RIEMANN SURFACES

8. Week 9: Harmonic forms, meromorphic forms, meromorphic functions

8.1. Existence of harmonic differentials.

8.1.1. **Theorem.** *Let X have a closed differential $\omega \in L^2(X)$ which is not exact. Then $H \neq 0$. The converse is true if X is compact.*

Proof. If ω is closed, $\omega = \alpha + h$ where $\alpha \in E$ and $h \in H$. If ω is not exact, there exists a closed curve γ such that $\int_\gamma \omega \neq 0$. Note that $\alpha = \omega - h$ is C^1, so $\int_\gamma \alpha = (\alpha, *\eta_\gamma) = 0$. Therefore, $\int_\gamma h \neq 0$ which implies that $h \neq 0$.

If X is compact and $h \neq 0$ then h itself is closed and not exact. □

Let now γ be a simple closed curve in X such that $X - \gamma$ is connected. Then there exists another simple closed curve γ^* intersecting γ in exactly one point $P \in X$.

One has

$$\int_{\gamma^*} \eta_\gamma = \lim_{Q \to P^-} f(Q) - \lim_{Q \to P^+} f(Q) = 1. \tag{1}$$

This yields the following

8.1.2. **Theorem.** *Let a Riemann surface X admits a closed curve γ such that $X - \gamma$ is connected. Then X admits a nonzero harmonic one-form.*

8.2. Decomposition of smooth forms.

Let $\omega \in L^2(X)$. We have an orthogonal decomposition

$$\omega = \alpha + \beta + h$$

where $\alpha \in E$, $\beta \in E^*$ and $h \in H$. We also know that if $\omega \in C^1 \cap E$ then ω is exact and if $\omega \in E^* \cap C^1$ then it is coexact. However, we can not deduce smoothness of α and β from the smoothneww of ω. We intend to fix this.

In this subsection we will prove the existence of a decomposition

$$\omega = df + *dg + h, \tag{2}$$

for $\omega \in L^2 \cap C^3$ where $h \in H$ and f, g are function of class C^2.

As a first step we will look for a function g such that the difference $\omega - *dg$ is closed.

The latter is equivalent to saying that $d\omega - d * dg = 0$ that is

$$\Delta g = d\omega. \tag{3}$$

8.2.1. **Proposition.** *Let f be a C^2 function with a compact support on $\mathbb{C}$. Then the Poisson equation*

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = f$$

has a solution $g \in C^2$.

Proof. We will prove that the formula

$$g(z) = -\frac{i}{4\pi} \int_{\mathbb{C}} \log|z-\zeta| f(\zeta) d\zeta \wedge d\bar{\zeta} \tag{4}$$

gives a C^2 solution.

Passing to the polar coordinates $\zeta - z = r(\cos\phi + i\sin\phi)$, one gets $d\zeta \wedge d\bar{\zeta} = -2ir dr \wedge d\phi$. This ensures that the integral (4) converges since $\log(r)r \to 0$ for $r \to 0$.

In order to apply a differential operator to the function given by (4), it suffices to apply it to the function f. Thus, the Laplace operator of g is given by the formula

$$\Delta g = -\frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{\infty} \log r \left[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} \right] r dr \right) d\phi. \tag{5}$$

Recall that f has a compact support, so the limits of integration along r can be talen as $[0, R]$. Then

$$\Delta g = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_0^{2\pi} \left(\int_{\epsilon}^{R} \left\{ \frac{\partial}{\partial r}\left[r \log r \frac{\partial f}{\partial r} - f \right] + \frac{\partial}{\partial \phi}\left(\frac{\log r}{r} \frac{\partial f}{\partial \phi} \right) \right\} dr \right) d\phi. \tag{6}$$

We can now use Stokes formula. The first summand will disappear, so we get

$$\Delta g = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_0^{2\pi} \left[\epsilon \log \epsilon \frac{\partial f}{\partial r}|_{r=\epsilon} - f|_{r=\epsilon} \right] d\phi. \tag{7}$$

The expression $\epsilon \log \epsilon \to 0$ when $\epsilon \to 0$ and the absolute value of $\frac{\partial f}{\partial r}$ is bounded, so the first summand tends to zero; the second summand yields what we need. □

Using Proposition 8.2.1 we can get a decomposition for smooth forms. The first result below is a local claim; the second one is what we really intended to get.

