calculated. In fact, τ is the same along a_j and a_j^{-1} and the difference of values of f at different sides is constant, equal to $\int_{b_i}\theta$. Thus, the above sum is equal to

$$\int_{a_j}\tau\int_{b_j}\theta.$$

Making the similar calculation with the second pair of summands, we get 9.2.3.

Let now θ be holomorphic and let τ be a meromorphic differential. We will assume that τ has no poles on the boundary $\partial\mathcal{P}$.

We can still write

(4)

$$\int_{\partial\mathcal{P}} f\tau = \sum_{j=1}^{g}(\int_{a_j} f\tau + \int_{b_j} f\tau + \int_{a_j^{-1}} f\tau + \int_{b_j^{-1}} f\tau) = \sum_{j=1}^{g}(\int_{a_j}\theta\int_{b_j}\tau - \int_{a_j}\tau\int_{b_j}\theta).$$

The left-hand side can be calculated using Residue theorem, and the right-hand side has the values we are interesed in.

9.3.1. Let $\tau_{x,y}$ be a meromorphic differential having two poles with residues 1 and -1 at points x, y of X. Normalize $\tau_{x,y}$ so that it has vanishing a-periods.

Apply the formula (4) to $\theta := \zeta_k$ and $\tau = \tau_{x,y}$. The left-hand side yields $2\pi i(f(x) - f(y)) = 2\pi i \int_y^x \zeta_j$. Thus,

$$2\pi i \int_y^x \zeta_j = \int_{b_j} \tau_{x,y}. \tag{5}$$

9.3.2. Another possibility is to take $\tau = \tau_x^{(n)}$ to be a meromorphic differential with the only pole at $x \in X$, behaving as $\frac{dz}{z^n}$ near x. We normalize $\tau_x^{(n)}$ so that it has zero a-periods and we wish to express its b-periods.

Similarly to the above we choose $\theta = \zeta_j$ and we get

$$\int_{b_j} \tau_x^{(n)} = \int_{\partial \mathcal{P}} f \tau_x^{(n)}.$$

If $\zeta_j = \sum_{k \geq 0} c_k^{(j)} z^k dz$ at the neighborhood of x, so that $f = \sum_{k \geq 0} \frac{c_k^{(j)}}{k+1} z^{k+1}$, we get

$$\int_{b_j} \tau_x^{(n)} = \frac{2\pi i}{n-1} c_{n-2}^{(j)}. \tag{6}$$

Home assignment.

1. Let a canonical basis $a_1, \ldots, a_{2g}$ of H_1 correspond to a polygonal presentation of X. Write down explicit formulas expressing α_i via η_{a_j}.

2. Let $X = \mathbb{C}/L$ be an elliptic curve. Find a holomorphic differential on X and calculate the period matrix in this case. Show explicitly how the answer depends on the choice of a canonical basis.