8.2.2. **Lemma.** *Any form $\omega \in C^3$ can be locally decomposed as*

$$\omega = df + *dg$$

where f, g are functions of class C^2.

Proof. Let $x \in X$. Choose a coordinate chart around x so that $z = 0$ corresponds to the point x. Choose a smooth function $s(z)$ in D which is 1 for $|z| < \frac{1}{3}$ and 0 for $|z| > \frac{2}{3}$. The product $s\omega$ is a C^3 one-form with compact support; therefore, by Proposition 8.2.1 ensures there exists a function g in D such that $s\omega - *dg$ is closed and, therefore, exact. □

8.2.3. **Theorem.** *Let $\omega \in L^2(X)$ be of type C^3. Then in the orthogonal decomposition*

$$\omega = h + \alpha + \beta$$

with $h \in H,\ \alpha \in E$ and $\beta \in E^*$ the forms α and β are of class C^1.

Note that the theorem implies that $\alpha = df,\ \beta = *dg$ for f, g of class C^2.

Proof. Let $\omega = h + \alpha + \beta$ with $h \in H,\ \alpha \in E$ and $\beta \in E^*$.

Let $x \in X$. Choose a local coordinate around x so that the unit disc $D = \{z||z| < 1\}$ identifies with an open neighborhood of x.

By Lemma 8.2.2 the restriction of ω to D can be presented as $df + *dg$. Note that df is closed and therefore belongs to $E(D) \oplus H(D)$; similarly $*dg$ belongs to $E^*(D) \oplus H(D)$. At the same time $h|_D + \alpha|_D$ belongs to $E(D) \oplus H(D)$ and $\beta|_D$ belongs to $E^*(D) \oplus H(D)$.

Thus, the element $h|_D + \alpha|_D - df = *dg - \beta|_D$ is harmonic in D. This implies that $\alpha|_D$ and $\beta|_D$ are of class C^1.

□

8.3. **Harmonic differentials with poles.** Harmonic differentials on a compact Riemann surface X can not be exact: otherwise they would belong to the intersection $E \cap H$.

We will construct now harmonic differentials having a pole .

Let X be any Riemann surface and $x \in X$. Choose a chart in a neighborhood of x with a local coordinate so that the neighborhood identifies with the unit disc D and x corresponds to $z = 0$.

Fix $n \geq 1$.

We denote $\delta = d\left(\frac{1}{z^n}\right)$. This is an exact differential on $D - \{x\}$.

8.3.1. **Theorem.** *There exists a unique differential ω on $X - \{x\}$ satisfying the following properties.*

(a) *ω is harmonic and exact on $X - \{x\}$.*
(b) *$\omega - \delta$ is harmonic in a neighborhood N of x.*
(c) $||\omega_{X-N}|| < \infty$.
(d) *For any function ϕ on X of class C^2 such that $||d\phi|| < \infty$ and $\phi = 0$ at a neighborhood of x, one has $(\omega, d\phi) = 0$.*

Proof. Let us start with uniqueness. Assume two differentials, ω_1 and ω_2 satisfy the conditions of the theorem. Their difference is harmonic since its restriction to $X - \{x\}$ and to N is harmonic. It is exact, therefore,

$$\omega_1 - \omega_2 = dh$$

where h is a harmonic function on X.

Choose a smooth function ρ on $[0, 1]$ so that $\rho(t) = 1$ for $t < \frac{1}{3}$ and $\rho(t) = 0$ for $t > \frac{2}{3}$.

Define a new function g on X by the formula

$$g = \begin{cases} \rho(|z|)h(z), & z \in D \\ 0, & \text{in } X - D. \end{cases} \tag{8}$$

Then g has a compact support and $g = h$ at $N := \{z||z| < \frac{1}{3}$. Then

$$||d(h - g)|| \leq ||dh||_{X-N} + ||dg||_{X-N} \leq ||\omega_1||_{X-N} + ||\omega_1||_{X-N} + ||dg|| < \infty,$$

so by the property (4) applied to $\phi = g - h$ we get

$$||\omega_1 - \omega_2||^2 = (\omega_1 - \omega_2, dh) = (\omega_1 - \omega_2, d(h - g)) + (\omega_1 - \omega_2, dg) = 0$$

since $dg \in E$ and $\omega_1 - \omega_2 \in H$.

Let us now prove the existence.

Define

$$\theta = \begin{cases} d\left(\frac{\rho(|z|)}{z^n}\right), & z \in D \\ 0, & \text{in } X - D. \end{cases}$$

We will define ω be the formula

$$\omega = \theta - df$$

where the function $f \in C^2$ will be determined later on.

The form ω so defined is automatically exact since θ and df are exact. Harmonicity of ω in $X - \{x\}$ is equivalent therefore to the condition

$$0 = d * \theta - d * df. \tag{9}$$

Since θ and δ coincide in N, the property (b) is satisfied automatically if f is as in (9): $\omega - \delta = df$ in N and f satisfying (9) is harmonic in N since $d*\theta = d*\delta = 0$ in N.

The property (c) is satisfied automatically if $df \in L^2$ since $||\omega||_{X-\bar{N}} \leq ||\theta||_{X-\bar{N}} + ||df||_{X-\bar{N}}$.

The formula (9) defines f uniquely up to a harmonic function. The easiest way would be to decompose θ into $h + \alpha + \beta$ with $\alpha \in E$ and $\beta \in E^*$; we do not do this literally since θ has a pole. However, the difference $\theta - i * \theta$ vanishes in N since it coincides there with $\delta - i * \delta = 0$ since δ is holomorphic in $D - \{x\}$.

Thus, $\theta - i * \theta = 0$ in N as well as in $X - D$. Therefore, $\theta - i * \theta \in \Lambda^1_{fin}(X)$. In particular, it can be decomposed as in Theorem 8.2.3 as

$$\theta - i * \theta = h + \alpha + \beta \tag{10}$$

where $h \in H,\ \alpha \in E,\ \beta \in E^*$, and $\alpha = du, \beta = *dv$ for some $u, v \in \Lambda^0(X)$. Note that α and β belong to $\Lambda^1_{fin}(X)$ by the construction.

Applying $d*$ to (10) we get

$$d * \theta = d * du$$

which is a possible solution for f. That is we will have to choose f different from u by a harmonic function χ, so that now $\omega = \theta - du - d\chi = i * \theta + h - d\chi + *dv$. The only requirement is that we have to satisfy property (d).

Let ϕ be a function as in (d). We have

$$(\omega, d\phi) = i(*\theta, d\phi) + (h - d\chi, d\phi)$$

since $(*dv, d\phi) = 0$ (any exact form is orthogonal to E^*). Since ϕ vanishes in a neighborhood of x,

$$(*\theta, d\phi) = \int_A \theta \wedge d\phi = \int_A \phi d\theta + \int_{\partial A} \phi\theta = 0$$

where A is the annulus $\epsilon < |z| < 1$.

Thus, we have to find a harmonic function χ such that $h - d\chi$ is orthogonal to all exact forms in $L^2(X)$. This is left as an exercise. □

Note that we were unable to construct a harmonic one-form with a first order pole (compare to our discussion of elliptic functions).

The following variation of the above theorem allows one to construct a harmonic form with *two* simple poles.

8.3.2. **Proposition.** *Let $x \in X$ be as in Theorem 8.3.1 and let $y \in N$ where N is defined as above. Let now*

$$\delta = (\frac{1}{z-x} - \frac{1}{z-y})dz.$$

There exists a unique form ω on $X - \{x, y\}$ such that

(a) *ω is harmonic in $X - \{x, y\}$ and exact in $X - \bar{N}$.*
(b) *$\omega - \delta$ is harmonic in N; ω and δ have the same periods in N.*
(c) *$||\omega_{X-N}|| < \infty$.*
(d) *For any function ϕ on X of class C^2 such that $||d\phi|| < \infty$ and $\phi = 0$ in N, one has $(\omega, d\phi) = 0$.*

Proof. One that $\delta = d(\log \frac{z-x}{z-y})$. Since the resudues at x and at y cancel, the function $\log \frac{z-x}{z-y})$ is single-valued outside N. Thus, δ is exact outside N.

The construction goes as in the previous case, but with a new form θ defined by the formula

$$\theta = \begin{cases} d\left(\rho(|z|) \log \frac{z-x}{z-y}\right), & z \in D \\ 0, & \text{in } X - D. \end{cases}$$

□

8.4. **Meromorphic differentials and meromorphic functions.** Recall that one of our declared aims has been to construct a nonconstant meromorphic function on any Riemann surface. We are now very close to fulfilling the promise.

8.4.1. Meromorphic one-forms

A meromorphic differential on X can be defined as a holomorphic one-form on $X - S$ where S is a discrete subset of "singular points", with an extra condition that near each singular point it looks in the local coordinates as $f(z)dz$ where $f(z)$ has a pole at 0.

Another way of defining a meromorphic one-form is to mimic the definition of a one-form, using meromorphic functions as coefficients.

In any case, if ω is the harmonic form constructed in Theorem 8.3.1, the expression $\frac{1}{2}(\omega + i * \omega)$ is holomorphic outside X and coincides with $\delta = d(\frac{1}{z^n})$ inside N. Thus, it is meromorphic, with a pole of degree $n+1$ at x.

Similarly, we can construct a meromorphic differential having two order 1 poles at close points x, y with residues differing by signs.

8.4.2. Meromorphic one-forms versus meromorphic functions

If ω is a meromorphic one-form and f is a meromorphic function on X, the product $f\omega$ is a meromorphic one-form. It turns out that, vice versa, any two (non-zero) meromorphic differentials can be obtained one from another by multiplying to a meromorphic function.

Lemma. *Let ω_1 and ω_2 be two non-zero meromorphic differentials. There exists a unique meromorphic function f on X such that $\omega_2 = f\omega 1$.*

Proof. In any coordinate chart we have $\omega_i = g_i dz$ where g_i are nonzero meromorphic functions. We define $f = \frac{g_2}{g_1}$. This gives a meromorphic function in a chart. Now it is enough to prove that this collection of meromorpihc functions is compatible with the coordinate change.

If $z = z(w)$ is the transition function, we have

$$\omega_i = g_i(z)dz = g_i(z(w))z'_w dw,$$

so that the fraction $\frac{g_2(z(w))z-_w}{g_1(z(w))z-_w}$ is equal to $\frac{g_2(z(w))}{g_1(z(w))}$. This proves the assertion. □

Corollary. *Any Riemann surface admits a nonconstant meromorphic function.*

Proof. We know there exists a meromorphic differential with a pole at a given point x. Divide it to a meromorphic differential with a pole at another point. The meromorphic function we get cannot be constant. □

8.4.3. Residues

If ω is a meromorphic one-form and $x \in X$, we define the residue

$$\mathrm{Res}_x(\omega) = \frac{1}{2\pi i} \int_\gamma \omega$$

where γ is a small circle around x (properly oriented). As we know, $\mathrm{Res}_x(\omega) = a_{-1}$ where $\omega = f(z)dz$ and $f(z) = \sum a_i z^i$ is the Laurent expansion near z.

Proposition. *Let X be compact and let ω be a meromorphic one-form. Then*

$$\sum_{x \in X} \mathrm{Res}_x(\omega) = 0.$$

Proof. We can triangulate X so that all poles of ω are in the interiors of the triangles. Let $\Delta_1, \ldots, \Delta_n$ are the triangles of our triangulation. Then

$$\sum_{x \in X} \mathrm{Res}_x(\omega) = \sum_{i=1}^{n} \frac{1}{2\pi i} \int_{\partial \Delta_i} \omega.$$

The right-hand side of the equation vanishes since each edge of the triangulation appears in it twice with the opposite signs. □

8.4.4. Triangulability

We can now deduce that the compact Riemann surfaces admit a triangulation.

We can interpret a meromorphic function f on X as a holomorphic map

$$f : X \to \widehat{\mathbb{C}}.$$

If f is non-comstant and X is compact, it defines a *ramified covering* of $\widehat{\mathbb{C}}$. Recall that there is a finite set S of ramification points and f is a nonramified covering outside S.

Choose a triangulation of the Riemann sphere $\widehat{\mathbb{C}}$ so that the points of S are (a part of) the vertices of the triangulation. The inverse image of his triangulation yields a triangulation of X: its vertices are preimages of the vertices of $\widehat{\mathbb{C}}$, its edges and triangles are closures of the connected components of the preimages of (the interiors of) the edges and the trianges downstairs.

8.5. **Riemann-Hurwitz formula.** Riemann-Hurwitz formula calculates the genus of a compact Riemann surface X in terms of the ramification data corresponding to a nonconstant meromorphic function

$$f : X \longrightarrow \widehat{\mathbb{C}}.$$

Let $S \subset \widehat{\mathbb{C}}$ be the ramification locus and let $R = f^{-1}(S)$. For each point $x \in R$ we define its *branch number* $b(x)$ to be $n - 1$ if n is the order of zero of the function $f(z) - f(x)$ at x (the dafinition has to be adjusted if $f(x) = \infty \in \widetilde{\mathbb{C}}$, see Exercise 6).

Then we have

8.5.1. **Theorem.** *One has the equality*

$$g(X) = 1 - n + \frac{B}{2}$$

where n is the degree of f and $B = \sum_{x \in R} b(x)$.

The proof is easy one one knows the notion of Euler characterstic.

8.5.2. Euler characteristic Let X be a finite simplicial complex of dimension n, so that X_i denotes the set of i-simplices. Its Euler characteristic $\chi(X)$ is defined by the formula

$$\chi(X) = \sum_{i=0}^{n} (-1)^i |X_i|.$$

The notion of Euler characteristic would have no sense of it depended on a triangulation. Fortunately, this is not so.

Proposition. *Let X be a finite simplicial complex and let $h_i = rkH_i(X, \mathbb{Q})$ (it is called i-th Betti number of X). Then one has*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i h_i.$$

Proof. Recall that the homology of X is defined as the homology of a certain *chain complex* $C_*(X)$. This means that $H_i = Z_i/B_i$ where $Z_i = \{z \in C_i | dz = 0\}$ and $B_i = d(C_{i+1})$. In our case we are interested in homology with rational coefficiens, which means that C_i is the $\mathbb{Q}$-vector space generated by X_i (rather than the free abelian group).

Thus, C_* is a finite complex of finite dimensional vector spaces over $\mathbb{Q}$. By definition of C_i, Z_i, B_i and H_i we have the equalities

$$h_i = \dim H_i = \dim Z_i - \dim B_i.$$

$$\dim B_i = \dim C_{i+1} - \dim Z_{i+1}.$$

Taking the alternating sum of these equalities we finally get

$$\sum (-1)^i \dim C_i = \sum (-1)^i h_i.$$

This proves Proposition. □

Note that we have not proven the independence of $H^i(X)$ of a triangulation. However, in our spacial case of compact Riemann surfaces, we know the answer:

$$h_0(X) = h_2(X) = 1, \; h_1(X) = 2g$$

where g is the genus of X. Thus, $\chi(X) = 2 - 2g$.

8.5.3. Proof of Theorem 8.5.1

Choose a triangulation of $\widehat{\mathbb{C}}$ so that the ramification points are its vertices. Let $V,\ E,\ F$ are the number of vertices, edges and faces in this triangulation.

We have $2 = \chi(\widehat{\mathbb{C}}) = V - E + N$. Lift the triangulation to X; The number of edges and faces in it will be nE and nF where n is the degree of $f : X \to \widehat{\mathbb{C}}$. However, the number of vertices V_X can be calculated by the formula

$$nV = \sum (b(x) + 1) = B + V_X$$

. Thus, we get

$$2 - 2g = V_X - E_X + F_X = n(V_E + F) - B = 2n - B.$$

This proves the theorem.

Home assignment.

1. Check the equality (1).

2. Let $\widetilde{E}$ be the closure in $L^2(X)$ of the space of differentials df with $f \in C^2$ and $||df|| < \infty$. Check that $E \subset \widetilde{E} \subset E \oplus H$. Define $\widetilde{H} = \widetilde{E}^{\perp} \cap (E \oplus H)$. Prove $E \oplus H = \widetilde{E} \oplus \widetilde{H}$ and $\widetilde{H} \subset H$.

3. Complete the proof of Theorem 8.3.1.